A note on Leinster groups

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Introduction For the course "Computer Algebra" that was taught by Wieb Bosma on the Radboud University Nijmegen in the spring of 2020, I studied Leinster groups: Finite groups such that the orders of its normal subgroups sum to twice its group order. Here, I summarize my results I could not find in the literature. All computations were performed in Magma [1]. From here on, all groups are assumed to be finite.

The notation $D(G) := \sum_{N \trianglelefteq G} |N|$ is used to denote the sum of the orders of the normal subgroups of a group G, and $\delta(G)$ for its quotient, that is $\frac{D(G)}{|G|}$. So a group G is Leinster if and only if $\delta(G) = 2$. In [4], it was proven that both D and δ are multiplicative with respect to direct products of two groups with disjoint Jordan-Hölder decompositions. Similarly, $D(n) := \sum_{d|n} d$ and $\delta(n) := \frac{D(n)}{n}$.

Odd Leinster groups There is one known Leinster group of odd order [2] and is due to François Brunault in reply to Tom Leinster's question on the existence of such groups [5]. The group is $(C_{127} \rtimes C_7) \times C_{3^4 \cdot 11^2 \cdot 19^2 \cdot 113}$ and is of order 355433039577. I found another example:

$$(C_7 \rtimes C_{3^2}) \times (C_{19^2} \rtimes C_5) \times C_{11^2 \cdot 197} \cong \mathrm{SmallGroup}(63,1) \times \mathrm{SmallGroup}(1805,2) \times C_{23837}.$$

It is Leinster due to the multiplicative property of δ and as

$$\delta(C_7 \rtimes C_{3^2})\delta(C_{19^2} \rtimes C_5)\delta(C_{11^2 \cdot 197}) = \frac{95}{63} \cdot \frac{2167}{1805} \cdot \frac{133}{121} \cdot \frac{198}{197} = \frac{5 \cdot 19}{3^2 \cdot 7} \cdot \frac{11 \cdot 197}{5 \cdot 19^2} \cdot \frac{7 \cdot 19}{11^2} \cdot \frac{2 \cdot 3^2 \cdot 11}{197} = 2.$$

This group is of order 2710624455 and thus more than 100 times smaller than the known example. Here, SmallGroup(a, b) denotes the bth group of order a according to the database of GAP [3].

ZM-groups Assume one wants to construct a Leinster group from a group G such that $\delta(G) < 2$. Then, due to δ's multiplicativity, one can keep taking direct products with cleverly chosen groups until a Leinster group is reached. Cyclic groups of prime power order are a good choice as they have few normal subgroups and are easy to enumerate. In [4], this method was first performed and in [2] implemented in an algorithm: the Cyclic Extension Method. In [2], extending Zassenhaus Metacyclic groups (ZM-groups) of the form ZM$(m, 2^t, \text{-}1)$ with m odd and $t > 0$ tended to be the most successful. According to [2, Corollary 5.10],

$$\delta\big(\mathrm{ZM}(m, 2^t, \text{-}1)\big) = 1 + \delta(m)\Big(1 - \frac{1}{2^t}\Big),$$

which simplifies to $1 + \delta(m2^{t-1})/2$. Now, if G and ZM$(m, 2^t, \text{-}1)$ have a disjoint Jordan-Hölder decomposition and ZM$(m, 2^t, \text{-}1) \times G$ is Leinster, then

$$\delta(m2^{t-1})\frac{D(G)}{2|G| - D(G)} = 2. \qquad (0.1)$$

For a given group G, one can use the Cyclic Extension Method to find an $n = m2^{t-1}$ and connect this to the correct ZM-group, inducing a Leinster group.

In [2], it was noted that for many ZM-groups, the order of the cyclic group found by the Cyclic Extension Method has a few small prime factors and is much smaller than the order of the ZM-group, and in the dihedral case, often is a single, relatively small prime. Hence, I performed the algorithm for $G = C_p$ with p prime and $n = m2^{t-1}$, for which 0.1 simplifies to

$$\delta(n)\frac{p+1}{p-1} = 2. \tag{0.2}$$

Using this, with $p < 6 \cdot 10^8$ prime, I found 1086 Leinster groups of which most did not appear in the lists of [2]. For example, ZM(59278264911515460287033648022416l3297, 1, -1) $\times C_{458752001}$. Note that for some p, multiple n satisfy 0.2, inducing distinct Leinster groups.

Recursion One can also construct Leinster groups of the form ZM$(m, 2^t, \text{-}1) \times C_p$ recursively. Let n and k be positive integers satisfying 0.2: $\delta(n)\frac{k+1}{k-1} = 2$. Then this formula can be rewritten into $\frac{4n}{2n-D(n)} = (k+1)$, so $\big(2n - D(n)\big)|4n$. New solutions can be obtained by replacing n with $n \cdot q$ for a suitable prime q. So $\big(2nq - D(nq)\big) = \big(q(2n - D(n)) - D(n)\big)|4nq$, and

$$q := \frac{d + D(n)}{2n - D(n)},$$

where $d|4n$ and q is prime, suffices. This only produces a solution to $\delta(n)\frac{k+1}{k-1} = 2$, and for a Leinster group, $k = \frac{4nq}{d} - 1$ needs to be a prime too. So, if for a given n and a divisor d of $4n$, q is prime then that solution can be saved and used as a new n to find Leinster groups. The only expensive steps are factorizing the initial n and the primality testing of q and k.

Taking 2^t with $t > 0$ as initial n implies that q is always an integer in the first step as $2n - D(n) = 1$. I applied this algorithm recursively for all $n = 2^t$ with $t = 1, \ldots, 250$, $q < 10^{100}$ and $k < 10^{150}$ to find a total of 381 Leinster groups including many new examples. For example, ZM$(5, 4, \text{-}1) \times C_{19}$, ZM$(5 \cdot 11, 4, \text{-}1) \times C_{109}$, ZM$(5 \cdot 11 \cdot 59, 4, \text{-}1) \times C_{1297}$, ZM$(5 \cdot 11 \cdot 59 \cdot 659, 4, \text{-}1) \times C_{77761}$, ZM$(5 \cdot 11 \cdot 59 \cdot 659 \cdot 77761, 4, \text{-}1) \times C_{155521}$, and ZM$(5 \cdot 11 \cdot 59 \cdot 659 \cdot 77761 \cdot 155521, 4, \text{-}1) \times C_{311041}$ are all Leinster.

Perfect Groups Finally, I applied the cyclic extension method on perfect groups, groups that equal their commutator subgroup. Note that these are not the Leinster groups as described in [4]. As Leinster groups are uncommon, it surprised me that with the 493 perfect groups of order at most 50000 in Magma's perfect group database [1], I found 293 Leinster groups with the Cyclic Extension Method. Again, for some groups, multiple extensions exist.

References

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