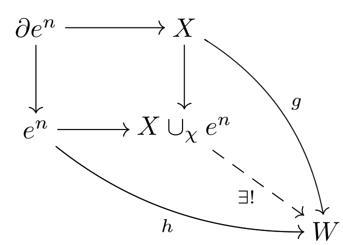
ALGEBRAIC TOPOLOGY, EXERCISE SHEET 7, 29.10.2014

Exercise 1.

(1) Show that the maps $X \to X \cup_\chi e^n$ and $e^n \to X \cup_\chi e^n$ are continuous and make the following square commutative. Moreover, prove that the triple consisting of the space $X \cup_\chi e^n$ and these two maps is initial with respect to this property. In other words, if we are given a triple (W, g, h) consisting of a topological space W and continuous maps $g\colon X \to W$ and $h\colon e^n \to W$ such that the outer square in the following diagram commutes



then there is a unique dashed arrow $X \cup_\chi e^n \to W$ such that the two triangles commute.

(2) Define more generally the notion of a pushout for two arbitrary maps $A \to X$ and $A \to Y$ of spaces with a common domain. Show that the pushout exists and is unique up to a unique isomorphism in a way which is compatible with the structure maps.

(3) Recall the notion of a pullback from a previous lecture and compare the two notions. These two notions are dual to each other. Compare also the actual constructions of pushouts and pullbacks in the category of spaces and see in which sense they are dual.

(4) The notion of a pushout makes sense in every category but does not necessarily exist. To familiarize yourself with the concept, show that the categories Set and Ab have pushouts by giving an explicit construction.

Exercise 2.

(1) The torus can be obtained from the square by identifying opposite sides. Use an adapted CW decomposition of the square to also turn the torus into a CW complex.

(2) Similarly we can obtain the Klein bottle from the unit square by identifying $(0, t) \sim (1, t)$ and $(s, 0) \sim (1 - s, 1)$. Show that there is a similar CW decomposition of the Klein bottle.

(3) Can you come up with CW decompositions of the torus and the Klein bottle which have the same number of cells in each dimension? In particular this shows the obvious fact that the number of cells does *not* determine the space.

Exercise 3. Let (X, Y) be a CW pair. Then the quotient space X/Y can be turned into a CW complex such that the quotient map $X \to X/Y$ is cellular.

Recall from the lecture that we sketched a strategy on how to try to endow the product of two CW complexes with the structure of a CW complex. For finite CW complexes this strategy is successful and you are asked to use this fact in the following exercise.

Exercise 4. Let W be a finite CW complex whose cells are parametrized by sets J_n. We define the **Euler characteristic** $\chi(W)$ of W to be the integer

$$\chi(W) = \sum_{n\in\mathbb{N}} (-1)^n |J_n|.$$

Note that the sum is finite by our assumptions on W. Show that the Euler characteristic has the following properties (all spaces are finite CW complexes):

(1) For subcomplexes A and B of X such that $A \cup B = X$ we have:

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cup B) \qquad \text{(additivity)}$$

(2) For CW complexes X and Y we have:

$$\chi(X \times Y) = \chi(X) \times \chi(Y) \qquad \text{(multiplicativity)}$$

(3) A combination of some of the constructions introduced so far allows us to deduce that the cone CX of a finite CW complex X is again a finite CW complex (show this using, for example, the CW structure on the interval consisting of two 0-cells and one 1-cell). Use this explicit cell structure on CX to show that we have

$$\chi(CX) = 1.$$

Exercise 5. Let X be a CW complex, Y a topological space, and $H\colon X \times I \to Y$ a map of sets. Then H is continuous if and only if each composition

$$H \circ (\chi_\sigma \times id_I)\colon e^n_\sigma \times I \to X \times I \to Y$$

is continuous for each cell e^n_σ of X.

Exercise 6 (**Hawaiian Earrings**)**.** Let $E_n \subset \mathbb{R}^2$ be the circle with radius $1/n$ centered at $(1/n, 0)$ and radius $1/n$. Moreover, let $E = \bigcup_{n\in\mathbb{N}} E_n$ be topologized as subspace of $\mathbb{R}^2$. Show that there is no CW structure on E.

Exercise 7. Let X be a CW complex and for every $n > 0$ let $\{e_\sigma\}_{\sigma\in J_n}$ be the set of n-cells of X. Let us set $J = \bigcup_{n>0} J_n$ and let $A \subseteq X$ be a subspace.

(1) Given a tuple $\varepsilon = \{\varepsilon_\sigma\}_{\sigma\in J}$ of numbers $\varepsilon_\sigma \in (0, 1]$ we want to inductively construct an 'ε-small' neighborhood $N_\varepsilon(A)$ of A in X. Begin by setting $N^0_\varepsilon(A) = A \cap X^{(0)}$. For the induction step, show that from $N^{n-1}_\varepsilon(A)$ you can build a neighborhood $N(A)^n_\varepsilon$ of $A \cap X^n$ in X^n which has the following two properties:
- For the intersection with $X^{(n-1)}$ we obtain $N(A)^n_\varepsilon \cap X^{(n-1)} = N(A)^{n-1}_\varepsilon$.
- For every n-cell $\sigma \in J_n$ and every $x \in \chi^{-1}_\sigma(N(A)^n_\varepsilon)$ the distance from x to the preimage $\chi^{-1}_\sigma(A \cup N(A)^{n-1}_\epsilon)$ is less than ε_σ.

Now observe that the union $N_\varepsilon(A) = \bigcup_{n\in\mathbb{N}} N^n_\varepsilon(A)$ is a neighborhood of A in X. (Hint: it might be useful to identify use "polar coordinates", i.e., identify D^n with the cone over S^{n-1}).

(2) Show that every CW complex is normal. Thus show that disjoint closed subsets have disjoint open neighborhoods, and that points are closed.