# ALGEBRAIC TOPOLOGY, EXERCISE SHEET 2, 24.09.2014

**Exercise 1.** Let $X$ and $Y$ be pointed spaces and let $i_X \colon X \to X \vee Y$ and $i_Y \colon Y \to X \vee Y$ be the canonical inclusions.

(1) Given two further pointed maps $f \colon X \to W$ and $g \colon Y \to W$ then there is a unique pointed map $(f, g) \colon X \vee Y \to W$ such that:
$$(f, g) \circ i_X = f \qquad \text{and} \qquad (f, g) \circ i_Y = g$$

(2) Use (1) to conclude that the wedge product is associative. More precisely, show that if $X, Y$, and $Z$ are pointed spaces then there is a unique pointed homeomorphism
$$(X \vee Y) \vee Z \overset{\cong}{\to} X \vee (Y \vee Z)$$
which is compatible with the inclusions.

**Exercise 2.** Show that for a Hausdorff space $X$ the following are equivalent:

(1) Every point of $X$ has a compact neighbourhood.
(2) Every point of $X$ has a local base of compact neighbourhoods.

A space satisfying one of these equivalent conditions is called a **locally compact Hausdorff space**.

**Exercise 3.** In the notes there is a proof of the following statement. Let $K, X$, and $Y$ be spaces and let $K$ be compact and Hausdorff. Then there is a bijective correspondence between maps
$$Y \overset{f}{\to} X^K \qquad \text{and maps} \qquad Y \times K \overset{g}{\to} X.$$
Show that the same proof also applies under the weaker additional assumption on $K$ to be locally compact Hausdorff.

**Exercise 4.** Let $X, Y$, and $Z$ be spaces and let $f \colon X \to Y$ be a map.

(1) Show that the maps
$$f^* \colon Z^Y \to Z^X \colon g \mapsto g \circ f \qquad \text{and} \qquad f_* \colon X^Z \to Y^Z \colon h \mapsto f \circ h$$
are continuous. Conclude that for every $K \in \mathsf{Top}$ there is a **mapping space functor**:
$$(-)^K \colon \mathsf{Top} \to \mathsf{Top} \colon X \mapsto X^K$$

(2) Let $Y$ be a locally compact Hausdorff space. Show that the composition map
$$\circ \colon Z^Y \times Y^X \to Z^X \colon (g, f) \mapsto g \circ f$$
is continuous.

(3) Prove a similar result for pointed spaces. More precisely, construct a **pointed mapping space functor**
$$(-)^{(X,x_0)} \colon \mathsf{Top}_* \to \mathsf{Top}_*$$
for every pointed space $(X, x_0)$. This includes, in particular, the construction of a loop space functor $\Omega \colon \mathsf{Top}_* \to \mathsf{Top}_*$.

**Exercise 5.** Let $K, X$, and $Y$ be pointed spaces and assume that $K$ is locally compact Hausdorff.

(1) Show that the function
$$\mathsf{Top}_*(X, Y^K) \to \mathsf{Top}_*(X \wedge K, Y) \colon f \mapsto g$$
defined by
$$g([x,k]) = f(x)(k), \quad x \in X, \quad k \in K,$$
is a bijection.

(2) Can you give sense to the following slogan 'the bijections in (1) are nicely behaved with respect to maps $X \to X'$, $K \to K'$, and $Y \to Y'$'? (Hint: Given, say, such a map $X \to X'$ is there a precise sense in which the bijections for $X$ and $X'$ are compatible?)

(3) Try to prove that the function defined in (1) induces a bijection of homotopy classes:
$$[X, Y^K] \cong [X \wedge K, Y]$$
(Hint: Consider the cylinder construction $(-) \times I$ on spaces and use results similar to the ones of (2) but in $\mathsf{Top}$. Note that for a space $W$ there are two natural maps $W \to W \times I$ – the inclusion at the 'top' and at the 'bottom'. What properties of the product construction are you using?)