HOMOTOPY EXCISION AND THE FREUDENTHAL SUSPENSION THEOREM

The homotopy excision theorem is a result about relative homotopy groups, or homotopy groups of pairs. Recall that for a pair (X, A), we defined $\pi_i(X, A)$ as the set of homotopy classes of maps $I^i \to X$ which send the top face $I^{i-1} \times \{1\}$ to A and the rest of the boundary $J^{i-1} = I^{i-1} \times \{0\} \cup \partial I^{i-1} \times I$ to the base point x_0 (which we will consistently omit from the notation). In other words,

$$\pi_i(X, A) = [(I^i, \partial I^i, J^{i-1}), (X, A, x_0)].$$

This is a pointed set for $i = 1$, a group for $i = 2$, and an abelian group for $i \geq 3$. Moreover, these groups fit into a long exact sequence

$$\ldots \to \pi_i(A) \to \pi_i(X) \to \pi_i(X, A) \xrightarrow{\partial} \pi_{i-1}(A) \to \ldots$$

We didn't define $\pi_0(X, A)$, but we can set $\pi_0(X, A) = \mathrm{cok}(\pi_0(A) \to \pi_0(X))$ so that the long exact sequence can be prolonged so as to end as $\ldots \to \pi_0(A) \to \pi_0(X) \to \pi_0(X, A) \to 0$.

Recall that the pair (X, A) is n-connected if $\pi_i(X, A) = 0$, $i \leq n$. By the long exact sequence, this is the same as asking that $\pi_i(A) \to \pi_i(X)$ is an isomorphism for $i < n$ and a surjection for $i = n$ (in other words, that the inclusion $A \to X$ is an n-equivalence). Recall further that the pair (X, A) is called a relative CW complex if X is obtained from A by successively attaching cells.

The first part of the title refers to the following statement in which we consider a situation as depicted in the following diagram

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & X. \end{array}$$

Theorem 1. ***(Homotopy excision theorem)***
Let C be any space, and let (A, C) and (B, C) be relative CW complexes. Write $X = A \cup_C B$ for the union of A and B (the pushout under C). If (A, C) is m-connected and (B, C) is n-connected, then

$$\pi_i(A, C) \to \pi_i(X, B)$$

is an isomorphism for $i < m + n$, and a surjection for $i = m + n$.

To put it differently, if $C \to A$ is an m-equivalence and $C \to B$ an n-equivalence, then the induced map of pairs $(A, C) \to (X, B)$ is an $(m + n)$-equivalence. The proof of this theorem will occupy this lecture and part of the next. But before we go into the proof, we will mention the following important application. Recall the suspension functor $\Sigma(X) = X \wedge S^1$ which can also described by the following pushout

$$\begin{array}{ccc} X & \longrightarrow & C'X \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

where CX and $C'X$ are two copies of the (reduced) cone of X. In the special case of $X = S^p$, we have $\Sigma(S^p) \cong S^{p+1}$ so that the suspension induces a suspension homomorphism (see also Lecture 4)

$$S\colon \pi_i(X) \to \pi_{i+1}(\Sigma X).$$

Theorem 2. ***(Freudenthal suspension theorem)***
Let (X, x_0) be an $(n-1)$-connected CW complex. Then the suspension homomorphism

$$S\colon \pi_i(X) \to \pi_{i+1}(\Sigma X)$$

is an isomorphism for $i < 2n - 1$, and a surjection for $i = 2n - 1$.

Proof. Notice that the long exact sequence of the pair (Y, A) gives an isomorphism $\pi_i(Y) \to \pi_i(Y, A)$ for any i if A is contractible, and an isomorphism $\partial\colon \pi_{i+1}(Y, A) \to \pi_i(A)$ if Y is contractible. Now consider the square:

$$\begin{array}{ccc} \pi_{i+1}(CX, X) & \longrightarrow & \pi_{i+1}(\Sigma X, C'X) \\ \cong \big\downarrow \partial & & \big\uparrow \cong \\ \pi_i(X) & \longrightarrow & \pi_{i+1}(\Sigma X) \end{array}$$

Since the two copies CX and $C'X$ of the cone are contractible, we have the two vertical isomorphisms which are induced by the respective long exact sequences. The upper horizontal map is induced by the inclusion $(CX, X) \to (\Sigma X, C'X)$ while the bottom horizontal map can be identified with the suspension homomorphism, the map showing up in the statement of the theorem. We leave it as an exercise to verify that the diagram commutes. To prove the theorem, it thus suffices to check that the upper map is an isomorphism in an appropriate range of i's. To this end, apply the excision theorem to $\Sigma(X) = CX \cup_X C'X$. Indeed, since CX is contractible, the long exact sequence of the pair (CX, X) shows that (CX, X) is n-connected if X is $(n-1)$-connected. So the upper horizontal map is an isomorphism for $i + 1 < 2n$, and a surjection for $i + 1 = 2n$, exactly as stated in the theorem. □

Example 3. The n-sphere S^n is a CW complex with one 0-cell and one n-cell ($n > 0$), so is surely $(n-1)$-connected by the cellular approximation theorem. So by the Freudenthal suspension theorem,

$$S\colon \pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is an isomorphism for $i < 2n - 1$. In particular,

$$S\colon \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$$

is an isomorphism if $n < 2n - 1$, i.e. if $n \geq 2$. We already know that $\pi_1(S^1) \cong \mathbb{Z}$, while the long exact sequence of the Hopf fibration

$$S^1 \to S^3 \to S^2$$

together with the fact that $\pi_i(S^3) = 0$ for $i < 3$ readily shows that $\partial\colon \pi_2(S^2) \to \pi_1(S^1)$ is an isomorphism. Thus,

$$\pi_n(S^n) \cong \mathbb{Z}, \qquad \text{for all } n \geq 1.$$

Perspective 4. For an arbitrary pointed CW complex X, the Freudenthal suspension theorem and the connectivity of ΣX give that $\Sigma^n X$ is always $(n-1)$-connected. Thus, the map

$$S\colon \pi_i(\Sigma^n X) \to \pi_{i+1}(\Sigma^{n+1} X)$$

is an isomorphism for $i < 2n-1$. This implies that for a *fixed* value of k, the maps in the sequence

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \ldots \to \pi_{k+i}(\Sigma^i X) \to \pi_{k+i+1}(\Sigma^{i+1} X) \to \ldots$$

eventually become isomorphisms. (More precisely, $\pi_{k+i}(\Sigma^i X) \to \pi_{k+i+1}(\Sigma^{i+1} X)$ is certainly an isomorphism for $k+i < 2i-1$, or $k+1 < i$.) The eventual value of this sequence is called the k-th **stable homotopy group** of X, denoted

$$\pi_k^s(X).$$

These (abelian) stable homotopy groups are still extremely informative, while being more computable than the ordinary ('unstable') ones. In a sense, they sit between the unstable homotopy groups and the homology groups and form a central subject of study in algebraic topology.

More generally, given a sequence of pointed spaces $X_0, X_1, X_2, \ldots$ related by structure maps $\sigma_k \colon \Sigma X_k \to X_{k+1}$, one can form a sequence

$$\pi_k(X_0) \to \pi_{k+1}(X_1) \to \pi_{k+2}(X_2) \to \ldots \to \pi_{k+i}(X_i) \to \pi_{k+i+1}(X_{i+1}) \to \ldots$$

(by using the suspension homomorphisms together with the homomorphism induced by the structure maps). Such sequences of spaces, called **spectra**, are the main objects of 'stable homotopy theory'. Any pointed space X gives rise to a spectrum by taking $X_n = \Sigma^n X$, the **suspension spectrum** $\Sigma^\infty X$ of X. In a specific sense, the passage to the (homotopy) category of spectra is a good approximation of the (homotopy) category of spaces, which has more structure and is more tractable.

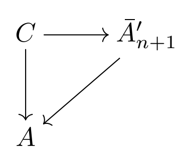
We will now turn to the proof of the excision theorem, and start with a few reductions to simpler cases. The first one is concerned with the dimension of the cells to be added to C in an m-connected pair (A, C). Notice that if A is obtained from C by attaching cells of dimension larger than m only, then the pair (A, C) is automatically m-connected (see an earlier lecture). In fact, the proof of the CW approximation theorem shows that the converse is also true as the following lemma shows.

Lemma 5. *Let $i \colon C \to A$ be an inclusion defining an n-connected pair of spaces (A, C). Then there is a relative CW complex $i' \colon C \to A'$ and a weak homotopy equivalence $e \colon A' \to A$ for which $ei' = i$, and where A' is obtained from C by attaching cells of dimension $> n$ only.*

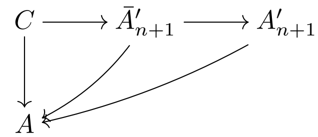
Proof. We follow the same strategy as in the CW approximation theorem, and build up a CW complex by adding cells which represent elements of $\pi_i(A)$ respectively kill elements which should not be there. Now all of $\pi_i(A)$ for $i < n$ is already represented by maps $S^i \to C$ since $\pi_i(C) \xrightarrow{\cong} \pi_i(A)$ for $i < n$, so the first step in this process consists of killing the kernel of the surjection $\pi_n(C) \to \pi_n(A)$ by attaching cells to A. This gives an extension

$$C \to \bar{A}'_{n+1}$$

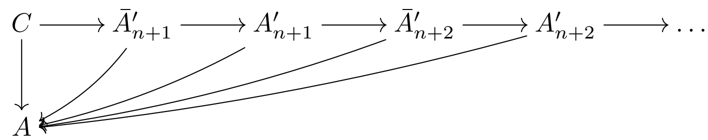
by $(n+1)$-cells, together with a map $\bar{A}'_{n+1} \to A$ making the diagram



commute. In the next step, we attach $(n+1)$-cells to the base point of $\bar{A}'_{n+1}$ to represent all elements (or a set of generators) of $\pi_{n+1}(A)$, giving a space $A'_{n+1} \supseteq \bar{A}'_{n+1}$ and an extension of the diagram by a map:

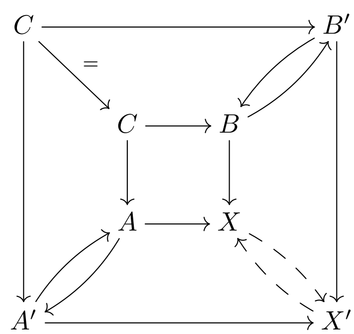


Next, we attach $(n+2)$-cells to A'_{n+1} to kill the kernel of $\pi_{n+1}(A'_{n+1}) \to \pi_{n+1}(A)$, giving an extension $\bar{A}'_{n+2}$ together with a map to A. Continuing like this, we obtain a sequence

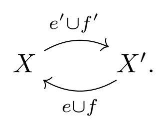


and we let A' be the union (colimit) of this sequence with the weak topology. This space A' together with the induced maps $C \to A' \to A$ verifies the assertion in the lemma. $\square$

If in Lemma **??** we start with a relative CW complex $C \to A$, then by the relative version of the Whitehead Theorem, there exists a map $e' \colon A \to A'$ with $e'i = i'$ and homotopies relative to C between ee' and 1_A, and between $e'e$ and $1_{A'}$. Thus, if we apply Lemma **??** to both $C \to A$ and $C \to B$ as in the statement of the excision theorem, we get homotopy equivalences e, e' and f, f' relative to C,



and hence for $X' = A' \cup_C B'$ well-defined homotopy equivalences



Thus we conclude:

Reduction 1. It suffices to prove the excision theorem for extensions $C \to A$ by cells of dimension larger than m and $C \to B$ by cells of dimension larger than n.

The next reduction concerns the number of cells one attaches to C to obtain A and B respectively. Let us say that a pair of extensions $C \to A$ and $C \to B$ as in Reduction 1 is of size (p, q) if A is obtained by attaching p cells (of dimension $> m$) to C, and B by attaching q cells (of dimension $> n$).

Reduction 2. If the excision theorem holds for extensions of size $(1,1)$, then it holds for extensions of arbitrary size (p,q) with $p,q \geq 1$.

Proof. Let us show by induction that the excision theorem holds for all extensions of type $(p,1)$. Given such an extension with $p > 1$ let us write

$$A = A' \cup e, \qquad X' = A' \cup_C B$$

so that we have two pushout squares

$$\begin{array}{ccc} C & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

in which the upper square is an extension of size $(p-1,1)$ and the lower one an extension of size $(1,1)$. By induction assumption the two maps of pairs $(A',C) \to (X',B)$ and $(A,A') \to (X,X')$ are $(m+n)$-equivalences and we want to conclude the same for $(A,C) \to (X,B)$. The above diagram gives us a map of triples $(A,A',C) \to (X,X',B)$ so that we obtain by Lemma **??** the following diagram

$$\begin{array}{ccccccccc} \pi_{i+1}(A,A') & \xrightarrow{\partial} & \pi_i(A',C) & \xrightarrow{i_*} & \pi_i(A,C) & \xrightarrow{j_*} & \pi_i(A,A') & \xrightarrow{\partial} & \pi_{i-1}(A',C) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ \pi_{i+1}(X,X') & \xrightarrow[\partial]{} & \pi_i(X',B) & \xrightarrow[u_*]{} & \pi_i(X,B) & \xrightarrow[v_*]{} & \pi_i(X,X') & \xrightarrow[\partial]{} & \pi_{i-1}(X',B). \end{array}$$

(Note that we needed the naturality statement of that lemma to get this commutative ladder.) To show that γ is an isomorphism in dimensions $i < n+m$ it suffices to observe that our induction assumption guarantees that β, δ and ϵ are isomorphisms while α is surjective. Thus, by the 5-Lemma (Lemma **??**), we conclude that γ is an isomorphisms. Similarly, for $i = m+n$ our induction assumption implies that β and δ are surjective and that ϵ is an isomorphism. Thus, again by the 5-Lemma, also the map γ is surjective and the map $(A,C) \to (X,B)$ is hence an $(m+n)$-equivalence.

By a second induction over q we can now show that the excision theorem holds for all extensions of size (p,q). In fact, if $q > 1$ then let us write

$$A = B' \cup e, \qquad X'' = A \cup_C B'$$

so that we have two pushout squares

$$\begin{array}{ccccc} C & \longrightarrow & B' & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & X'' & \longrightarrow & X. \end{array}$$

The map $(A,C) \to (X,B)$ factors as $(A,C) \to (X'',B') \to (X,B)$. By induction assumption, both maps are $(m+n)$-equivalences so that the same is the case for $(A,C) \to (X,B)$. $\square$

Lemma 6. *For pointed inclusions of subspaces $A \subseteq B \subseteq X$ the sequence of inclusions of pairs*

$$(B,A) \xrightarrow{k} (X,A) \xrightarrow{l} (X,B)$$

induces a long exact sequence

$$\dots \to \pi_i(B,A) \stackrel{k_*}{\to} \pi_i(X,A) \stackrel{l_*}{\to} \pi_i(X,B) \stackrel{\partial}{\to} \pi_{i-1}(B,A) \stackrel{k_*}{\to} \dots$$

with ∂ being defined by $\pi_i(X,B) \stackrel{\partial}{\to} \pi_{i-1}(B) \to \pi_{i-1}(B,A)$. The connecting homomorphism is natural in maps of triples in the obvious sense.

Proof. We will just prove exactness at $\pi_i(X,A)$, and leave the other cases and the naturality as exercise. Also, we will write the proof for abelian groups, and leave as an exercise that the lemma also holds in those low degrees where one just has groups or even pointed sets (exercises!). It is clear that the composition

$$\pi_i(B,A) \stackrel{k_*}{\to} \pi_i(X,A) \stackrel{l_*}{\to} \pi_i(X,B)$$

is the zero map. To prove that $\ker(l_*) \subseteq \operatorname{im}(k_*)$, expand the diagram to

$$\begin{array}{ccccc}
\pi_i(B) & \xrightarrow{i_*} & \pi_i(X) & \xrightarrow{v_*} & \pi_i(X,B) \\
\downarrow{\scriptstyle w_*} & & \downarrow{\scriptstyle u_*} & & \downarrow{\scriptstyle =} \\
\pi_i(B,A) & \xrightarrow{k_*} & \pi_i(X,A) & \xrightarrow{l_*} & \pi_i(X,B) \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
\pi_{i-1}(A) & \xrightarrow[=]{} & \pi_{i-1}(A) & \xrightarrow[j_*]{} & \pi_{i-1}(B)
\end{array}$$

where ∂, ∂, and d are boundary maps for pairs, while we use the following names for inclusions:

$$\begin{array}{ccccc}
A & \xrightarrow{=} & A & \xrightarrow{j} & B \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle =} & & \downarrow \\
B & \xrightarrow{i} & X & \xrightarrow{v} & (X,B) \\
\downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle =} \\
(B,A) & \xrightarrow[k]{} & (X,A) & \xrightarrow[l]{} & (X,B)
\end{array}$$

Now suppose $x \in \pi_i(X,A)$ with $l_*(x) = 0$. Then $j_*\partial x = \partial l_* x = 0$, so $\partial x = d(y)$ for some $y \in \pi_i(B,A)$. Then $\partial(k_*(y) - x) = dy - \partial x = 0$, so $k_*(y) - x = u_*(z)$ for some $z \in \pi_i(X)$. But then $v_* z = l_* u_* z = l_* k_* y - l_* x = 0$, so $z = i_* t$ for some $t \in \pi_i(B)$. Now

$$x = k_* y - u_*(z) = k_* y - u_* i_* t = k_*(y - w_* t),$$

showing that any $x \in \ker(l_*)$ lies in the image of k_*. □

We refer to the sequence of Lemma **??** as the **long exact sequence of the (pointed) triple** (X,B,A). Note that the definition of the connecting homomorphism of this sequence uses the connecting homomorphism of the long exact sequence of the pair (X,B).

Lemma 7. ***(5-lemma)***
Consider a diagram of abelian groups (or groups) and homomorphisms with exact rows

$$\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & E \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\
A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \xrightarrow[h']{} & D' & \xrightarrow[i']{} & E'.
\end{array}$$

(1) *If β and δ are surjective while ϵ is injective then γ is surjective.*
(2) *If β and δ are injective while α is surjective then γ is injective.*

In particular, β and δ are isomorphisms, ϵ is injective, and α is surjective, then also γ is an isomorphism.

Proof. Again, we write the proof for abelian groups and leave the verification for the relevant cases of groups and pointed sets as an exercise.

(1) To check that γ is onto consider $c' \in C'$. Since δ is onto, there is a $d \in D$ such that $\delta(d) = h'(c')$. Then $\epsilon(i(d)) = i'(\delta(d)) = i'(h'(c')) = 0$. Since ϵ is injective it follows that $i(d) = 0$, i.e., that $d \in \ker(i) = \operatorname{im}(h)$. Thus, there is a $c \in C$ such that $h(c) = d$. Now,

$$h'(c' - \gamma(c)) = h'(c') - \delta(h(c)) = h'(c') - \delta(d) = h'(c') - h'(c') = 0,$$

hence $c' - \gamma(c) = g'(b')$ for some $b' \in B'$. Since β is surjective there exists $b \in B$ such that $\beta(b) = b'$. We conclude that $c' \in \operatorname{im}(\gamma)$ by the final calculation

$$\gamma(c + g(b)) = \gamma(c) + g'(\beta(b)) = \gamma(c) + g'(b') = \gamma(c) + (c' - \gamma(c)) = c'.$$

(2) Let $c \in C$ be such that $\gamma(c) = 0$. Then $h'(\gamma(c)) = \delta(h(c)) = 0$. Since δ is injective, we deduce that $h(c) = 0$ and hence $c = g(b)$ for some $b \in B$. It suffices to show that $b \in \operatorname{im}(f)$ since then $gf = 0$. Now $g'(\beta(b)) = \gamma(g(b)) = \gamma(c) = 0$ tells us that $\beta(b) = f'(a')$ for some $a' \in A'$. Using the surjectivity of α we conclude that $a' = \alpha(a)$ for some $a \in A$. Thus, the element $f(a)$ satisfies $\beta(f(a)) = f'(\alpha(a)) = \beta(b)$ and the injectivity of β implies $f(a) = b$ as intended.

Combining these two statements immediately implies the remaining claim. □

As a further special case, if the four morphisms α, β, δ, and ϵ are isomorphisms then so is the fifth γ.

After all these preparations, the proof of the Excision Theorem will be completed by the following proposition.

Proposition 8. *Let C be a space, and define spaces $A = C \cup e$, $B = C \cup e'$, and $X = A \cup B$ by attaching cells of dimension $> m$ and $> m'$, respectively. Then $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < n + m$ and a surjection for $i \leq n + m$.*

Proof. We will only prove surjectivity for $i \leq n+m$. The proof of injectivity of $i < n+m$ proceeds in exactly the same way, and is left as an exercise. If $x \in e^\circ$ and $y \in e'^\circ$ are points in the interior of the cells e and e', there is a diagram

$$\begin{array}{ccc}
\pi_i(A, C) & \longrightarrow & \pi_i(X, B) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\
\pi_i(X - \{y\}, X - \{x, y\}) & \longrightarrow & \pi_i(X, X - \{x\})
\end{array}$$

where the vertical maps are isomorphisms. Indeed, the space $X - \{x\}$ is homotopy equivalent to B because one can contract $e - \{x\}$ to its boundary; and similarly for $X - \{y\}$ and $X - \{x, y\}$. Consider an arbitrary map $f\colon I^i \to X$ which represents an element of $\pi_i(X, B)$ for $i \leq m + n$. This means that f maps the top face $I^{i-1} \times \{1\}$ into B and sends the rest of the boundary $J^{i-1} = I^{i-1} \times \{0\} \cup \partial(I^{i-1}) \times I$ to the base point x_0. By the above diagram, it suffices to prove that f is homotopic to a map $h = h_1$ through a homotopy h_s, $s \in [0, 1]$ such that

(a) h avoids the point y, i.e., $h\colon I^i \to X - \{y\}$,
(b) in addition, for every $s \in [0, 1]$, the restriction of h_s to the top face of I^i avoids the point x,
(c) for each $s \in [0, 1]$, h_s maps J^{i-1} to the base point.

Let $e_{1/2}$ and $e'_{1/2}$ be small balls of radius 1/2 (or for that matter, any non-empty open subsets whose closures are contained in the interior of e and e', respectively), and let

$$U = f^{-1}(e^\circ_{1/2} \cup e'^\circ_{1/2}).$$

We will now use the basic fact that any continuous map between manifolds can be approximated by a homotopic smooth map (see e.g. the book by Bott, Tu). Since $\bar{U}$ is disjoint form J^{i-1}, this gives a map $g\colon I^i \to X$ such that $g = f$ on J^{i-1}, and the restriction of g is smooth as a map $g\colon U \to e^\circ_{1/2} \cup e'^\circ_{1/2}$. Moreover, by choosing g as well as the homotopy close to f, we can assume that the restriction of g as well as that of the homotopy $g \simeq f$ to the top face of I^i both avoid x. Let us write $V = g^{-1}(e^\circ_{1/4})$ and $V' = g^{-1}(e'^\circ_{1/4})$. Then, since g is close to f we can assume that the closure of $V \cup V'$ is contained in U. In other words, g is smooth over the entire preimage of these two small balls of diameter 1/4. Also, write $\pi\colon I^i = I^{i-1} \times I \to I^{i-1}$ for the projection away from the last coordinate. We claim that there exist points x and y with

$$x \in e^\circ_{1/4}, \quad y \in e'^\circ_{1/4}, \quad \text{and} \quad \pi g^{-1}(x) \cap \pi g^{-1}(y) = \emptyset. \tag{1}$$

Indeed, let $V \times_{I^{i-1}} V'$ be the pullback along π, consisting of pairs (v, v') with $v \in V, v' \in V'$ and $\pi(x) = \pi(v')$, and consider the smooth map

$$g \times g\colon V \times_{I^{i-1}} V' \to e^\circ_{1/4} \cup e'^\circ_{1/4}. \tag{2}$$

Then a pair of points (x, y) satisfies (**??**) if and only if it is *not* in the image of the map (**??**). So we only need to check that $g \times g$ is not surjective. This is simply a matter of counting dimensions: $V \times_{I^{i-1}} V'$ is a manifold of dimension $i + 1$, and since $i \leq m + n$ we have

$$i + 1 < (m + 1) + (n + 1) \leq \dim(e) + \dim(e') = \dim(e^\circ_{1/4} \times e'^\circ_{1/4}).$$

So any regular value (x, y) of (**??**) cannot be in its image (cf. Guillemin-Pollack, page 21). Since $\pi g^{-1}(x)$ and $\pi g^{-1}(y)$ are disjoint closed subsets of I^{i-1}, there exists a continuous (even smooth) map $\theta\colon I^{i-1} \to I$ with

$$\theta\,|_{\pi g^{-1}(x)} = 0 \qquad \text{and} \qquad \theta\,|_{\pi g^{-1}(y)} = 1.$$

Now define the homotopy

$$h_s\colon I^{i-1} \times I \to X, \qquad s \in [0, 1],$$

by

$$h_s(z, t) = g(z, t - st\theta(z)).$$

Notice that $h_0 = g$. We claim that h_s and $h_1 = h$ satisfy requirements (a)-(c).

For (a), suppose to the contrary that $h_1(z, t) = y$, i.e., $g(z, 1 - \theta(z)) = y$. Then $z \in \pi g^{-1}(y)$ so $\theta(z) = 1$, whence $g(z, 0) = y$. But g (like f) maps the bottom face of $I^i = I^{i-1} \times I$ to the base point x_0 which does not lie in $e'^\circ_{1/2}$, so this is impossible.

For (b) we argue similarly: Suppose $h_s(z,1) = x$. Then $z \in \pi g^{-1}(x)$ so $\theta(z) = 0$, whence $g(z, 1 - s\theta(z)) = g(z,1) = x$. This contradicts that g avoids x on the top face of I^i.

Finally, for (c), take $(z,t) \in J^{i-1}$. Then either $t = 0$ whence $h_s(z,t) = g(z,t)$, or $z \in \partial I^{i-1}$, and in both cases $h_s(z,t) = x_0$ because g agrees with f on J^{i-1}. $\square$