ALGEBRAIC TOPOLOGY, EXERCISE SHEET 3, 01.10.2014

Exercise 1. For $(X, x_0) \in \mathsf{Top}_*$ we recall the definition of the Moore loop space $M(X, x_0)$ given in the lecture. Its points are pairs (t, α) with $t \in \mathbb{R}$ and a map $\alpha \in \mathsf{Top}([0,t], X)$ satisfying $\alpha(0) = \alpha(t) = x_0$. We can topologize this set as a subspace of $\mathbb{R} \times X^{[0,\infty)}$ identifying a path $\alpha\colon [0,t] \to X$ with the map $[0,\infty) \to X$ which coincides with α on $[0,t]$ and is constant on $[t,\infty)$. The base point of $M(X, x_0)$ is $(0, \kappa_{x_0})$, where $\kappa_{x_0}(0) = x_0$.

(1) Show that the function

$$\begin{aligned} \bullet\colon M(X,x_0) \times M(X,x_0) &\longrightarrow M(X,x_0) \\ ((t,\beta),(s,\alpha)) &\longmapsto (t+s, \beta *_M \alpha) \end{aligned}$$

where

$$(\beta *_M \alpha)(r) = \begin{cases} \alpha(r), & 0 \le r \le s \\ \beta(r-s), & s \le r \le t+s \end{cases}$$

is continuous and defines a stricly associative and unital multiplication on $M(X, x_0)$.

(2) Given an inclusion of spaces $i\colon A \hookrightarrow X$, recall that A is a **deformation retract** of X if and only if there exists a map $r\colon X \to A$ (called the retraction map) such that $r \circ i = \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_X$.

Show that $\Omega(X, x_0)$ is a deformation retract of $M(X, x_0)$ by using the retraction map $\varphi\colon M(X, x_0) \to \Omega(X, x_0)$ defined by the formula:

$$\varphi(t,\alpha)(r) \quad = \quad \alpha(tr), \qquad 0 \le r \le 1$$

Exercise 2. Define the notion of a **commutative H-space** and a **commutative H-group**. We know already that the loop space $\Omega(X, x_0)$ of a pointed space (X, x_0) is an H-group. Is this H-group also commutative?

Exercise 3. Let G be an (associative) H-space and let X be a space which is homotopy equivalent to G. Use this homotopy equivalence to show that X has the structure of an H-space as well. If G is a (commutative) H-group, show that X is a (commutative) H-group.

Let us now introduce one more concept from category theory which was already implicit in the lectures. Let $\mathcal{C}, \mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors between them. A **natural transformation** $\eta\colon F \to G$ from F to G is a collection of morphisms in $\mathcal{D}$,

$$\eta_C\colon F(C) \to G(C), \qquad C \in \mathcal{C},$$

such that for every morphism $f\colon X \to Y$ in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\eta_Y]{} & G(Y) \end{array}$$

is commutative, i.e., we have $G(f) \circ \eta_X = \eta_Y \circ F(f)$.

Exercise 4. Let Ring be the category of commutative rings and ring homomorphisms.

(1) Given a commutative ring R then we can assign to it the group of units $R^{\times}$ and the group $GL_n(R)$ of invertible $(n \times n)$-matrices with coefficients in R. Show that these two assignments can be extended to functors:

$$(-)^{\times} \colon \mathsf{Ring} \to \mathsf{Grp} \qquad \text{and} \qquad GL_n \colon \mathsf{Ring} \to \mathsf{Grp}$$

Show that the determinant operations $\det_R \colon GL_n(R) \to R^{\times}$ assemble into a natural transformation $\det \colon GL_n \to (-)^{\times}$.

(2) Try to find examples of natural transformations in the notes of the first three lectures. Do you know other examples?

(3) Check the details of the proof omitted in the lecture that π_n actually defines a functor

$$\pi_n \colon \mathsf{Top}_* \to \mathsf{Grp}, \qquad n \geq 1.$$

Show that the natural transformations from π_n to π_m are in bijective correspondence with elements of $\pi_m(S^n, *)$. (Hint: Given such a natural transformation, consider its component at S^n and apply it to $[id_{S^n}]$.)