On autoequivalences of the $(\infty, 1)$-category of ∞-operads

Javier J. Gutiérrez
(joint work with Dimitri Ara and Moritz Groth)

Institute for Mathematics, Astrophysics and Particle Physics
Department of Algebra and Topology
Radboud University Nijmegen

Category Theory 2014
Cambridge, 29 June – 5 July 2014

Introduction

Higher category theory and higher operad theory can be formalized by means of different approaches.

$(\infty, 1)$-categories

- Quasi-categories (Joyal, Lurie).
- Simplicial categories (Bergner).
- Segal categories (Hirschowitz–Simpson).
- Complete Segal spaces (Rezk).

All these models admit *model structures* and all of them are *Quillen equivalent* (Bergner, Joyal–Tierney).

Introduction

∞-operads

- Dendroidal sets (Moerdijk–Weiss).
- Simplicial operads (Cisinski–Moerdijk).
- Complete dendroidal Segal spaces (Cisinski–Moerdijk).
- ∞-operads (Lurie).

All these models admit *model structures* and all of them are *Quillen equivalent* (Cisinski–Moerdijk, Heuts–Hinich–Moerdijk).

Introduction

Question: What is the $(\infty, 1)$-category of autoequivalences of these models? In how many ways can we compare these models?

Theorem (Toën)

Aut($(\infty, 1)$-*categories*) $\cong \mathbb{Z}/2\mathbb{Z}$ *and the non-trivial element corresponds to passage to the opposite category.*

Any two possibly different ways of comparing two models for $(\infty, 1)$-categories differ at most by passage to opposites.

Goal: Compute Aut(∞-operads).

Main strategy

Let $\mathcal{C}$ be a quasi-category. We want to compute $\text{Aut}(\mathcal{C})$. Find a small category A inside $\mathcal{C}$ such that:

(i) $A \hookrightarrow \mathcal{C}$ is **dense**.

(ii) The autoequivalences of $\mathcal{C}$ **restrict** to autoequivalences of A.

Then it follows that $\text{Aut}(\mathcal{C}) \to \text{Aut}(A)$ is fully faithful.

To compute $\text{Aut}(\mathcal{C})$ it is enough to compute $\text{Aut}(A)$ and to check that the previous functor is essentially surjective.

If $\mathcal{C}$ is a localization of a category of simplicial presheaves on A, then (under good conditions) it satisfies the two conditions above.

Main strategy

For a small category A, we denote by:

- $\mathrm{Pr}(A)$ the category of preheaves on A.
- $\mathrm{sPr}(A)$ the category of simplicial preheaves on A.
- $\mathcal{P}(A)$ the quasi-category of preheaves on A.

Proposition

Let A be a small category and S a set of morphisms of $\mathrm{Pr}(A)$. Suppose that:

(i) Representable presheaves in $\mathrm{Pr}(A)$ are S-local.

(ii) The autoequivalences of $S^{-1}\,\mathrm{Pr}(A)$ restrict to autoequivalences of A.

Then $A \to S^{-1}\mathcal{P}(A)$ induces a fully faithful functor

$$\mathrm{Aut}(S^{-1}\mathcal{P}(A)) \to \mathrm{Aut}(A).$$

A category of trees

The only model proposed so far for ∞-operads based on simplicial presheaves is **Ω-spaces**.

Let Oper denote the category of symmetric coloured operads.

The *category of trees* Ω is the category whose objects are trees and whose morphisms are given by

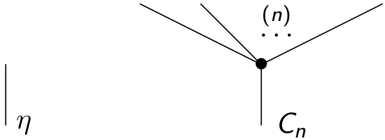
$$\Omega(S, T) = \mathrm{Oper}(\Omega(S), \Omega(T)),$$

where $\Omega(T)$ denotes the operad generated by the tree T.

There is an inclusion $\Omega \to \mathrm{Oper}$ that induces a fully faithful *dendroidal nerve functor* $N_d \colon \mathrm{Oper} \to \mathrm{Pr}(\Omega)$. The category $\mathrm{Pr}(\Omega)$ is called the category of *dendroidal sets*.

Canonical decomposition of trees

Let $\mathbb{C}$ denote the full subcategory of Ω consisting of η and the corollas.

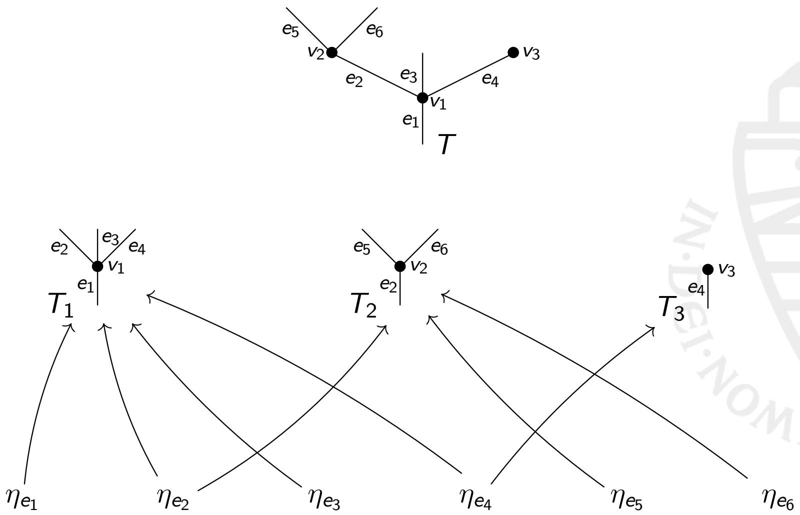


For every tree T let $\mathcal{D}_T$ denote the functor $\mathbb{C}/T \to \mathbb{C} \hookrightarrow \Omega$. Then the canonical morphism

$$\operatorname{colim} \mathcal{D}_T = \operatorname*{colim}_{(C, C\to T)\in\mathbb{C}/T} C \longrightarrow T$$

is an isomorphism in Ω.

Canonical decomposition of trees



Canonical decomposition of trees

Let T be a tree. The *spine* of T is the dendroidal set

$$I_T = \operatorname*{colim}_{(C, C\to T)\in\mathbb{C}/T} C,$$

where the colimit is taken in Pr(Ω). There is a canonical morphism of dendroidal sets $i_T \colon I_T \to T$. We will denote by $\mathcal{I}$ the set

$$\mathcal{I} = \{i_T \mid T \in \Omega\}.$$

Let J be the simply connected groupoid on two objects and let $J \to \eta$ be the unique map of operads. For any tree T we obtain an induced map of operads

$$j_T \colon J \otimes T \longrightarrow \eta \otimes T \xrightarrow{\cong} T.$$

We will denote by $\mathcal{J}$ the set

$$\mathcal{J} = \{N_d(j_T) \mid T \in \Omega)\}.$$

Dendroidal spaces

The category of *dendroidal spaces* is $\mathrm{sPr}(\Omega) \cong \mathrm{Pr}(\Omega \times \Delta)$.

Theorem (Cisinski–Moerdijk)

There is a (generalized Reedy) model structure on $\mathrm{sPr}(\Omega)$ *whose weak equivalences are the objectwise simplicial weak homotopy equivalences. The model category of complete dendroidal Segal spaces is the left Bousfield localization of* $\mathrm{sPr}(\Omega)_{Reedy}$ *with respect to the set* $\mathcal{I} \cup \mathcal{J}$.

The *quasi-category* of Ω-spaces Ω-Sp is a localization of the quasi-category $\mathcal{P}(\Omega)$ by the set $\mathcal{I} \cup \mathcal{J}$.

Rigid operads

An operad is *rigid* if every invertible unary operation is an identity. For example, operads induced by trees are rigid.

Proposition

A dendroidal set is $(\mathcal{I} \cup \mathcal{J})$-local if and only if it is the dendroidal nerve of a rigid operad

Proposition

If F is an autoequivalence of rigid operads, then $F(T) \cong T$. In particular, F induces an autoequivalence of Ω.

Computing Aut(Ω)

Let $\Sigma_\Omega = \prod_{T\in\Omega} \mathrm{aut}_\Omega(T)$ and let $(\Sigma_\Omega)_{sc}$ be the simply connected groupoid on Σ_Ω.

Given an element $\sigma = (\sigma_T)_{T\in\Omega}$ in Σ_Ω, we define an autoequivalence F_σ by setting:

(i) $F_\sigma(T) = T$.

(ii) For a map $f\colon S \to T$, we set $F_\sigma(f) = \sigma_T f \sigma_S^{-1}$.

This assignment defines a functor $\Phi\colon (\Sigma_\Omega)_{sc} \to \mathrm{Aut}(\Omega)$

Computing Aut(Ω)

Let F be an autoequivalence of Ω. We define $\sigma(F)$ in Σ_Ω in the following way: $\sigma(F)_T$ is the unique automorphism of T such that

$$\sigma(F)_T \circ c = F(c)$$

for every morphism $c\colon \eta \to T$ of Ω.

Theorem

The functor $\Phi\colon (\Sigma_\Omega)_{sc} \to \mathrm{Aut}(\Omega)$ is an isomorphism of categories. In particular, $\mathrm{Aut}(\Omega)$ is a contractible groupoid.

Theorem

The quasi-category $\mathrm{Aut}(\Omega\text{-}Sp)$ is a contractible Kan complex.

Summary

∞-**category**	A	$\mathrm{sPr}(A)$	$S^{-1}\mathcal{P}(A)$	$\mathrm{Aut}(A)$
(∞, n)-categories (Barwick–Schommer-Pries)	Θ_n	n-cellular spaces	Θ_n-spaces	$(\mathbb{Z}/2\mathbb{Z})^n$
∞-operads	Ω	dendroidal spaces	Ω-spaces	$*$
non-symmetric ∞-operads	Ω_p	planar dendroidal spaces	Ω_p-spaces	$\mathbb{Z}/2\mathbb{Z}$