ALGEBRAIC TOPOLOGY, EXERCISE SHEET 12, 17.12.2014

Exercise 1. Consider a sequence of cofibrations

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$$

and a sequence of maps $v_n \colon X_n \to T$ so that the triangles

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & X_{n+1} \\ & {\scriptstyle v_n}\searrow \quad \swarrow{\scriptstyle v_{n+1}} & \\ & T & \end{array}$$

commute up to homotopy. Conclude the proof of Proposition 13.1 in the notes by showing that we can replace the maps v_n by homotopic maps $w_n \colon X_n \to T$ such that the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & X_{n+1} \\ & {\scriptstyle w_n}\searrow \quad \swarrow{\scriptstyle w_{n+1}} & \\ & T & \end{array}$$

commutes strictly.

Exercise 2. Consider two sequences of cofibrations

$$\begin{array}{ccccccc} X_0 & \xrightarrow{i_0} & X_1 & \xrightarrow{i_1} & X_2 & \xrightarrow{i_2} & \dots \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \\ Y_0 & \xrightarrow[j_0]{} & Y_1 & \xrightarrow[j_1]{} & Y_2 & \xrightarrow[j_2]{} & \dots \end{array}$$

and let $\phi_i \colon X_i \to Y_i$ be maps between these sequences making all squares commute. There is a natural map $\phi\colon \text{colim } X_i \to \text{colim } Y_i$ between the colimits of these two sequences.

Show that ϕ is a homotopy equivalence if all ϕ_i are homotopy equivalences. You can use the following proof outline:

(1) The colimit of the sequence

$$X_0 \times I \xrightarrow{i_0 \times 1} X_1 \times I \xrightarrow{i_1 \times 1} X_2 \times I \xrightarrow{i_2 \times 1} \dots$$

is homeomorphic to $\big(\text{colim } X_i\big) \times I$. Consequently, the map ϕ has a left homotopy inverse if there is a sequence of maps $\psi_i \colon Y_i \to X_i$ making the relevant squares commute, as well as homotopies $\psi_i \circ \phi_i \simeq \text{id}_{X_i}$ making the relevant squares commute (similarly for the right homotopy inverse).

(2) Consider a square

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow f \\ C & \longrightarrow & D \end{array}$$

commuting up to homotopy. If i is a cofibration, then there is a map homotopic to f which makes the diagram commute strictly.

(3) Let $i\colon A \to B$ be a cofibration and suppose we have a map $f\colon B \to B$ such that $f \circ i = i$ and $f \simeq \mathrm{id}_B$. Show that f admits a left homotopy inverse relative to A: there is a map $g\colon B \to B$ with $g \circ i = i$ and a homotopy $g \circ f \simeq \mathrm{id}_B$ relative to A.

(4) Use (2) and (3) to inductively construct *natural* left homotopy inverses ψ_i to ϕ_i, as well as natural homotopies witnessing that $\psi_i \circ \phi_i \simeq \mathrm{id}_{X_i}$. This shows that the map ϕ has a left homotopy inverse $\psi := \mathrm{colim}\ \psi_i$.

Hint: given ψ_n and a homotopy $H_n : \psi_n \circ \phi_n \simeq \mathrm{id}_{X_n}$, first construct a map $\tilde{\psi}_{n+1}$ which is left homotopy inverse to ϕ_{n+1} such that $\tilde{\psi}_{n+1} \circ j_n = i_n \circ \psi_n$. Extend the homotopy H_n to a homotopy between $\tilde{\psi}_{n+1} \circ \phi_{n+1}$ and some map $f\colon X_{n+1} \to X_{n+1}$ which is the identity on X_n. Use (3) to find a left homotopy inverse g to f relative to X_n and consider the map $\psi_{n+1} := g \circ \tilde{\psi}_{n+1}$. Show that this map is left homotopy inverse to ϕ_{n+1} via a homotopy that extends H_n.

As an application, show that two sequences

$$X_0 \underset{j_0}{\overset{i_0}{\rightrightarrows}} X_1 \underset{j_1}{\overset{i_1}{\rightrightarrows}} \dots$$

with each $i_n \simeq j_n$ have homotopy equivalent colimits.

Exercise 3. Let G be a (topologically discrete) group. We define a principal G-bundle over a pointed space (X, x_0) to be a principal G-bundle $\pi\colon P \to X$ (in the sense of Lecture 6), together with a point $p_0 \in \pi^{-1}(x_0)$.

We say that two principal G-bundles $(P, p_0) \to (X, x_0)$, $(Q, q_0) \to (X, x_0)$ are isomorphic if there is a homeomorphism $\phi\colon P \to Q$ such that $\phi(p \cdot g) = \phi(p) \cdot g$ and $\phi(p_0) = q_0$.

(1) If $f\colon Y \to X$ is a map of pointed spaces and $P \to X$ is a principal G-bundle, show that the pullback

$$f^*(P) := P \times_X Y \longrightarrow Y$$

is a principal G-bundle as well (you have already shown that it is a fiber bundle with fiber G in Exercise sheet 6).

(2) Show that the pullbacks of two isomorphic G-bundles over X are isomorphic over Y. Conclude from this that there is a functor

$$G\mathrm{Bund}(-)\colon \mathsf{Top}_*^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

sending a space X to the set of isomorphism classes of principal G-bundles over X.

From now on we consider the restriction of $G\mathrm{Bund}(-)$ to the category of pointed (connected) CW-complexes.

(3) Let $P \to X$ be a principal G-bundle and consider a map $h\colon Y \times I \to X$ such that $H(y_0, t) = x_0$ for all $t \in I$. Show that the restrictions $h^*(P)\big|_{Y\times\{0\}}$ and $h^*(P)\big|_{Y\times\{1\}}$

determine isomorphic G-bundles over Y.
Hint: use that $h^*(P) \to Y \times I$ is a covering map and that a CW complex is locally path-connected.

(4) Conclude from this that $G\mathrm{Bund}(-)$ determines a functor from the homotopy category of pointed connected CW complexes to Set.

Exercise 4. (**Classifying space of a group**)
The goal of this exercise is to show that $G\mathrm{Bund}(-)$ satisfies the conditions of the Brown representability theorem. It follows that there is a CW-complex BG such that $[X, BG] \simeq G\mathrm{Bund}(X)$. This space is called the *classifying space* of the group G.

(1) Show that the map
$$G\mathrm{Bund}\Big(\bigvee X_i\Big) \to \prod G\mathrm{Bund}(X_i)$$
is a bijection.
Hint: given a set of G-bundles $(P_i, p_i) \to (X_i, x_i)$, show that the space
$$\coprod_i P_i \Big/ p_i \cdot g \sim p_j \cdot g$$
forms a principal G-bundle over $\bigvee X_i$.

(2) Show that it suffices to check condition (ii) from the Brown representability theorem only in the case that $A \to B$ and $A \to C$ are CW subcomplexes (in which case B and C are subcomplexes of the CW complex $B \cup_A C$ with intersection A).
Hint: use that taking the pushout of two homotopic maps along cofibrations produces two homotopy equivalent spaces. Furthermore, we have seen that any cellular map factors as the inclusion of a CW subcomplex followed by a strong deformation retract.

(3) Show that for the union of two CW subcomplexes $B, C \subseteq B \cup_A C$, we have that
$$G\mathrm{Bund}\big(B \cup_A C\big) \longrightarrow G\mathrm{Bund}\big(B\big) \times_{G\mathrm{Bund}(A)} G\mathrm{Bund}\big(C\big)$$
is a surjection.
Hint: use a variation of (1), together with the fact that the topology on A is the subspace topology with respect to B, but also with respect to C.

Exercise 5.

(1) Show that the classifying space of a group G is a $K(G, 1)$.

(2) Recall that the universal cover of BG is a principal G-bundle. Show that the corresponding element in $G\mathrm{Bund}(BG)$ corresponds to the homotopy class of the identity map $BG \to BG$.

(3) Show that any principal G-bundle $P \to X$ fits in a pullback square
$$\begin{array}{ccc} P & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG \end{array}$$
where $EG \to BG$ is the universal cover of BG (note that EG is a contractible space). Because of this, the bundle $EG \to BG$ is called the *universal* G-bundle.