ALGEBRAIC TOPOLOGY, EXERCISE SHEET 10, 03.12.2014

Exercise 1. Conclude the proof of the existence of CW approximations (Lecture 10, Theorem 10.7) by establishing the following three steps (we use the notation of the proof):

(1) The map ${f'_{n+1}}_*\colon \pi_n(K'_{n+1},*) \to \pi_n(X,*)$ is injective (and hence a bijection).
(2) The map $f_{n+1}\colon K_{n+1} \to X$ constructed in the induction step is an $(n+1)$-equivalence.
(3) The map $f\colon K \to X$ is a weak equivalence.

Exercise 2. A close inspection of the proof of the existence of CW approximations (Lecture 10, Theorem 10.7) shows that we also have a refined version as given in statement (1). Use it to also deduce statement (2).

(1) Let X be a n-connected space. Then there is a CW approximation $K \to X$ such that K has a trivial n-skeleton, i.e., such that $K^{(n)} = *$.
(2) A n-connected CW complex is homotopy equivalent to a CW complex with trivial n-skeleton.

Exercise 3. Let (X, x_0) and (Y, y_0) be pointed CW-complexes. Suppose that X is n-connected and Y is m-connected for $n, m \geq 0$. Show that $X \wedge Y$ is an $(n+m+1)$-connected CW-complex (assuming that the product of CW-complexes is again a CW-complex).

Exercise 4. (Double Comb Space) In this exercise we will see that the assumptions made in Whitehead's theorem are essential. More precisely, you construct a space Z –necessarily not a CW complex– which is weakly contractible but not contractible.

For every natural number $n > 0$ consider the following segments in the real plane $\mathbb{R}^2$

$$A_n = \{(x,y) \in \mathbb{R}^2\colon (x,y) = t(-1,0) + (1-t)(0,\frac{1}{n}) \text{ for some } t \in [0,1]\}$$

$$B_n = \{(x,y) \in \mathbb{R}^2\colon (x,y) = t(1,0) + (1-t)(0,-\frac{1}{n}) \text{ for some } t \in [0,1]\}$$

$$C = \{(x,y) \in \mathbb{R}^2\colon (x,y) = t(1,0) + (1-t)(-1,0) \text{ for some } t \in [0,1]\}$$

and let us also form the following subspaces of $\mathbb{R}^2$:

$$A = \bigcup_{n\in\mathbb{N}_{>0}} A_n, \qquad B = \bigcup_{n\in\mathbb{N}_{>0}} B_n, \qquad \text{and} \qquad Z = A \cup C \cup B$$

Use the following steps to show that Z is weakly contractible, i.e., that $\pi_n(Z)$ is trivial for every $n \in \mathbb{N}$.

(1) Every map $\alpha\colon S^n \longrightarrow Z$ is homotopic to a map with image in $C \cup B$ via the homotopy $H\colon S^n \times I \longrightarrow Z$ defined as follows:

$$H(s,t) = \begin{cases} \alpha(s) \text{ if } \alpha(s) \in C \cup B \\ t\alpha(s) + (1-t)(-1,0) \text{ otherwise} \end{cases}$$

$$H(s,t) = \begin{cases} \alpha(s), & \alpha(s) \in C \cup B \\ t\alpha(s) + (1-t)(-1,0), & \text{otherwise} \end{cases}$$

(2) Prove similarly that every map $\alpha\colon S^n \longrightarrow C \cup B$ is homotopic to a map with image in C and conclude that all homotopy groups of Z are trivial.

Using the following steps you will now show that Z is not contractible. Observe that Z is contractible if and only if there exist a map $H\colon Z \times I \longrightarrow Z$ such that

$$H(-,0) = \mathrm{id}_Z \qquad \text{and} \qquad H(z,1) = (0,0) =: p$$

for every $z \in Z$. We now suppose that such a map exists and deduce a contradiction.

(1) Consider the following two sequences in Z

$$\{a_n = (0, \frac{1}{n})\colon n \in \mathbb{N}_{>0}\} \qquad \text{and} \qquad \{b_n = (0, -\frac{1}{n})\colon n \in \mathbb{N}_{>0}\}$$

which both converge to p. For every $n > 0$ there exist a t_n such that $H(a_n, t_n) = (-1, 0)$. Conclude that the subset of the interval

$$\{t \in I\colon H(t,p) = (-1,0)\}$$

is non empty and admit a minimal element t_0.

(2) Observe that $\lim_n H(b_n, t_0) = (-1, 0)$ and conclude that $H(b_n, t_0) \in A \cup C$ for n big enough. Use an argument similar to the one used in the previous point to conclude that there exist a $t'_0 < t_0$ such that $H(t'_0, p) = (1, 0)$.

(3) Argue again similar to the previous point to show that there exist a $t''_0 < t_0$ such that $H(t''_0, p) = (-1, 0)$ and conclude that we deduced a contradiction.

Thus, there are connected spaces with trivial homotopy groups in all dimensions which are nevertheless not contractible spaces.