ALGEBRAIC TOPOLOGY, EXERCISE SHEET 1, 17.09.2014

Exercise 1. (1) Show that every convex subspace of $\mathbb{R}^n$, $n \in \mathbb{N}$, is homotopy equivalent to a point. Provide examples of non-convex subspaces of $\mathbb{R}^n$ which are homotopy equivalent to the point.

(2) Show that the two subspaces of n-dimensional Euclidean space

$$S^{n-1} = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + ... + x_n^2 = 1\} \qquad \text{and} \qquad \mathbb{R}^n - \{(0,0)\}$$

are homotopy equivalent for all $n \geq 1$.

(3) Let M be the Möbius strip, obtained as the quotient space of the square $[0,1] \times [0,1]$ by identifying

$$(x, 0) \sim (1-x, 1) \qquad \text{for all} \qquad 0 \leq x \leq 1.$$

Show that the Möbius strip is homotopy equivalent to S^1.

(4) Show that if X_1, X_2, Y_1, and Y_2 are space such that $X_1 \simeq X_2$ and $Y_1 \simeq Y_2$ then also $X_1 \times Y_1 \simeq X_2 \times Y_2$.

Exercise 2. (Fundamental groupoid)
Let X be a space, let $x, y, z \in X$, let α be a path from x to y and let β be a path from y to z.

(1) The concatenation $\beta * \alpha$ is again continuous and defines a path from x to z.

(2) The assignment $([\beta], [\alpha]) \mapsto [\beta * \alpha]$ defines a well-defined composition map

$$\circ\colon \pi(X)(y,z) \times \pi(X)(x,y) \to \pi(X)(x,z).$$

(3) The composition is associative.

(4) Prove that the homotopy classes of constant paths give identity morphisms. Thus we already know that $\pi(X)$ defines a category.

(5) Given a path α from x to y then we the **inverse path** α^{-1} from y to x is defined by the formula

$$\alpha^{-1}(t) = \alpha(1-t).$$

Show that every morphism $[\alpha]$ in $\pi(X)$ is an isomorphism by verifying that $[\alpha^{-1}]$ defines a two-sided inverse of $[\alpha]$.

Exercise 3. (Unpointed homotopies versus pointed homotopies.)
Given a space X then we can assign to it a further (but uninformative!) category $\pi'(X)$. The objects of $\pi'(X)$ are again just the points of X. The set of morphisms $\pi'(X)(x_0, x_1)$ between two such points is the set of *unpointed* homotopy classes of paths from x_0 to x_1. Show that we have:

$$\pi'(X)(x,y) = \begin{cases} \{*\} & \text{if } x \text{ and } y \text{ lie in the same path component} \\ \emptyset & \text{otherwise} \end{cases}$$

Conclude that there is a unique way to extend this to a category $\pi'(X)$ (which is even a groupoid).

Exercise 4. (1) Recall (or prove!) that given a pair of spaces $(X, A) \in \mathsf{Top}^2$ we can form the quotient space X/A. This quotient space together with the quotient map $p\colon X \to X/A$ has the following universal property: for every space Z and every continuous map $f\colon X \to Z$ such that

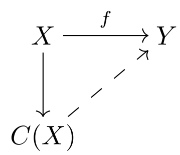
$$f(a) = f(a') \qquad \text{for all } a, a' \in A$$

there exists a unique map $f'\colon X/A \to Z$ such that $f' \circ p = f$. Use this to show that the construction of quotient spaces defines a functor $\mathsf{Top}^2 \to \mathsf{Top}_*$.

(2) The **cone** $C(X)$ of a space X is the space given by:

$$C(X) = (X \times I)/(X \times \{0\}).$$

Note that there is a canonical inclusion $X \to \mathrm{C}(X)$ of X in the cone. Show that a map $f\colon X \to Y$ is homotopic to a constant map if and only if it extends over the cone $C(X)$ i.e., there exists a map $C(X) \to Y$ such that the following diagram commutes:



(3) For $n \geq 1$, let $i\colon S^{n-1} \to S^n$ be the inclusion of the $(n-1)$-sphere in the n-sphere given by

$$i(x_1, ..., x_n) = (x_1, ..., x_n, 0).$$

Show that the map i is homotopic to a constant map.

(4) Can you come up with the notion of a **reduced cone** of a pointed space which has properties similar to the ones of the cone construction in (2)?