ALGEBRAIC TOPOLOGY, EXERCISE SHEET 6, 22.10.2014

Exercise 1.
(1) Let G be a group acting on a space E. Show that the quotient map $E \to E/G$ is open.
(2) Let $p\colon E \to X$ be a fiber bundle. Show that p is an open map.
(3) Let $p\colon E \to X$ be a fiber bundle with fiber F and let $f\colon X' \to X$ be any map. Show that the projection
$$f^*(p)\colon X' \times_X E \to X'$$
is again a fiber bundle with fiber F.

Exercise 2. In Exercise 6 of sheet 5 we proved that given an Hurewicz fibration $p\colon E \to X$ we can define a functor from $\pi(X)$ to $\mathsf{Ho}(\mathsf{Top})$ sending every point $x \in X$ to the corresponding fiber $p^{-1}(x)$.

Here we want to show that in the case of a fiber bundle p there is an equivalent functor sending all the points $x \in X$ to the same object of $\mathsf{Ho}(\mathsf{Top})$. Let $p\colon E \to X$ be a fiber bundle with fiber F.

(1) Define a functor $\mathbf{F}\colon \pi(X) \to \mathsf{Ho}(\mathsf{Top})$ such that $\mathbf{F}(x) = F$ for every $x \in X$ and F is naturally equivalent to the functor defined in Exercise 6 of sheet 5 (i.e., there is a natural equivalence of functors between them).
(2) Suppose that F is discrete. Use the previous point to define a left action of $\pi_1(X, x_0)$ on F.

Exercise 3. Let E, F be spaces and let G be a group. Suppose that we have a right action of G on E and a left action of G on F. Then we obtain an induced right action of G on $E \times F$ by setting:
$$\begin{aligned} \cdot\colon E \times F \times G &\to E \times F \\ (e, f, g) &\mapsto (e \cdot g, g^{-1} \cdot f) \end{aligned}$$
The orbit space $(E \times F)/G$ will be denoted by $E \times_G F$. There is a map $p\colon E \times_G F \to E/G$ induced by the first projection of the product. Show that if $E \to E/G$ is a principal bundle then p is a fiber bundle with fiber F.

(Warning: the notation for the orbit space of $E \times F$ is not meant to indicate that it would be a pullback! Instead the notation emphasizes a similarity between this construction and the tensor products of modules.)

Exercise 4. Let X be a connected and locally simply connected space and let $p\colon \widetilde{X} \to X$.

(1) Let $[\kappa_{x_0}] \in \widetilde{X}$ be the homotopy class of the constant path at x_0. Show that the map $\epsilon_1\colon \widetilde{X} \to X$ gives a homeomorphism between an open neighbourhood of $[\kappa_{x_0}]$ in $\widetilde{X}$ and an open neighbourhood of x_0 in X.
(2) Show that p is a principal bundle induced by the action of $\pi_1(X, x_0)$ on $\widetilde{X}$;
(3) Let $e\colon Y \to X$ be a fiber bundle with discrete fiber F. Combine Exercises 2 and 3 to conclude that $Y \cong \widetilde{X} \times_{\pi_1} F$ where $\pi_1 = \pi_1(X, x_0)$ is the fundamental group and that this isomorphism is compatible with the maps to X.

Exercise 5. Let G be a compact Hausdorff topological group and let H, K be closed subgroups of G such that $K \subset H$. Prove that the canonical map $p\colon G/K \to G/H$ is a fiber bundle with fiber H/K.

(Hint: define actions of H on G and H/K in such a way that $G/K \cong G \times_H (H/K)$ and then conclude by Exercise 3.)

Exercise 6. Let $\cdot\colon S^3 \times S^1 \to S^3$ be the action of S^1 on S^3 defined in Lecture 3, Example 3.11(iii). Use the fact that the homotopy groups $\pi_k(S^n)$ are trivial for all $i < n$.

(1) Prove that this action defines a principal bundle and $S^3/S^1 \cong S^2$.
(2) Conclude that $\pi_3(S^3) \cong \pi_3(S^2)$ and $\pi_2(S^2) \cong \mathbb{Z}$. (We will later see that $\pi_3(S^3) \cong \mathbb{Z}$. Thus, this exercise shows that the homotopy groups of a space can be non-trivial even in dimensions *larger* than the dimension of the space: $\pi_3(S^2) \cong \mathbb{Z}$.)

Exercise 7. For $n \in \mathbb{N}_{>0}$ and $k \leq n$ let us denote by $O(n) \subseteq GL(n, \mathbb{R})$ the real orthogonal group, by $V_{k,n}$ the (n,k)-Stiefel variety, and by $G_{k,n}$ the (n,k)-Grassmann variety.

(1) Prove that for every n the real orthogonal group $O(n)$ defined in the lecture has two connected components.
(2) Observe that the Grassmanian $G_{1,n}$ can be identified with $\mathbb{P}^{n-1}(\mathbb{R})$, the real projective space of dimension $n-1$.
(3) Prove that for $i < n-k$ we have isomorphisms $\pi_i(G_{k,n}) \cong \pi_{i-1}(O(k))$. Conclude from this that, in particular, there is an isomorphism $\pi_1(\mathbb{P}^n(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$ for every $n > 1$.