# *A wonderful stream*

*for Jaco*

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This note is written for Jaco de Bakker, in the hope that it may entertain him, and by way of thanks for all the years that I was working in his department or cluster, in a stimulating and productive environment created by Jaco's calm and effective leadership.

As I noticed recently, Jaco is well aware of the existence of the stream figuring in this note, namely the Thue-Morse sequence, to be called M henceforth. So the main aspects of this sequence mentioned below will not surprise him. One feature is maybe not well-known, namely the plane tiling that the stream M induces, and of which a figure is included. Otherwise the sequence M is frequently discovered, and studied in a variety of contexts, including formal languages, combinatorics on words, group theory and symbolic dynamics. Browsing around through the literature, it is amazing how widely and deep this sequence is studied. The references included in this note also contain CWI-connected authors; the current president of ERCIM; and the former dutch chess-world champion Max Euwe. It has even been used to generate some minimal music, see Figure 1, given in Allouche and Johnson [96]. Some quite heavy mathematics is devoted to it. As a disclaimer, it should be said that this note does not add to that more serious matter. But the sequence is a delightful example to illustrate in class-room various notions in term rewriting, process algebra, and algebraic data types. In fact this note arose out of a search for some simple example to treat in a process algebra class, in order to show a certain expressivity result - see below. There are several ways to introduce the sequence M.

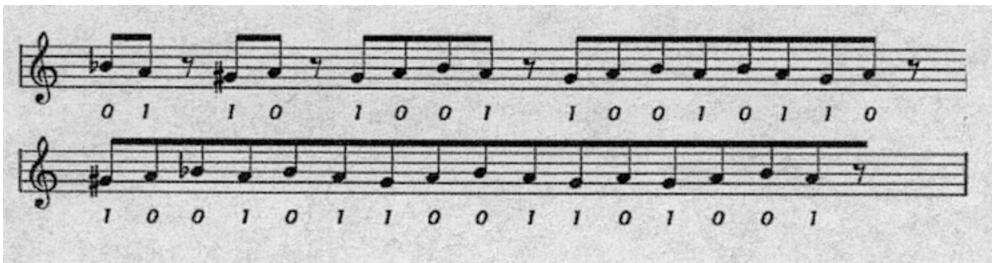


Figure 1. Thue-Morse music

**1. Four definitions of M, and some properties.**The sequence which is the subject of this note was discovered in 1912 by Axel Thue, one of the founding fathers of the theory of formal languages. It was rediscovered in 1917 by Marston Morse. Actually it occurred already in Prouhet [1851]. My fascination with this sequence started by playing around with symbolic expressions:

$\alpha$
$\alpha - \alpha$
$(\alpha - \alpha) - (\alpha - \alpha)$
$(\alpha - \alpha) - (\alpha - \alpha) - ((\alpha - \alpha) - (\alpha - \alpha))$
$(\alpha - \alpha) - (\alpha - \alpha) - ((\alpha - \alpha) - (\alpha - \alpha)) - ((\alpha - \alpha) - (\alpha - \alpha) - ((\alpha - \alpha) - (\alpha - \alpha)))$
…,

so in each step subtracting the result of the previous step. Working out the brackets, the +'s and -'s follow a pattern that is in fact the Thue-Morse sequence. It is much easier to write 1 and 0 instead of + and -, and so we obtain the first definition of M:

(i) Start with 1 and append in each generation step the 'negative' of the sequence obtained thus far, where 'negative' means changing a 1 into 0 and 0 into 1. We get

1
10
1001
10010110
1001011001101001
….

and the limit is the infinite stream known as the Thue-Morse sequence M. From this definition it is easy to see that M is, let's call it, an *infinitary palindrome*: each initial part can be extended to a possibly larger segment that is a palindrome. Or otherwise said, M is the limit of palindromes.

(ii) The second definition: M is the result of iterating the morphism $1 \to 10$, $0 \to 01$, starting with 1. In another terminology (see Saloma [81]) the rules of such a morphism (together with mention of the alphabet $\Sigma$ and a starting word) form a *DOL system*.

(iii) The third definition: count the number of 1's in the binary representation of n, and take this modulo 2; then we get the negative of M; see Table 1.

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 01 | 1 | 1 |
| 2 | 10 | 1 | 1 |
| 3 | 11 | 2 | 0 |
| 4 | 100 | 1 | 1 |
| 5 | 101 | 2 | 0 |
| 6 | 110 | 2 | 0 |
| 7 | 111 | 3 | 1 |
| 8 | 1000 | 1 | 1 |
| 9 | 1001 | 2 | 0 |
| 10 | 1010 | 2 | 0 |
| 11 | 1011 | 3 | 1 |

Table 1

(iv) The fourth definition gives the n-th entry in the sequence M, call it $\varepsilon_n$, by the recurrence equations

$$\varepsilon_0 = 1$$

$$\varepsilon_{2n} = \varepsilon_n$$

$$\varepsilon_{2n+1} = 1 - \varepsilon_{2n}$$

The definitions are easily proved equivalent. Now some properties of M, other than the palindrome property noted above. The main property is that M is *cube-free*: it does not contain a subword of the form www. A detailed proof is in Saloma [81]. As explained in Saloma [81], a stronger statement is true: the Thue-Morse sequence does not even contain a subword of the form wwa, where a is the first symbol of the word w. This is called *strongly cube-free*. Strongly cube-free in turn is equivalent to *overlap-free*, meaning that there is no subword x having two overlapping occurrences.)

From the cube-freeness it follows immediately that M is *not eventually periodic*.

M is *self-similar*: each finite subword occurs infinitely many often in the sequence. Also this is easily proved.

Of course M is not *square-free*; a square-free stream of two symbols 0, 1 does not exist. But with three symbols 0,1,2 there are square-free streams, and we obtain one from the negative of M, 0110100110010110... as follows: 01 yields 0, 10 yields 1, 00 yields 2 and 11 yields 2, where 01 yields 0 means that we write a 0 under the first symbol of 01, etc. Thus we obtain:

0110100110010110...
021012021....

and this sequence 021012021.... is according to Morse and Hedlund [1944] square-free. (A proof is in Saloma [81].)

Much more sophisticated properties are in Allouche and Shallit [99], where also the following curious way to obtain the sequence M is mentioned. Let A be the lexicographically smallest set of integers that starts with 0, 1 and for $x \geq 1$, if $x \in A$ then $2x \notin A$. So A =

0, 1, 3, 4, 5, 7, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, ...

of which the sequence of differences is

1 2 1 1 2 2 2 1 1 2 1 1 2 1 1 2 , .....

Using these differences as exponents of alternatingly 0 and 1 yields

$$0^1 1^2\ \ 0^1 1^1 0^2\ \ 1^2\ \ 0^2\ \ 1^1 0^1 1^2\ \ 0^1 1^1 0^2 1^1 0^1 1^2 \ldots$$

which indeed is the ubiquitous stream M:

0 11 0 1 00 11 00 1 0 11 0 1 001 0 11...

Table 2 gives the first 256 digits of M, by reading the 16-digits lines consecutively. As this table suggests, we can also consider 'two-dimensional DOL-systems', and consider the result of iterating the morphism with rules

$$1 \to \begin{matrix} 10 \\ 01 \end{matrix} \qquad\qquad 0 \to \begin{matrix} 01 \\ 10 \end{matrix}$$

starting from a single 1. We will return to the 'two-dimensional stream' thus obtained later.

**2. M and rewriting.** It is a nice exercise to define the stream M by rewriting. Actually we need *infinitary rewriting*. There are several solutions to this simple exercise in functional programming.

(i) For the first solution we need auxiliary symbols $\underline{1}$, $\underline{0}$. A finite word is denoted by x. Now consider rules

$$1x \to \underline{1}x10,$$

$$0x \to \underline{0}x01.$$

Then we have $\underline{1}\ 0 \to \underline{1}\ \underline{0}\ 01 \to \underline{1}\ \underline{0}\ \underline{0}\ 1\ 0\ 1 \to \underline{1}\ \underline{0}\ \underline{0}\ \underline{1}\ 0\ 1\ 1\ 0 \to \underline{1}\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{0}\ 1\ 1\ 0\ 0\ 1 \to \ldots$

| **M** | | | |
|---|---|---|---|
| 1 0 0 1 | 0 1 1 0 | 0 1 1 0 | 1 0 0 1 |
| 0 1 1 0 | 1 0 0 1 | 1 0 0 1 | 0 1 1 0 |
| 0 1 1 0 | 1 0 0 1 | 1 0 0 1 | 0 1 1 0 |
| 1 0 0 1 | 0 1 1 0 | 0 1 1 0 | 1 0 0 1 |
| | | | |
| 0 1 1 0 | 1 0 0 1 | 1 0 0 1 | 0 1 1 0 |
| 1 0 0 1 | 0 1 1 0 | 0 1 1 0 | 1 0 0 1 |
| 1 0 0 1 | 0 1 1 0 | 0 1 1 0 | 1 0 0 1 |
| 0 1 1 0 | 1 0 0 1 | 1 0 0 1 | 0 1 1 0 |
| | | | |
| 0 1 1 0 | 1 0 0 1 | 1 0 0 1 | 0 1 1 0 |
| 1 0 0 1 | 0 1 1 0 | 0 1 1 0 | 1 0 0 1 |
| 1 0 0 1 | 0 1 1 0 | 0 1 1 0 | 1 0 0 1 |
| 0 1 1 0 | 1 0 0 1 | 1 0 0 1 | 0 1 1 0 |
| | | | |
| 1 0 0 1 | 0 1 1 0 | 0 1 1 0 | 1 0 0 1 |
| 0 1 1 0 | 1 0 0 1 | 1 0 0 1 | 0 1 1 0 |
| 0 1 1 0 | 1 0 0 1 | 1 0 0 1 | 0 1 1 0 |
| 1 0 0 1 | 0 1 1 0 | 0 1 1 0 | 1 0 0 1 |

Table 2. First 256 digits of Thue-Morse stream M.

This works, but the two rewrite or reduction rules do not yet constitute a proper term rewriting system (TRS). However, this can be remedied by conceiving the 0, 1, $\underline{0}$, $\underline{1}$ as unary symbols, employing a constant nil, and the function append, for which rewrite rules are easy to give. Then e.g. the first rule reads

$$1(x) \to \underline{1}(\text{append}(x, 1(0(\text{nil}))$$

which has the proper TRS format. We note that the infinite reduction sequence of which the first four steps are displayed, satisfies the fundamental requirement in infinitary rewriting, namely that the depth of the contracted redexes tends to infinity. In other words, the reduction sequence is strongly convergent, which guarantees the existence of the infinite limit term.

(ii) The second solution begins with finding a TRS for the function $\varepsilon$ in the recurrence rules in 1.(iv) above. Let E be the symbol defining $\varepsilon$, so $E(\mathbf{n}) \twoheadrightarrow \boldsymbol{\varepsilon}_{\mathbf{n}}$. (Here $\mathbf{n}$ is the numeral corresponding to n, and $\twoheadrightarrow$ denotes a finite reduction.) Next, we define the stream $E(\mathbf{0}) : E(\mathbf{1}) : E(\mathbf{2}) : \ldots$ where : is the usual prefix operation, as follows: $H(x) \to E(x) : H(succ(x))$. Then $H(0)$ has as infinite normal form the desired sequence.

It is interesting that instead of employing the mechanism of infinitary rewriting and infinitary normal forms, we can obtain the same sequence with an appeal to coinductive techniques.

(iii) The third solution employs a self-similarity property of M, namely

$$M = 10(t(M) \square i(t(M))).$$

Here t is the *tail* operation that removes the first element of a stream, and i is the operation *'invert'* that takes the 'negative' of a 0-1-stream. Further, $\square$ is the *zip* operation that alternates elements from its left argument with elements from its right argument. (Many more of these properties are mentioned in Table 6.)

A definition of M exploiting this equation can now easily be given; in Table 3 it is rendered as a functional program in Clean (with thanks to Peter Achten.).

```
Start = thue_morse
where
thue_morse = [1,0:zipp (tl thue_morse) (map inv (tl thue_morse))]
inv 0 = 1; inv 1 = 0
zipp [a:as] bs = [a:zipp bs as]
```

Table 3: Definition in Clean of Thue-Morse stream

**3. M and process algebra.** We can also view the stream M as a *process* performing steps (or actions) 0 and 1. It is an infinite state process; it is not hard to prove that all the tails $t^n(M)$, arising by removing the first n elements, are different. Since equality on streams is the same as bisimilarity, we have that the process M proceeds through infinitely many different states. How can we define M in process algebra, ACP? A theorem in Bergstra and Klop [1984] shows that we cannot define M in PA, that is, without communication. Namely, that theorem states that a process having an infinite branch, and recursively definable in PA (so only with operators $+$, $\cdot$, $\|$ and $\lfloor\!\lfloor$) must have a branch which is eventually periodic. Since M has only one branch, which is not eventually periodic, it follows that M is not PA-definable. We do need communication. Indeed, it is not hard to define M in *ACP with renaming*, as follows. We start again from the self-similarity equation above. This yields the guarded system of recursion equations

$$X = 1.Y$$
$$Y = 0.\,(Y \,\square\, i(Y))$$

Next, we express $\square$ by a zip process defined by

$$\square = \text{blue}.\text{red}.\square,$$

and we color (i.e., rename) the left and right argument to be zipped in order to avoid confusion; we take along the operation i (invert) in one stride:

$$X = 1.Y$$
$$Y = 0.\,(Y^{\text{blue}} \,\square\, Y^{\text{invred}})$$

where

$$\text{blue}(0) = 0_{\text{blue}},\ \text{blue}(1) = 1_{\text{blue}},$$
$$\text{invred}(0) = 1_{\text{red}},\ \text{invred}(1) = 0_{\text{red}},$$

and communications are

$$\text{blue} \mid 0_{\text{blue}} = 0$$
$$\text{blue} \mid 1_{\text{blue}} = 1$$
$$\text{red} \mid 0_{\text{red}} = 0$$
$$\text{red} \mid 1_{\text{red}} = 1$$

Finally we have

$$X = 1.Y$$
$$Y = 0.\,\partial_H(Y^{\text{blue}} \parallel \square \parallel Y^{\text{invred}})$$

where H contains the communication actions.The resulting process X is just M.

There is an interesting question arising here (the answer is not known to the author): can M be defined *in ACP with handshaking communication*? (So without renaming.) We can eliminate the renamings in favour of some more communicating processes, but the catch is that in this way ternary communications arise, while handshaking communication is binary.

A next exercise is to define in process algebra the square-free stream 021012021.... obtained above by applying the "stream transforming rules" 01 yields 0, 10 yields 1, 00 yields 2 and 11 yields 2, on the negative of M. We can capture these rules, also depicted in Figure 2(a), by the process

$$X = 0.(0.\underline{2} + 1.\underline{0})X + 1.(0.\underline{1} + 1.\underline{2})X$$

this equation is wrong

where 0 means ‘read 0’, and $\underline{0}$ means ‘write 0’, likewise for 1.Now it is easy to obtain the square-free sequence sqf(M) by letting i(M) communicate with X, and applying abstraction:

$$\mathrm{sqf}(M) = (\tau_I \circ \partial_H)(X \parallel i(M))$$

with communications 0|0 = 0’, 1|1 = 1’, and abstracting the communication results 0’, 1’ away into $\tau$. Again, it is not known to the author whether we can avoid using abstraction.

**4. The Toeplitz stream T.** Another interesting stream originates from M by taking the difference sequence (see Figure 2(c).It is called the Toeplitz stream, or the ‘period doubling sequence’. We will refer to it as T.

M = 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 ...
T = 1 0 1 1 1 0 1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 0 1 0 1 1 1 0 1 0 1 0 1 1 1 0 1 0 1 0 1 1 1 0 1 ......

It can also be generated by iterating a morphism: 1 → 10, 0 → 11, starting from 1.

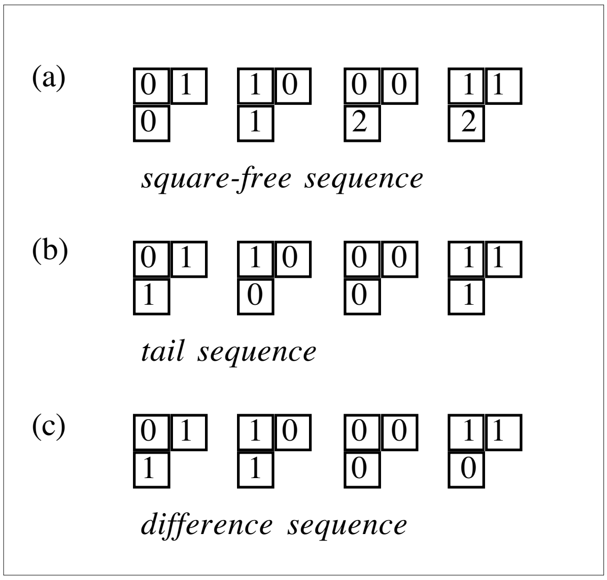


Figure 2: stream transfomation rules

Clearly, T is not cube-free. But it is also self-similar in the sense that every finite part of it is repeated infinitely many times in the sequence. A question that arises is whether the Toeplitz stream is also an infinitary palindrome. Indeed the prefixes of length 1, 3, 7, 15, 31 are palindromes:

**1**
1 **0** 1
1 0 1 **1** 1 0 1
1 0 1 1 1 0 1 **0** 1 0 1 1 1 0 1
1 0 1 1 1 0 1 0 1 0 1 1 1 0 1 **1** 1 0 1 1 1 0 1 0 1 0 1 1 1 0 1

The midpoints of these palindromes (in boldface) are 1, 0,1,0,1, ... Let us establish that T is indeed the limit of palindromes. The simple inductive proof is suggested by considering the generation tree of T as in Figure 2. Let the morphism $\phi$ be defined on the set of non-empty 0,1-words $\{0,1\}^+$ by $\phi(1) = 10$, $\phi(0) = 11$, $\phi(uv) = \phi(u)\phi(v)$.Define words $\alpha_n$, $\beta_n$ by

$$\alpha_0 = 1,\ \alpha_{n+1} = \phi(\alpha_n)$$
$$\beta_0 = 0,\ \beta_{n+1} = \phi(\beta_n).$$

Then $\alpha_{n+1} = \alpha_n\beta_n$. Further, define words $\gamma_n$ $(n \geq 1)$:

$$\gamma_{2n} = \gamma_{2n-1}0\gamma_{2n-1} \qquad (n \geq 1)$$
$$\gamma_{2n+1} = \gamma_{2n}1\gamma_{2n} \qquad (n \geq 1)$$

so the words $\gamma_n$ are palindromes. Now we prove for all $n \geq 1$:

$$\alpha_{2n} = \gamma_{2n}1 \qquad \beta_{2n} = \gamma_{2n}0$$
$$\alpha_{2n+1} = \gamma_{2n+1}0 \qquad \beta_{2n+1} = \gamma_{2n+1}1.$$

It follows that the limit of the $\alpha_n$ is also the limit of the palindromes $\gamma_n$, QED.

We continue with establishing another property of T. Unzipping T yields

*odd*(T) = 1 * 1 * 1 * 1 * 1 * 1 * 1 *1 * 1 * 1 * 1 * 1 * 1 * 1 * 1 * 1 * 1 * 1 * 1 * 1 * 1 * 1 * 1 * 1 ......

*even*(T) = * 0 * 1 * 0 * 0 * 0 * 1 * 0 * 1 * 0 * 1 * 0 * 0 * 0 * 1 * 0 * 0 * 0 * 1 * 0 * 0 * 0 * 1 * 0 * ......

T = 1 0 1 1 1 0 1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 ...

So we observe that $T = \mathbb{1} \square\, i(T)$, where $\mathbb{1}$ is the stream of ones and i is as before the operation inverting 0,1 to 1,0, and $\square$ is the zip operation also introduced earlier.

*Question.* Are all the the tails $t^n(T)$, different? In other words, is T an infinite state process, like M? What are recurrence equations for the entries of T?

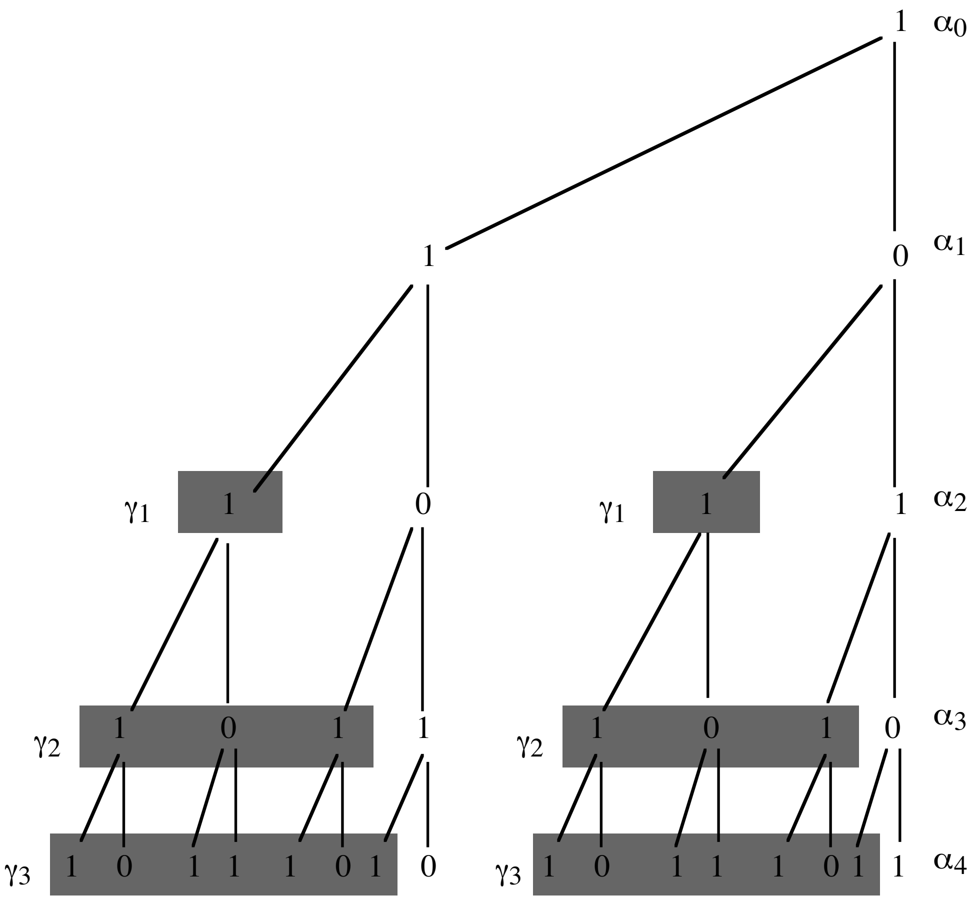


Figure 2. Generation tree of Toeplitz stream

*Remark.* For M we can determine the analogous generation tree. In the one for T, we have that right branches (i.e. branches taking the right successor of a node each time) alternate 1 and 0, and left branches are constant 1. In the analogous generation tree for M, we have that left branches starting in a 0-node are constant 0, left branches starting in a 1-node are constant 1, while right branches alternate.

Note that the generation tree for T is in fact a regular tree, defined by the recursive equations

$$\alpha = 1(\alpha, \beta)$$
$$\beta = 0(\alpha,\alpha)$$

where we conceive 0,1 as binary operators, $\alpha$, $\beta$ as recursion variables. Analogously for the generation tree of M.

**5. The algebra of M and T.** It is also interesting to look at *algebraic* aspects of the stream M. ('Algebraic' as in algebraic data types, or abstract data types.) To start with, there is the set of all streams of natural numbers. Here we have unary operations *tail*, removing the first element of the stream; *odd*, taking the entries at odd places 1,3,5,...; *even*, taking the entries at even places 0,2,4,...; the binary operation *zip*, taking two streams and zipping them up alternatingly to become one stream. We abbreviate the operations tail, odd, even by their first letter, and write $x \square y$ for zip(x,y). Some obvious equations holding in this algebra are in Table 4.

$$o(t^2(x)) = t(o(x))$$
$$e(t^2(x)) = t(e(x))$$
$$o(x) \square e(x) = x$$
$$o(x \square y) = x,$$
$$e(x \square y) = y$$
$$t(x \square y) = y \square t(x)$$
$$t(x) \square t(y) = t^2(x \square y)$$

Table 4. Algebra of streams

Second, we consider the subalgebra of *boolean streams*: infinite sequences of 0's and 1's. In addition to the operations for the whole stream algebra, we now have operations +, adding two streams element-wise modulo 2; *invert*, replacing 0 by 1 and vice versa; the constants ● (the stream of 0's) and $\mathbb{1}$, (the stream of 1's). We abbreviate invert by its first letter. Another interesting operation is the operation *dif*, giving for a stream x the stream of differences (modulo 2) of consecutive elements of x. Also this operation is denoted by its first letter. So in fact, T = d(M). Now we have in addition to the equations for all streams, the following equations for the boolean streams, in Table 5. It is just a handful of obvious equations, without any attempt for completeness in whatever sense.

$x + x = \bullet$
$x + \mathbb{0} = x$
$x + \mathbb{1} = i(x)$
$i(i(x)) = x$
$d(\mathbb{0}) = \mathbb{0},\ d(\mathbb{1}) = \mathbb{0}$
$d(\mathbb{0} \square \mathbb{1}) = \mathbb{1}$
$i(\mathbb{1}) = \mathbb{0},$
$t(\mathbb{1}) = \mathbb{1},\ t(\mathbb{0}) = \mathbb{0},$
$\mathbb{0} = 0.\mathbb{0}$
$\mathbb{1} = 1.\mathbb{1}$
$i(x \square y) = i(x) \square i(y)$
$i(0.x) = 1.i(x)$
$i(1.x) = 0.i(x)$
$i(t(x)) = t(i(x))$
$t(x) = x + d(x)$
$d(x) = x + t(x)$
$d(i(x) = d(x)$
$t(x + y) = t(x) + t(y)$
$d(x+y) = d(x) + d(y)$
$(x + x') \square (y + y') = (x \square y) + (x' \square y')$
$d(x \square y) = t(x \square x) + (y \square y)$
$t(d(x)) = d(t(x))$

Table 5. Algebra of boolean streams

Third, we take the subalgebra of the boolean stream algebra generated by the Thue-Morse sequence M. Some of the extra equations that hold in this algebra are in Table 6. The equations for the difference streams give in fact recurrence equations for these streams, which were observed empirically from Table 7. The analogous recurrence equations for the tails of M, $t^n(M)$, are easily derived algebraically from the equations mentioned in the tables. They can be checked in Table 8.

$$T = d(M)$$
$$M = M \square\, i(M)$$
$$T = \mathbb{1} \square\, i(T)$$
$$M = 10(M \square\, i(M))$$
$$t^{2n}(M) = t^n(M) \square\, t^n(i(M))$$
$$t^{2n+1}(M) = t^n(i)M)) \square\, t^{n+1}(M)$$
$$d^{2n}(M) = d^n(M) \square\, d^n(M)$$
$$d^{2n+1}(M) = \mathbb{0} \square\, d^{n+1}(M)$$

Table 6: Some equations for the streams generated by M

*Questions*. How about the word problem, complete axiomatizations, ω-complete axiomatizations? M and T are infinitary palindromes. How about the other $d^n(M)$? M and T can be obtained by iterating a morphism. How about the other $d^n(M)$? Some of the sequences $d^n(M)$ are homomorphic images of each other. For which i,j is $d^i(M)$ a homomorphic image of $d^j(M)$? It seems (empirically) that the only homorphisms that we have are those given by the recurrence equations for the $d^n(M)$, namely $d^n(M) \to d^{2n}(M)$ by the homomorphism $0 \to 00$, $1 \to 11$, and $d^{n+1}(M) \to d^{2n+1}(M)$ by the homomorphism $0 \to 00$, $1 \to 01$, for all $n \geq 1$.

**6. Plane tilings for M and T: connecting the dots.** Each 0,1-stream induces in a natural way a tiling of the plane, or rather, of a quadrant of the plane. For two 0,1-words x,y we define a kind of product that we denote with $x \otimes y$, as follows. The word x is written horizontally, the word y is written vertically. See Figure 3, with x = 10101010 and y = 110110110. Now we construct a matrix of 0,1's by copying x in each row where y has an entry 1, and taking the word i(x) (in the notation employed earlier, so the 'negative' of x) in a row where y has an entry 0. Next, we connect the 0's that are *adjacent* by connection lines, disregarding the 1's. We can do this for finite but also for infinite words x and y. Experiment shows that we will often get a tiling built from some basic tiles that are easily identified. However, in order to get a tiling for $x \otimes y$ from these basic tiles, we have to impose the restriction on x and y that they do not contain $\mathbb{0}$ or $\mathbb{1}$ as a subword, or equivalently, that in x and y each 1 is eventually followed by a 0 and vice versa. Otherwise we may get some degenerate 'tilings', e.g. for $1110\mathbb{1} \otimes \mathbb{1}$, just resulting in a single point, or $(10)^\omega \otimes \mathbb{1}$, consisting of vertical lines. Another esthetic detail is that we do not connect adjacent 0's when the connection would be a diagonal of a unit square tile.

| | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **1** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **0** | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| **1** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **1** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **0** | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| **1** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **1** | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| **0** | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |

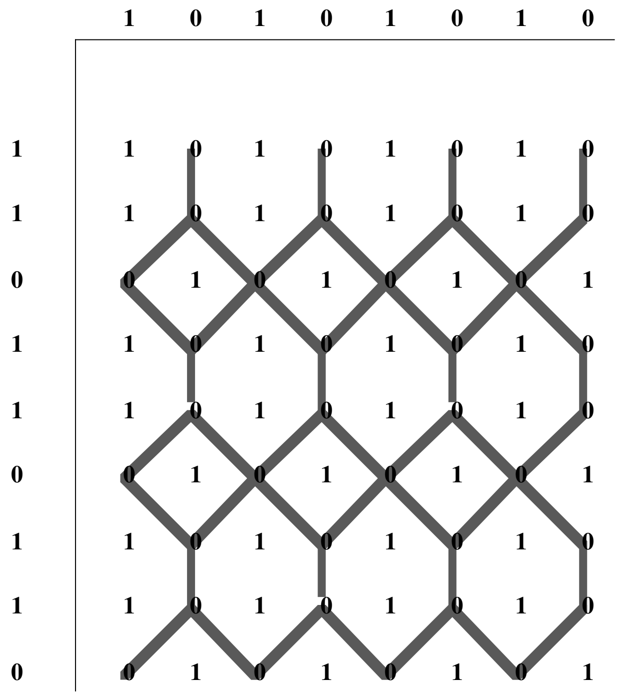


Figure 3. Connecting the dots.

In the figures we have taken the products of (an initial segment of) M with itself, and likewise for T. Thus Figure 4 contains the tiling $(M)_{64} \otimes (M)_{64}$, where $(M)_{64}$ is the prefix of M of length 64, a palindrome. Figure 5 contains an initial part of the tiling for the stream T, namely $(T)_{31} \otimes (T)_{31}$. As we saw $(T)_{31}$ is also a palindrome. The two tilings are coloured so that their structure is more easily seen. Even for these small initial parts of the tilings one can clearly see something of the self-similarity of M and T, now in a "two-dimensional way".

Note that the total tilings $M \otimes M$ and $T \otimes T$ are also self-similar in the sense that each finite part is present infinitely many times in the plane tiling.

| | |
|---|---|
| **0.** | 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * |
| **1.** | 1 0 1 1 1 0 1 0 1 0 1 1 1 0 1 1 1 0 1 1 1 0 1 0 1 0 1 1 1 0 1 0 1 0 1 1 1 0 1 0 1 0 1 1 1 0 1 * * |
| **2.** | 1 1 0 0 1 1 1 1 1 1 0 0 1 1 0 0 1 1 0 0 1 1 1 1 1 1 0 0 1 1 1 1 1 1 0 0 1 1 1 1 1 1 0 0 1 1 * * * |
| **3.** | 0 1 0 1 0 0 0 0 0 1 0 1 0 1 0 1 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 * * * * |
| **4.** | 1 1 1 1 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 * * * * * |
| **5.** | 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 * * * * * * |
| **6.** | 0 0 1 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 * * * * * * * |
| **7.** | 0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 1 * * * * * * * * |
| **8.** | 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 * * * * * * * * * |
| **9.** | 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * |
| **10.** | 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * |
| **11.** | 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * |
| **12.** | 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * |
| **13.** | 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * |
| **14.** | 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * |
| **15.** | 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * |
| **16.** | 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * |
| **17.** | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * |
| **18.** | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * |
| **19.** | 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * |
| **20.** | 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * |
| **21.** | 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * |
| **22.** | 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * |
| **23.** | 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * |
| **24.** | 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **25.** | 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **26.** | 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **27.** | 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **28.** | 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **29.** | 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **30.** | 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **31.** | 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **32.** | 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **33.** | 0 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **34.** | 0 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **35.** | 0 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **36.** | 0 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **37.** | 0 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **38.** | 0 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **39.** | 0 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **40.** | 0 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **41.** | 0 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **42.** | 0 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| 43. | 0 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **44.** | 0 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **45.** | 0 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **46.** | 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **47.** | * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |

Table 7: Difference streams $d^n(M)$ of the Thue-Morse sequence M

| | |
|---|---|
| **0.** | 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * |
| **1.** | 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * |
| **2.** | 0 1 0 1 1 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * |
| **3.** | 1 0 1 1 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * |
| **4.** | 0 1 1 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * |
| **5.** | 1 1 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * |
| **6.** | 1 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * |
| **7.** | 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * |
| **8.** | 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * |
| **9.** | 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * |
| **10.** | 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * |
| **11.** | 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * |
| **12.** | 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * |
| **13.** | 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * |
| **14.** | 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * |
| **15.** | 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * |
| **16.** | 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * |
| **17.** | 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * |
| **18.** | 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * |
| **19.** | 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * |
| **20.** | 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * |
| **21.** | 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * |
| **22.** | 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * |
| **23.** | 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * |
| **24.** | 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * |
| **25.** | 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **26.** | 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **27.** | 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **28.** | 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **29.** | 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **30.** | 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **31.** | 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **32.** | 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **33.** | 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **34.** | 1 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **35.** | 0 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **36.** | 1 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **37.** | 0 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **38.** | 0 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **39.** | 1 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **40.** | 1 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **41.** | 0 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **42.** | 0 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **43.** | 1 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **44.** | 0 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **45.** | 1 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **46.** | 1 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |
| **47.** | 0 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * |

Table 8: Tail streams $t^n(M)$ of the Thue-Morse sequence M

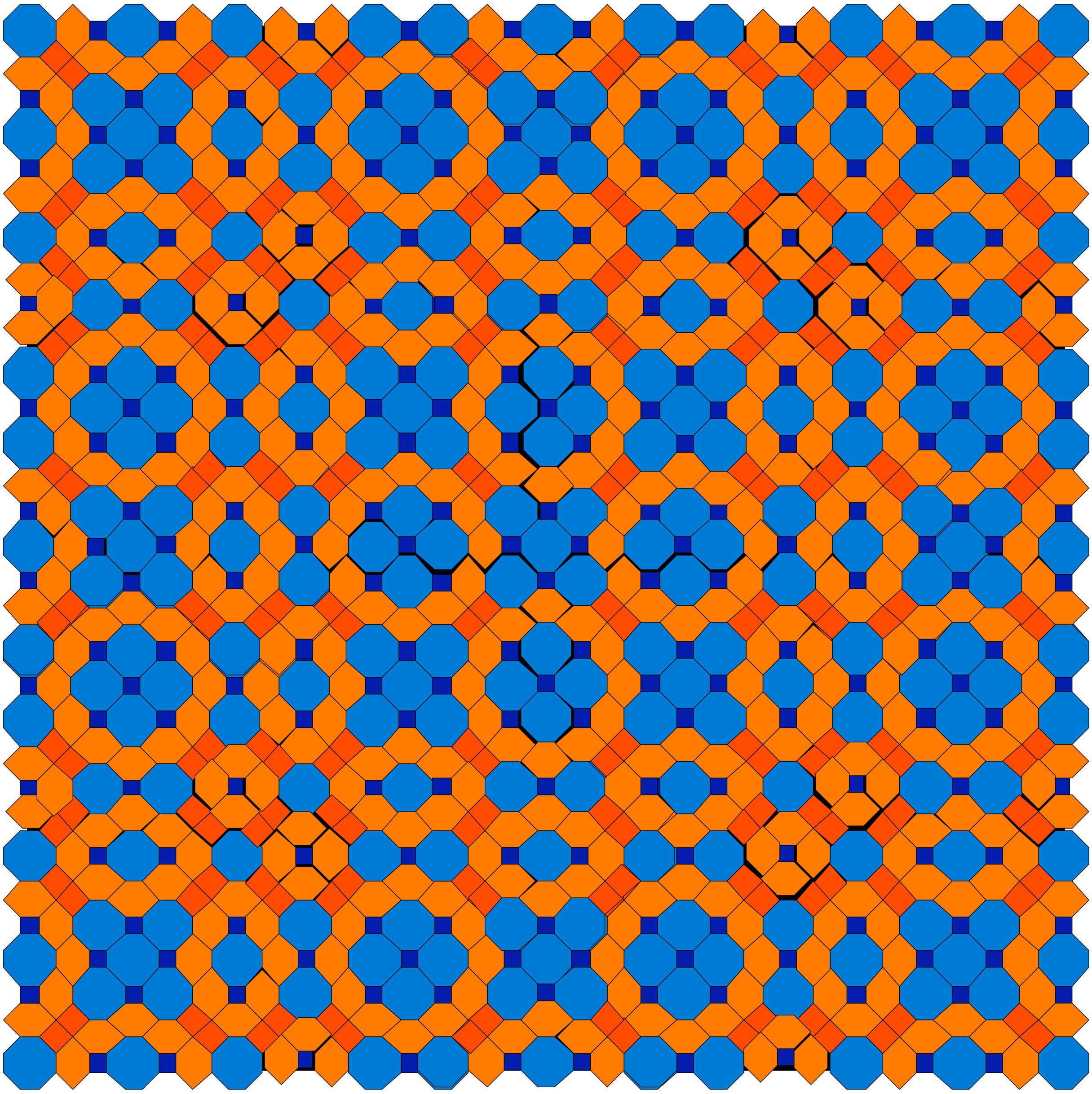

Figure 4: Plane tiling for the Thue-Morse sequence

Finally, we mention another intriguing stream, described by M. Keane in Alberts and van Zwet [2002] and called there the Mephistowals, the result of iterating the morphism $0 \to 001$, $1 \to 110$, starting with 0:

001 001 110 001 001 110 110 110 001 ...

It would be nice to see the plane tiling of this stream. It would be even nicer if someone developed an automated way of rendering these tilings graphically.

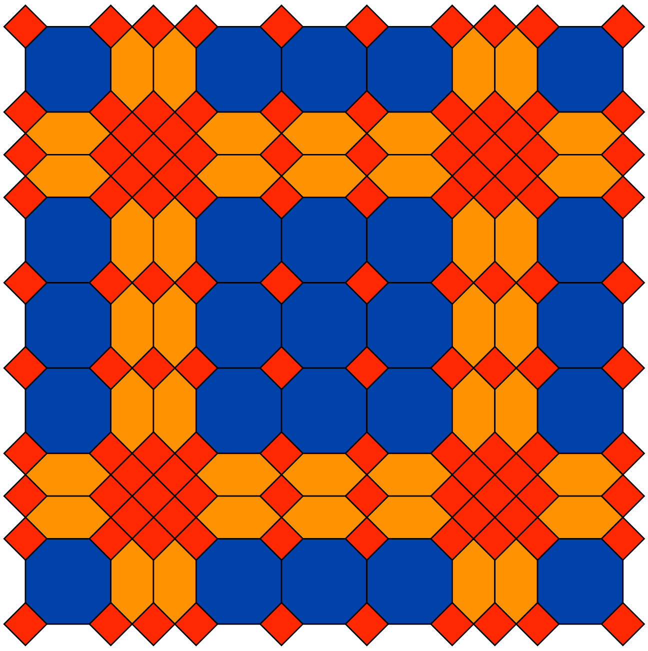

Figure 5: Tiling for the Toeplitz sequence

We conclude with wishing Jaco many hours of reflection at some wonderful stream!

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