ALGEBRAIC TOPOLOGY, EXERCISE SHEET 11, 10.12.2014

Exercise 1. (**Eilenberg-MacLane spaces**)

(1) Let X and Y be Eilenberg-MacLane spaces which have a nontrivial homotopy group in the same dimension. Show that $X \times Y$ is again an Eilenberg-MacLane space.
(2) The loop space of an Eilenberg-MacLane space is again an Eilenberg-MacLane space.
(3) Let $E \to B$ be a Serre fibration with fiber F such that the total space E is weakly contractible, i.e., such that $\pi_n(E, e_0) \cong *$ for all $e_0 \in E$ and all $n \geq 0$. Then F is an Eilenberg-MacLane space if and only if B is one.

Exercise 2. Let X' be obtained from X by attaching an $(n+1)$-cell, i.e., let us assume that there is a pushout of pointed spaces (we suppress base points from notation)

$$\begin{array}{ccc} \partial e^{n+1} & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle i} \\ e^{n+1} & \longrightarrow & X'. \end{array}$$

(1) The induced map $i_* \colon \pi_n(X, *) \to \pi_n(X', *)$ kills $\alpha = [f]$, i.e., $i_*(\alpha) = *$.
(2) Let $k \colon X \to Y$ be a further map such that $k_*(\alpha) = *$. Then there is a map $j \colon X' \to Y$ such that $j \circ i = k$. Let j and j' be two such extensions. Find conditions on Y which guarantee that j and j' are homotopic relative to X.

Exercise 3. (**Cofiber of a pointed map**)
Recall the definition of the (reduced) cone $CX = C(X, x_0)$ of a pointed space (X, x_0) and also that it comes with a natural inclusion $i \colon (X, x_0) \to (CX, *)$. The **cofiber** of a map $f \colon (X, x_0) \to (Y, y_0)$ of pointed spaces is the pointed space $(C_f, *)$ defined by the following pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} \\ CX & \longrightarrow & C_f. \end{array}$$

(1) Show that the composition $g \circ f$ is null-homotopic, i.e., that there is a pointed homotopy between $g \circ f$ and the constant map on the base point.
(2) Show that a map $h \colon (Y, y_0) \to (W, w_0)$ has the property that $h \circ f$ is null-homotopic if and only if it can be extended over the cofiber of f, i.e., if and only if there is a pointed map $h' \colon C_f \to W$ such that $h' \circ g = h$:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & \searrow^{\kappa_*} \; \simeq & \downarrow{\scriptstyle h} & \swarrow_{h'} & \\ & & W & & \end{array}$$

(3) Conclude that the sequence $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (C_f, *)$ of pointed spaces is **coexact** in the sense that for all pointed spaces (W, w_0) the following sequence of pointed sets is exact

$$[(C_f, *), (W, w_0)] \xrightarrow{g^*} [(Y, y_0), (W, w_0)] \xrightarrow{f^*} [(X, x_0), (W, w_0)].$$

(We introduced the notion of an exact sequence of maps of pointed sets in an earlier lecture.)

(4) Compare the statement of (2) to the universal property of the quotient of a pointed space by a pointed subspace. Note that the relation between the cofiber of an inclusion of a pointed subspace and its quotient is similar to the relation between the fiber and the homotopy fiber of a map of pointed spaces. (You are not expected to make this precise!)

Exercise 4. (**HELP**)
Let $p\colon E \to X$ be a map of spaces. We say a map of spaces $A \to B$ has the *homotopy extension and lifting property* ("HELP") with respect to p if any commuting diagram of the form

$$\begin{array}{ccc} A \times I \coprod_{A \times \{0\}} B \times \{0\} & \longrightarrow & E \\ \downarrow & & \downarrow p \\ B \times I & \longrightarrow & X \end{array}$$

admits a diagonal map $B \times I \to E$ making the top and bottom triangle commute.

(1) Suppose $A_j \to B_j$ are maps of spaces satisfying HELP with respect to p. Show that the map $\coprod_J A_j \to \coprod_J B_j$ also satisfies HELP with respect to p.

(2) Consider a pushout square of spaces

$$\begin{array}{ccc} A & \longrightarrow & A' \\ j \downarrow & & \downarrow j' \\ B & \longrightarrow & B' \end{array}$$

If the map j satisfies HELP with respect to p, then the map j' also satisfies HELP with respect to p.

(3) Show that the inclusion $\partial e^n \to e^n$ satisfies HELP with respect to any Serre fibration p.

(4) Conclude that a relative CW complex $A \to B$ has the homotopy extension and lifting property with respect to all Serre fibrations $p\colon E \to X$.

Exercise 5. Suppose $p\colon E \to X$ is a map having the right lifting property with respect to any relative CW-complex $A \to B$. Show that p is a Serre fibration and a weak equivalence.