ALGEBRAIC TOPOLOGY, EXERCISE SHEET 4, 08.10.2014

Exercise 1. Given a pointed pair of spaces (X, A, x_0) let us define $P(X; x_0, A) \subset X^I$ as the subspace given by the paths α in X such that $\alpha(0) = x_0$ and $\alpha(1) \in A$. This is canonically a pointed space if we choose $\kappa_{x_0}\colon I \to X$ as base point. Prove that there are natural isomorphisms

$$\pi_{n+1}(X, A) \cong \pi_n\big(P(X; x_0, A), \kappa_{x_0}\big), \qquad n \geq 1,$$

and conclude that $\pi_n(X, A)$ is a group for $n \geq 2$ which is abelian if $n \geq 3$.

Exercise 2. Complete the proof that there is a long exact sequence of homotopy groups for each pointed pair of spaces (X, A, x_0). It only remains to establish exactness at $\pi_n(X, x_0)$, i.e., to show that the following diagram is exact:

$$\pi_n(A, x_0) \to \pi_n(X, x_0) \to \pi_n(X, A), \qquad n \geq 1$$

Exercise 3. Given a space X and a natural number $n \geq 1$ we showed in the lecture that there is a functor

$$\pi_n(X, -)\colon \pi(X) \to \mathsf{Set}$$

from the fundamental groupoid of X to the category of sets. Show that this actually defines a functor

$$\pi_n(X, -)\colon \pi(X) \to \mathsf{Grp}.$$

(Since we already know that $\pi_n(X, x_0)$ always is a group it suffices to check that this functor sends elements of $\pi(X)(x_0, x_1)$ to group homomorphisms $\pi_n(X, x_0) \to \pi_n(X, x_1)$.)

Given a group G and a set X, recall that a **left action** of G on X is a function

$$\cdot\colon G \times X \to X, \qquad (g, x) \mapsto g \cdot x,$$

such that for all $g, g' \in G$ and $x \in X$ we have:

$$(gg') \cdot x = g \cdot (g' \cdot x) \qquad \text{and} \qquad 1_G \cdot x = x$$

Such an action induces an equivalence relation $\sim$ on X by setting $x \sim x'$ if and only if there is a group element $g \in G$ such that $g \cdot x = x'$. The equivalence classes with respect to this relation are the **orbits** of the action and the set of orbits is denoted by X/G.

Exercise 4.
1. Given an object C in a category $\mathcal{C}$, let us write $\mathrm{Aut}(C) = \mathrm{Aut}_{\mathcal{C}}(C)$ for the group of automorphisms of C in $\mathcal{C}$. Show that an arbitrary functor $F\colon \mathcal{C} \to \mathsf{Set}$ induces an action of $\mathrm{Aut}(C)$ on $F(C)$ for every $C \in \mathcal{C}$.
2. Use the previous point to conclude that for every pointed space (X, x_0) and $n \geq 1$ there is an action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.
3. Let X be a path-connected space and let $x_0 \in X$. Show that the forgetful map

$$\pi_n(X, x_0) = [(S^n, *), (X, x_0)] \to [S^n, X]$$

induces a bijection between the set of orbits $\pi_n(X, x_0)/\pi_1(X, x_0)$ and the set $[\mathrm{S}^n, X]$ of 'free homotopy classes'.

In the final exercise we want to examine the effect of some symmetry operations on n-cubes, n-discs or n-spheres at the level of homotopy groups. Using the 'obvious' relative homeomorphism $(I^n, \partial I^n) \cong (D^n, \partial D^n)$ and also the quotient map $(I^n, \partial I^n) \to (S^n, *)$ we can pass back and forth between the different descriptions of $\pi_n(X, x_0)$ as

$$[(S^n, *), (X, x_0)], \qquad [(I^n, \partial I^n), (X, x_0)], \qquad \text{or} \qquad [(D^n, \partial D^n), (X, x_0)].$$

Here, we choose to work with the one based on discs and, for convenience, center D^n at the origin.

Exercise 5. Let (X, x_0) be a pointed space and let $\alpha\colon (D^n, \partial D^n) \to (X, x_0)$ be a map for $n \geq 2$.

(1) If $\theta\colon D^n \to D^n$ is a rotation around the origin, then the assignment $\theta^*\colon \alpha \mapsto \alpha \circ \theta$ induces the identity on homotopy groups, i.e.,

$$\theta^* = id\colon \pi_n(X, x_0) \to \pi_n(X, x_0).$$

(2) Let $\phi_{(ij)}\colon D^n \to D^n$ be the map interchanging the i-th and the j-th coordinate (for $i \neq j$). Then the assignment $\phi^*_{(ij)}\colon \alpha \mapsto \alpha \circ \phi_{(ij)}$ induces multiplication by -1 at the level of homotopy groups. More generally, given a permutation $\sigma \in \Sigma_n$, let $\phi_\sigma\colon D^n \to D^n$ be the corresponding coordinate permutation. Then we have

$$\phi^*_\sigma = (-1)^{\mathrm{sign}(\sigma)}\colon \pi_n(X, x_0) \to \pi_n(X, x_0)$$

where $\mathrm{sign}(\sigma)$ denotes the signature of the permutation σ.

(3) Let $\rho\colon D^n \to D^n$ be the reflection at an arbitrary hyperplane in $\mathbb{R}^n$. Then we have

$$\rho^*([\alpha]) = [\alpha \circ \rho] = -[\alpha] \in \pi_n(X, x_0).$$