ALGEBRAIC TOPOLOGY, EXERCISE SHEET 5, 15.10.2014

Exercise 1.

(1) Show that compositions of Hurewitz and Serre fibrations are again Hurewitz and Serre fibrations respectively.

(2) Let X, Y, Z be spaces and let $f\colon X \to Z$ and $g\colon Y \to Z$ be continuous functions. We define the **pullback** of f and g as the subspace $X \underset{Z}{\times} Y$ of $X \times Y$ given by:

$$X \underset{Z}{\times} Y \quad = \quad \big\{(x, y) \in X \times Y \mid f(x) = g(y)\big\}$$

The restrictions to $X \underset{Z}{\times} Y$ of the two projections of the product give us two continuous maps $p_1\colon X \underset{Z}{\times} Y \to X$ and $p_2\colon X \underset{Z}{\times} Y \to Y$ making the following diagram commutative:

$$\begin{array}{ccc} X \underset{Z}{\times} Y & \xrightarrow{p_1} & X \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

Show that the triple $(X \underset{Z}{\times} Y, p_1, p_2)$ has the following universal property: given a further triple (P, a, b) consisting of a space P and continuous maps $a\colon P \to X$ and $b\colon P \to Y$ such that $fa = gb$, then there exists a unique continuous map

$$h\colon P \to X \underset{Z}{\times} Y$$

such that $p_1 h = a$ and $p_2 h = b$ (You should draw a diagram!).

(3) Show that Hurewicz fibrations and Serre fibrations are *stable under pullbacks*. More precisely, in the above pullback diagram, if f is a Serre fibration then so is p_2, and similarly for Hurewicz fibrations.

Exercise 2. Let $f\colon Y \to X$ be a map of spaces and let $p\colon P(f) \to X$ be the mapping fibration of f (see the lecture notes for more details). Prove that the map ϕ in

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & P(f) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

given by $\phi(y) = (\kappa_{f(y)}, y)$ is a homotopy equivalence which makes the diagram commute. Here, as usual, we denote by $\kappa_{f(y)}$ the constant path at $f(y)$.

Exercise 3. Show that the long exact sequence of a pointed pair (X, A, x_0) constructed in the fourth lecture can be obtained as a special case of the long exact sequence of a fibration. More precisely, consider the mapping fibration $P(i) \to X$ associated to the inclusion $i\colon (A, x_0) \to (X, x_0)$. (Hint: you might want to use Exercise 1 of the last exercise sheet.)

Exercise 4. Let $p\colon (E, e_0) \to (X, x_0)$ be a Serre fibration with fiber (F, e_0). Show that for every $n > 1$ the connecting map $\delta\colon \pi_n(X, x_0) \to \pi_{n-1}(F, e_0)$ is a group homomorphism. Conclude that the long exact sequence of a Serre fibration is a long exact sequence of groups (in positive degrees).

Exercise 5.

(1) Suppose we are given Serre fibrations $p\colon (E, e_0) \to (X, x_0)$ and $p\colon (E', e_0') \to (X', x_0')$. Let $i\colon (F, e_0) \to (E, e_0)$ and $i'\colon (F', e_0') \to (E', e_0')$ be the inclusion of the respective fibers. Show that given two maps of pointed spaces $f\colon (X, x_0) \to (X', x_0')$ and $g\colon (E, e_0) \to (E', e_0')$ such that $p'g = fp$ then there exists a unique map $h\colon (F, e_0) \to (F', e_0')$ such that $i'h = gi$.

$$\begin{array}{ccccc} (F, e_0) & \xrightarrow{\;i\;} & (E, e_0) & \xrightarrow{\;p\;} & (X, x_0) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ (F', e_0') & \xrightarrow{\;i'\;} & (E', e_0') & \xrightarrow{\;p'\;} & (X', x_0') \end{array}$$

(2) Give a precise meaning to the statement 'the long exact sequence of a Serre fibration is natural in the fibration'. Use the previous part to prove this statement.

(3) Let (X, x_0) be a pointed space. Use the long exact sequence of a Serre fibration to check that we have *natural* isomorphisms $\pi_n(X, x_0) \cong \pi_{n-1}(\Omega(X, x_0), \kappa_0)$ for every $n \geq 1$.

Exercise 6. Let $p\colon E \to X$ be a Hurewicz fibration, and let $\alpha\colon I \to X$ be a path from x to y. Use the right lifting property of $E \to X$ with respect to $p^{-1}(x) \times \{0\} \to p^{-1}(x) \times I$ to show that α induces a map $\alpha_*\colon p^{-1}(x) \to p^{-1}(y)$. Show that the homotopy class of α_* only depends on the homotopy class of α, and that this construction in fact defines a functor on the fundamental groupoid,

$$\pi(X) \to \mathsf{Ho}(\mathsf{Top}).$$

Conclude that any path $\alpha\colon I \to X$ induces a homotopy equivalence between the fibers over $\alpha(0)$ and $\alpha(1)$.