LECTURE 1: HOMOTOPY AND THE FUNDAMENTAL GROUPOID

You probably know the fundamental group $\pi_1(X, x_0)$ of a space X with a base point x_0, defined as the group of homotopy classes of maps $S^1 \to X$ sending a chosen base point on the circle to x_0. In this course we will define and study *higher homotopy groups* $\pi_n(X, x_0)$ by using maps $S^n \to X$ from the n-dimensional sphere to X, and see what they tell us about the space X.

Let us begin by fixing some preliminary conventions. By *space* we will always mean a Hausdorff topological space (later, we may narrow this down to various other important classes of spaces, such as *compact* spaces, *locally compact* spaces, *compactly generated* spaces, or *CW-complexes*). A *map* between two such spaces will always mean a *continuous map*.

A *pointed space* is a pair (X, x_0) consisting of a space X and a base point $x_0 \in X$. A *map between pointed spaces* $f\colon (X, x_0) \to (Y, y_0)$ is a map $f\colon X \to Y$ with $f(x_0) = y_0$. A *pair of spaces* is a pair (X, A) consisting of a space X and a subspace $A \subseteq X$. A *map of pairs* $f\colon (X, A) \to (Y, B)$ is a map $f\colon X \to Y$ such that $f(A) \subseteq B$.

1. Homotopy of maps

We now recall the central notion of a homotopy.

Definition 1.1. Two maps of spaces $f, g\colon X \to Y$ are called *homotopic* if there is a continuous map $H\colon X \times [0,1] \to Y$ such that:

$$H(x,0) = f(x) \qquad \text{and} \qquad H(x,1) = g(x), \qquad x \in X$$

Such a map H is called a *homotopy* from f to g. We write $f \simeq g$ to denote such a situation or $H\colon f \simeq g$ if we want to be more precise.

The homotopy relation enjoys the following nice properties.

Proposition 1.2. *Let $X, Y,$ and Z be spaces.*

(i) *The homotopy relation is an equivalence relation on the set of all maps from X to Y.*
(ii) *If $f, g\colon X \to Y$ and $k, l\colon Y \to Z$ are homotopic then also $kf, lg\colon X \to Z$ are homotopic.*

Proof. Let us prove the first claim. So, let us consider maps $f, g, h\colon X \to Y$. The homotopy relation is reflexive since we have the following *constant homotopy* at f:

$$\kappa_f\colon f \simeq f \qquad \text{given by} \qquad \kappa_f(x,t) = f(x)$$

Given a homotopy $H\colon f \simeq g$ then we obtain an *inverse homotopy* H^{-1} as follows:

$$H^{-1}\colon g \simeq f \qquad \text{with} \qquad H^{-1}(x,t) = H(x, 1-t)$$

Thus, the homotopy relation is symmetric. Finally, if we have homotopies $F\colon f \simeq g$ and $G\colon g \simeq h$ then we obtain a homotopy $H\colon f \simeq h$ by the following formula:

$$H(x,t) = \begin{cases} F(x,2t) & , \quad 0 \leq t \leq 1/2 \\ G(x,2t-1) & , \quad 1/2 \leq t \leq 1 \end{cases}$$

Thus we showed that the homotopy relation is an equivalence relation.

To prove the second claim let us begin by two special cases. Let us assume that we have a homotopy $H\colon f \simeq g$. Then we obtain a homotopy from kf to kg simply by post-composition with g:

$$kH\colon X \times [0,1] \xrightarrow{H} Y \xrightarrow{k} Z \quad \text{is a homotopy} \quad kH\colon kf \simeq kg.$$

The next case is slightly more tricky. Given a homotopy $K\colon k \simeq l$ then we obtain a homotopy from kg to lg as follows:

$$K \circ (g \times \mathrm{id})\colon X \times [0,1] \longrightarrow Y \times [0,1] \longrightarrow Z \quad \text{defines a homotopy} \quad K \circ (g \times \mathrm{id})\colon kg \simeq lg.$$

In order to obtain the general case it suffices now to use the transitivity of the homotopy relation since from the above two special cases we deduce $kf \simeq kg \simeq lg$ as intended. This concludes the proof. □

The equivalence classes with respect to the homotopy relation are called *homotopy classes* and will be denoted by $[f]$. Given two spaces X and Y then the set of homotopy classes of maps from X to Y is denoted by $[X, Y]$.

This proposition allows us to form a new category where the objects are given by spaces and where morphisms are given by homotopy classes of maps. Let us begin by recalling the notion of a category.

Definition 1.3. A *category* $\mathcal{C}$ consists of the following data:

(i) A collection $\mathrm{ob}(\mathcal{C})$ of *objects* in $\mathcal{C}$.
(ii) Given two objects X, Y there is a set $\mathcal{C}(X, Y)$ of *morphisms* in $\mathcal{C}$.
(iii) Associated to three objects X, Y, Z there is a *composition map*:

$$\circ\colon \mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \longrightarrow \mathcal{C}(X,Z)\colon (g,f) \mapsto g \circ f$$

This datum has to satisfy the following two properties:

- The composition is associative, i.e., we have $(h \circ g) \circ f = h \circ (g \circ f)$ whenever these expressions make sense.
- For every object X there is an *identity morphism* $id_X \in \mathcal{C}(X,X)$ such that for all morphisms $f \in \mathcal{C}(X,Y)$ we have:

$$id_Y \circ f = f = f \circ id_X$$

We use the following standard notation. Given a category $\mathcal{C}$ we write $X \in \mathcal{C}$ in order to say that X is an object of $\mathcal{C}$. If $f \in \mathcal{C}(X, Y)$ is a morphism from X to Y we write $f\colon X \to Y$. Moreover, the composition is often just denoted by juxtaposition, i.e., we write gf instead of $g \circ f$.

You know already a lot of examples of categories.

Example 1.4.

(i) The category of sets and maps of sets.
(ii) The category of groups with group homomorphisms.
(iii) Given a ring R we have the category of R-modules and R-linear maps.
(iv) The category of fields and field extensions.
(v) The category of smooth manifolds and differentiable maps.

Using the conventions introduced above we also have the following examples.

Example 1.5.

(i) The category Top of spaces and maps.
(ii) The category Top_* of pointed spaces and maps of pointed spaces.

(iii) The category Top^2 of pairs of spaces and maps of pairs.

Now, as a consequence of the above proposition we can form a new category with objects given by spaces and maps given by homotopy classes of maps. Two homotopy classes can be composed by forming the composition of representatives and then passing to the corresponding homotopy class. The above proposition guarantees that this is well-defined. It is easy to check that we get a category this way.

Corollary 1.6. *Spaces and homotopy classes of maps define a category* $\mathsf{Ho}(\mathsf{Top})$*, the (naive)* homotopy category of spaces*.*

There are variants of this for the case of Top_* and Top^2. In these two cases we are mainly interested in slightly different variants of the homotopy relation.

Definition 1.7. Two maps $f, g\colon (X, x_0) \to (Y, y_0)$ in Top_* are called *homotopic relative to base points*, notation

$$f \simeq g \quad \text{rel } x_0,$$

if there exists a homotopy $H\colon f \simeq g$ such that

$$H(x_0, t) = y_0, \qquad t \in [0, 1].$$

Thus, we are asking for the existence of a homotopy through pointed maps. Again, one checks that this is an equivalence relation which is compatible with composition. The equivalences classes $[f]$ are called *pointed homotopy classes* and the set of all such is denoted by $[(X, x_0), (Y, y_0)]$.

These pointed variants are in fact special cases of the following more general notion.

Definition 1.8. Let (X, A) be a pair of spaces, Y a space, and $f, g\colon X \to Y$ two maps such that $f(a) = g(a)$ for all $a \in A$. Then a *homotopy from f to g relative to A* is a homotopy $H\colon f \simeq g$ such that $H(a, -)\colon [0, 1] \to Y$ is constant for all $a \in A$. Thus the additional condition imposed is

$$H(a, t) = f(a) = g(a), \quad t \in [0, 1],\ a \in A.$$

If for two such maps f and g there is a relative homotopy H, then this will be denoted by

$$H\colon f \simeq g \quad \text{rel } A.$$

This notion specializes to pointed homotopy or homotopy, if A consists of a single point or is empty respectively.

Definition 1.9. Let us consider two maps $f, g\colon (X, A) \to (Y, B)$ in Top^2. A *homotopy of pairs* from f to g is a homotopy $H\colon f \simeq g$ which, in addition, satisfies

$$H(a, t) \in B, \qquad t \in [0, 1], \quad a \in A.$$

This is of course precisely the condition that for each $t \in [0, 1]$ the map $H(-, t)\colon X \to Y$ is actually a map of pairs $(X, A) \to (Y, B)$.

Also in this case we obtain a well-behaved equivalence relation and the equivalence classes are denoted as before. We will write $[(X, A), (Y, B)]$ for the set of homotopy classes of pairs. As a consequence of this discussion we obtain the following result.

Corollary 1.10.

(i) *Pointed spaces and pointed homotopy classes define a category* $\mathsf{Ho}(\mathsf{Top}_*)$*, the* homotopy category of pointed spaces*.*

(ii) *Pairs of spaces and relative homotopy classes define a category* $\mathsf{Ho}(\mathsf{Top}^2)$*, the* homotopy category of pairs of spaces.

Definition 1.11. A map $f\colon X \to Y$ of spaces is a *homotopy equivalence* if there is a map $g\colon Y \to X$ such that

$$g \circ f \simeq id_X \qquad \text{and} \qquad f \circ g \simeq id_Y.$$

The space X is *homotopy equivalent* to Y, notation $X \simeq Y$, if there is a homotopy equivalence $f\colon X \to Y$.

It is easy to see that being homotopy equivalent is an equivalence relation. The equivalence class of a space X with respect to this relation is called the *homotopy type* of X.

Definition 1.12. A morphism $f\colon X \to Y$ in a category $\mathcal{C}$ is an *isomorphism* if there is a morphism $g\colon Y \to X$ such that

$$g \circ f = id_X \qquad \text{and} \qquad f \circ g = id_Y.$$

An object X is *isomorphic* to Y if there is an isomorphism $X \to Y$.

Example 1.13.

(i) A morphism in the category of sets is an isomorphism if and only if it is bijective.
(ii) In many categories the objects are given by 'sets with additional structure' while morphisms are defined as morphisms of sets 'respecting this additional structure'. Frequently, it is true that a morphism in such a category is an isomorphism if and only if the underlying map of sets is a bijection. This is for example the case for groups, rings, fields, and modules.
(iii) In the category Top one has to be careful; a continuous bijection $f\colon X \to Y$ is, in general, not an isomorphism, i.e., a homeomorphism. For this to be true we have to impose additional conditions on the spaces (like compact and Hausdorff).
(iv) A morphism $[f]\colon X \to Y$ in $\mathsf{Ho}(\mathsf{Top})$ is an isomorphism if and only if any map $f\colon X \to Y$ representing this homotopy class is a homotopy equivalence.

Exercise 1.14.

(i) Define a notion of *pointed homotopy equivalence* (without using the concept of an isomorphism) and check that the pointed analog of Example 1.13(iv) holds.
(ii) Define a notion of *relative homotopy equivalence* (again, without using the concept of an isomorphism) and check that the relative analog of Example 1.13(iv) holds.
(iii) The notion of being isomorphic is an equivalence relation on the collection of objects in an arbitrary category. The corresponding equivalence classes are called isomorphism classes.

2. The fundamental groupoid

Recall that given a space X and an element $x_0 \in X$ we have the fundamental group $\pi_1(X, x_0)$ of homotopy classes of loops at x_0. If we take a different point $x_1 \in X$ we obtain a further such group $\pi_1(X, x_1)$ which a priori has nothing to do with $\pi_1(X, x_0)$. However, if x_0 and x_1 lie in the same path component of X then any path joining them induces an isomorphism between the two homotopy groups by conjugation with the given path. It is easy to check that paths homotopic to the boundary induce the same isomorphism. However, in general, the induced isomorphisms may be different. A convenient way of encoding all these different groups and isomorphisms in a single structure is given by the *fundamental groupoid* of a space. In order to discuss this we have to introduce one more notion from category theory.

Definition 1.15. A category $\mathcal{C}$ is a *groupoid* if every morphism in $\mathcal{C}$ is an isomorphism.

This terminology reflects the idea that a groupoid is like a group in a certain sense. In fact, for every object in a groupoid the set of endomorphisms is actually a group. The justification for this terminology is given by the first of the following examples.

Example 1.16.

(i) Every group G gives rise to a groupoid BG as follows. The category BG has a single object denoted by $*$. Hence, the only set of morphisms we have to specify is the set of endomorphisms of $*$ and this set is given by:

$$BG(*,*) = G.$$

The composition is given by the multiplication of the group. It is easy to check that all the axioms of a groupoid are precisely fulfilled because G is a group. In other words, a group is essentially the same thing as a groupoid with one object. Similarly, a monoid M is essentially the same thing as a category with a single object.

(ii) Every category has an underlying groupoid given by the same objects and the isomorphisms only. For example, we have the category of sets and bijections, spaces and homeomorphisms, smooth manifolds and diffeomorphisms, and so on.

We now give a description of the fundamental groupoid $\pi(X)$ of a space X. The collection of objects $\mathrm{ob}(\pi(X))$ is given by the points of X. Given two points $x, y \in X$ the set of morphisms

$$\pi(X)(x,y)$$

is given by the set of homotopy classes of paths from x to y relative to the boundary. To be completely specific, let us recall that a *path* α in X from x to y is given by a map

$$\alpha\colon [0,1] \to X \qquad \text{such that} \qquad \alpha(0) = x \qquad \text{and} \qquad \alpha(1) = y.$$

We want to consider two such paths α and β to be equivalent if they are homotopic relative to the boundary, i.e., if there is a map $H\colon [0,1]\times[0,1] \to X$ such that

$$H(0,-) = \alpha, \qquad H(1,-) = \beta, \qquad H(t,0) = x, \qquad \text{and} \qquad H(t,1) = y \quad \text{for all } 0 \le t \le 1.$$

Now, given a path α from x to y and a path β from y to z then the *concatenation* $\beta * \alpha$ is given by:

$$\big(\beta * \alpha\big)(t) = \begin{cases} \alpha(2t) & , \quad 0 \le t \le 1/2 \\ \beta(2t-1) & , \quad 1/2 \le t \le 1 \end{cases}$$

Exercise 1.17.

(i) The concatenation $\beta * \alpha$ is again continuous and defines a path from x to z.

(ii) Given three points $x, y, z \in X$ then the assignment $([\beta],[\alpha]) \mapsto [\beta * \alpha]$ defines a well-defined composition map

$$\circ\colon \pi(X)(y,z) \times \pi(X)(x,y) \longrightarrow \pi(X)(x,z).$$

(iii) The composition is associative.

(iv) Prove that the homotopy classes of constant paths give identity morphisms. Thus we already know that $\pi(X)$ defines a category.

(v) Given a path α from x to y then we the *inverse path* α^{-1} from y to x is defined by the formula

$$\alpha^{-1}(t) = \alpha(1-t).$$

Show that every morphism $[\alpha]$ in $\pi(X)$ is an isomorphism by verifying that $[\alpha^{-1}]$ defines a two-sided inverse of $[\alpha]$.

This exercise shows that $\pi(X)$ is a groupoid, the *fundamental groupoid* of X. Let (X, x_0) be a pointed space, then the *fundamental group* $\pi_1(X, x_0)$ of X at x_0 is defined as the group of automorphisms of x_0 in $\pi(X)$, i.e.,

$$\pi_1(X, x_0) = \pi(X)(x_0, x_0)$$

Let us introduce the following notation for the unit interval, its boundary, and the sphere:

$$I = [0,1], \qquad \partial I = \{0,1\}, \qquad \text{and} \qquad S^1 = \{x \in \mathbb{R}^2 \mid ||x|| = 1\}$$

Then, the fundamental group is given by

$$\pi_1(X, x_0) \quad = \quad [(I, \partial I), (X, x_0)] \quad \cong \quad [(S^1, *), (X, x_0)]$$

where the latter isomorphism comes from the homeomorphism $I/\partial I \cong S^1$.

Example 1.18. We assume you have learned about the fundamental group in your undergraduate topology, and know examples like

$$\pi_1(S^1, *) \cong \mathbb{Z} \qquad \text{and} \qquad \pi_1(\mathbb{T}, *) \cong \mathbb{Z} \times \mathbb{Z},$$

where

$$\mathbb{T} = S^1 \times S^1$$

is the torus. The latter formula is of course a special case of the following *product formula.* Given $(X, x_0), (Y, y_0) \in \mathsf{Top}_*$ then the product $(X \times Y, (x_0, y_0))$ is again a pointed space and the natural map

$$\pi_1(X \times Y, (x_0, y_0)) \xrightarrow{\cong} \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

is an isomorphism.