HOMOTOPY THEORY, EXERCISE SHEET 8, 05.11.2014

Exercise 1. Conclude the proof of Lemma 8.5 of Lecture 8, i.e., show that a map $i\colon A \to X$ is a cofibration if and only if for all spaces W and commutative diagrams of the form

$$\begin{array}{ccc} A & \longrightarrow & W^I \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle \epsilon_0} \\ X & \longrightarrow & W \end{array}$$

there is a diagonal filler as indicated, i.e., such a map making both triangles commutative. Do this by using the *naturality* of the bijection between maps $U \times I \to Y$ and maps $U \to Y^I$.

Exercise 2. (**Closure properties of cofibrations.**)

(1) Homeomorphisms are cofibrations. Similarly, if we have maps $i\colon A \to X$, $i'\colon A' \to X'$, and homeomorphisms $A \cong A'$, $X \cong X'$ such that

$$\begin{array}{ccc} A & \xrightarrow{\cong} & A' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ X & \xrightarrow[\cong]{} & X' \end{array}$$

commutes, then i is a cofibration if and only if i' is one.

(2) Cofibrations are closed under composition, i.e., if $i\colon A \to X$ and $j\colon X \to Y$ is a cofibration then so is $j \circ i\colon A \to Y$.

(3) Cofibrations are closed under coproducts, i.e., if we have a family $i_j\colon A_j \to X_j,\ j \in J$, of cofibrations then also the map $\sqcup_j i_j\colon \sqcup_j A_j \to \sqcup_j X_j$ is a cofibration.

(4) For two spaces X and Y, the inclusion $X \to X \sqcup Y$ is a cofibration. In particular, taking X to be the empty space, the map $\emptyset \to Y$ is cofibration for every space Y.

Exercise 3. Let $i\colon A \to X$ be a cofibration. Then i is injective.

Exercise 4. (1) Let $f\colon X \to Y$ be a homotopy equivalence. Then f is a weak equivalence. (Note that the functoriality of the homotopy groups does not suffice to solve this part!)

(2) Let $f\colon X \to Y$, $g\colon Y \to Z$ be maps of spaces, and let $h = gf\colon X \to Z$ be their composition. Show that if two of the maps f, g, and h are weak equivalences then so is the third one.

(3) Two spaces X and Y are called **weakly equivalent** if there are finitely many weak equivalences

$$X = X_0 \longrightarrow X_1 \longleftarrow X_2 \longrightarrow \ldots\ldots \longleftarrow X_{n-1} \longrightarrow X_n = Y$$

pointing possibly in different directions which 'connect' X and Y. Check that this is an equivalence relation. The equivalence classes with respect to this equivalence relation are called **weak homotopy types**.

(4) More generally, consider a relation $R \subseteq S \times S$. Define explicitly the equivalence relation $\sim_R$ on S generated by R, i.e., the smallest equivalence relation which contains R. Relate this to the previous part of the exercise (ignore set-theoretical issues for this comparison!).