LECTURE 2: SPACES OF MAPS, LOOP SPACES AND REDUCED SUSPENSION

In this section we will give the important constructions of loop spaces and reduced suspensions associated to pointed spaces. For this purpose there will be a short digression on *spaces of maps* between (pointed) spaces and the relevant topologies.

To be a bit more specific, one aim is to see that given a pointed space (X, x_0), then there is an entire pointed space of loops in X. In order to obtain such a *loop space*

$$\Omega(X, x_0) \in \mathsf{Top}_*,$$

we have to specify an underlying set, choose a base point, and construct a topology on it. The underlying set of $\Omega(X, x_0)$ is just given by the set of maps

$$\mathsf{Top}_*((S^1, *), (X, x_0)).$$

A base point is also easily found by considering the constant loop κ_{x_0} at x_0 defined by:

$$\kappa_{x_0} \colon (S^1, *) \to (X, x_0) \colon t \mapsto x_0$$

The topology which we will consider on this set is a special case of the so-called *compact-open* topology. We begin by introducing this topology in a more general context.

1. Function spaces

Let K be a compact Hausdorff space, and let X be an arbitrary space. The set $\mathsf{Top}(K, X)$ of continuous maps $K \to X$ carries a natural topology, called the *compact-open topology*. It has a *subbasis* formed by the sets of the form

$$B(T, U) = \{f \colon K \to X \mid f(T) \subseteq U\}$$

where $T \subseteq K$ is compact and $U \subseteq X$ is open. Thus, for a map $f \colon K \to X$, one can form a typical basis open neighborhood by choosing compact subsets $T_1, \ldots, T_n \subseteq K$ and small open sets $U_i \subseteq X$ with $f(T_i) \subseteq U_i$ to get a neighborhood O_f of f,

$$O_f = B(T_1, U_1) \cap \ldots \cap B(T_n, U_n).$$

One can even choose the T_i to cover K, so as to 'control' the behavior of functions $g \in O_f$ on all of K.

The topological space given by this compact-open topology on $\mathsf{Top}(K, X)$ will be denoted by:

$$X^K \in \mathsf{Top}$$

Proposition 2.1. *Let K be a compact Hausdorff space. Then for any $X, Y \in \mathsf{Top}$, there is a bijective correspondence between maps*

$$Y \xrightarrow{f} X^K \qquad \textit{and} \qquad Y \times K \xrightarrow{g} X.$$

Proof. Ignoring continuity for the moment, there is an obvious bijective correspondence between such functions f and g, given by

$$f(y)(k) = g(y, k)$$

for all $y \in Y$ and $k \in K$. We thus have to show that if f and g correspond to each other in this sense, then f is continuous if and only if g is.

In one direction, suppose g is continuous, and choose an arbitrary subbasic open $B(T, U) \subseteq X^K$. To prove that $f^{-1}(B(T, U))$ is open, choose $y \in f^{-1}(B(T, U))$, so $g(\{y \times T\}) \subseteq U$. Since T is compact and g is continuous, there are open $V \ni y$ and $W \supseteq T$ with $g(V \times W) \subseteq U$ (you should check this for yourself!). Then surely V is a neighborhood of y with $f(V) \subseteq B(T, U)$, showing that $f^{-1}(B(T, U))$ is open.

Conversely, suppose f is continuous, and take U open in X. To prove that $g^{-1}(U)$ is open, choose $(y, k) \in g^{-1}(U)$, i.e., $y \in f^{-1}(B(\{k\}, U))$. Now, in the space X^K, if a function $K \to X$ maps k into U, then it must map a neighborhood W_k of k into U, and if we choose W_k small enough it will even map the compact set $T = \bar{W_k}$ into U. This shows that:

$$B(\{k\}, U) = \bigcup \{B(T, U) \mid T \text{ is a compact neighborhood of } k\}$$

So $y \in f^{-1}(B(T, U))$ for some $T = \bar{W_k}$. By continuity of f, we find a neighborhood V of y with $f(V) \subseteq B(T, U)$, i.e., $g(V \times T) \subseteq U$; and hence surely $g(V \times W_k) \subseteq U$, showing that $g^{-1}(U)$ is open. □

As a special case we can consider the compact space $K = S^1$, for which maps out of K are loops.

Definition 2.2. Let $X \in \mathsf{Top}$ and let $(Y, y_0) \in \mathsf{Top}_*$.

(1) The space $\Lambda(X) = X^{S^1} \in \mathsf{Top}$ is the *free loop space* of X.
(2) The *loop space* $\Omega(Y, y_0) \in \mathsf{Top}_*$ of (Y, y_0) is the pair consisting of the subspace of $\Lambda(Y)$ given by the pointed loops $(S^1, *) \to (Y, y_0)$ together with the constant loop κ_{y_0} at y_0 as base point.

2. The reduced suspension

The above proposition applied to the compact space $K = S^1$ tells us that there is a bijective correspondence between maps $g\colon X \times S^1 \to Y$ and maps $f\colon X \to Y^{S^1} = \Lambda(Y)$. Let now (X, x_0) and (Y, y_0) be pointed spaces. We want to make explicit the conditions to be imposed on a map $g\colon X \times S^1 \to Y$ such that the corresponding map f actually defines a pointed map:

$$f\colon (X, x_0) \to \Omega(Y, y_0) = (\Omega(Y, y_0), \kappa_{y_0})$$

Since the correspondence is given by the formula

$$g(x, t) = f(x)(t)$$

it is easy to check that these conditions are:

$$g(x_0, t) = y_0, \quad t \in S^1, \qquad \text{and} \qquad g(x, *) = y_0, \quad x \in X$$

Thus the map g has to send the subspace $\{x_0\} \times S^1 \cup X \times \{*\} \subseteq X \times S^1$ to the base point $y_0 \in Y$ and hence factors over the corresponding quotient.

Definition 2.3. Let (X, x_0) be a pointed space. Then the *reduced suspension* $\Sigma(X, x_0) \in \mathsf{Top}_*$ is the pointed space

$$\Sigma(X, x_0) = X \times S^1 / (\{x_0\} \times S^1 \cup X \times \{*\})$$

where the base point is given by the collapsed subspace.

Using the quotient map $I \to I/\partial I \cong S^1$, a different description of the reduced suspension $\Sigma(X, x_0)$ is given by

$$\Sigma(X, x_0) = X \times I/(\{x_0\} \times I \cup X \times \partial I)$$

and in either description we will denote the base point by $*$. The quotient map $I \to I/\partial I \cong S^1$ induces maps of pairs

$$\big(X \times I, \{x_0\} \times I \cup X \times \partial I\big) \to \big(X \times S^1, \{x_0\} \times S^1 \cup X \times \{*\}\big) \to \big(\Sigma(X, x_0), *\big).$$

At the level of elements we allow us to commit a minor abuse of notation and simply write

$$(x, t) \mapsto (x, t) \mapsto [x, t], \quad x \in X, \quad t \in I.$$

Thus, we have the following descriptions of the base point $* \in \Sigma(X, x_0)$

$$[x_0, t] = [x, 0] = [x, 1] = *, \quad x \in X, \quad t \in I,$$

and similarly if the second factor is an element of S^1. Proposition 2.1 combined with the discussion preceding the definition of the reduced suspension gives us the following corollary.

Corollary 2.4. *Let* (X, x_0), $(Y, y_0) \in \mathsf{Top}_*$. *Then there is a bijective correspondence between pointed maps*

$$g\colon \Sigma(X, x_0) \to (Y, y_0) \qquad \textit{and} \qquad f\colon (X, x_0) \to \Omega(Y, y_0)$$

given by the formula $g([x, t]) = f(x)(t)$ *for all* $x \in X$ *and* $t \in S^1$.

This corollary turns out to be the special case of a pointed analog of Proposition 2.1. In order to understand this, we have to introduce a few constructions of pointed spaces. A pointed analog of spaces of maps is easily obtained (compare to the difference between the loop space and the free loop space!).

Definition 2.5. Let $(K, k_0), (X, x_0) \in \mathsf{Top}_*$ and assume that K is compact. The *pointed mapping space*

$$(X, x_0)^{(K, k_0)} \subset X^K$$

is the subspace of *pointed* maps $(K, k_0) \to (X, x_0)$. It has a natural base point given by the constant map κ_{x_0} with value x_0, and hence defines an object

$$(X, x_0)^{(K, k_0)} \in \mathsf{Top}_*.$$

3. The wedge sum and the smash product

From now on we begin to be a bit sloppy about the notation of base points. If we do not need a special notation for a base point of a pointed space we will simply drop it from notation. For example, we will write 'Let X be a pointed space', the suspension $\Sigma(X, x_0)$ will be denoted by $\Sigma(X)$, and similarly. Also the pointed mapping space of the above definition will sometimes simply be denoted by X^K. Moreover, we will sometimes generically denote base points by $*$. Whenever this simplified notation results in a risk of ambiguity we will stick to the more precise one.

As a next step, let us consider a pair of spaces (X, A). Then we can form the quotient space X/A by dividing out the equivalence relation $\sim_A$ generated by

$$a \sim_A a', \quad a, a' \in A.$$

The quotient space X/A is naturally a pointed space with base point given by the equivalence class of any $a \in A$. In the sequel this will always be the way in which we consider a quotient space as a pointed space. Note that this was already done in the above definition of the (reduced) suspension of a pointed space.

Definition 2.6. Let (X, x_0) and (Y, y_0) be two pointed spaces, then their *wedge* $X \vee Y$ is the pointed space

$$X \vee Y = X \sqcup Y / \{x_0, y_0\} \in \mathsf{Top}_*.$$

The wedge $X \vee Y$ comes naturally with pointed maps $i_X \colon X \to X \vee Y$ and $i_Y \colon Y \to X \vee Y$. In fact, this is the universal example of two such maps with a common target in the sense of the following exercise.

Exercise 2.7. Let X, Y, and W be pointed spaces and let $f \colon X \to W$, $g \colon Y \to W$ be pointed maps. Then there is a unique pointed map $(f, g) \colon X \vee Y \to W$ such that:

$$(f, g) \circ i_X = f \colon X \to W \qquad \text{and} \qquad (f, g) \circ i_Y = g \colon Y \to W$$

Thus, from a more categorical perspective the wedge is the categorical coproduct in the category of pointed spaces.

Example 2.8.

(i) The quotient space $I/\{0, 1/2, 1\}$ is homeomorphic to $S^1 \vee S^1$.
(ii) For any pointed space X we have $X \vee * \cong X \cong * \vee X$.
(iii) For two pointed spaces X and Y we have $X \vee Y \cong Y \vee X$.

The wedge $X \vee Y$ of two pointed spaces is naturally a subspace of $X \times Y$. In fact, this inclusion can be obtained by applying the above exercise as follows. For *pointed* spaces (!), the product $(X \times Y, (x_0, y_0))$ comes naturally with an inclusion map of (X, x_0) given by

$$(X, x_0) \to (X \times Y, (x_0, y_0)) \colon x \mapsto (x, y_0).$$

There is a similar map $(Y, y_0) \to (X \times Y, (x_0, y_0))$. Thus we are in the situation of Exercise 2.7 and hence obtain a pointed map $X \vee Y \to X \times Y$. This map can be checked to be the inclusion of a subspace. The corresponding quotient construction is so important that it deserves a special name.

Definition 2.9. Let X and Y be pointed spaces. Then the *smash product* $X \wedge Y$ of X and Y is the pointed space

$$X \wedge Y = X \times Y / X \vee Y \in \mathsf{Top}_*.$$

As it is the case for every quotient space, the smash product $X \wedge Y$ naturally comes with a quotient map

$$q \colon X \times Y \to X \wedge Y.$$

We will use the following notation for points in $X \wedge Y$:

$$[x, y] = q(x, y), \quad x \in X, \quad y \in Y$$

In the next example, we will use the *0-dimensional sphere* or *0-sphere* S^0 which is the two-point space:

$$S^0 = \{-1, +1\} \subseteq [-1, +1]$$

Let us agree that we consider S^0 as a pointed space with -1 as base point. Moreover, let I_+ be the disjoint union of $I = [0, 1]$ with a base point, i.e.,

$$I_+ = [0, 1] \sqcup * \in \mathsf{Top}_*$$

with $*$ as base point.

Example 2.10.

(i) For every $X \in \mathsf{Top}_*$ we have $X \wedge S^1 \cong \Sigma(X)$.

(ii) For every $X \in \mathsf{Top}_*$ we have a homeomorphism $X \wedge S^0 \cong X \cong S^0 \wedge X$.
(iii) For two pointed spaces X and Y we have $X \wedge Y \cong Y \wedge X$.
(iv) Let $X \in \mathsf{Top}_*$. The *reduced cylinder* of X is the smash product $X \wedge I_+$. Unraveling the definition of the smash product, we see that we have

$$X \wedge I_+ \cong X \times I/\{x_0\} \times I \in \mathsf{Top}_*.$$

Thus, pointed maps out of $X \wedge I_+$ are precisely the *pointed* homotopies.

Proposition 2.11. *Let K, X, and Y be pointed spaces and assume that K is compact, Hausdorff. Then there is a bijective correspondence between pointed maps*

$$g\colon X \wedge K \to Y \qquad \textit{and} \qquad f\colon X \to Y^K$$

given by the formula $g([x,k]) = f(x)(k)$ for all $x \in X$ and $k \in K$.

Using the cylinder construction on spaces one can also establish the following result. A proof will be given in the exercises.

Corollary 2.12. *Let K, X, and Y be pointed spaces and assume that K is compact, Hausdorff. Then there is a bijection:*

$$[X \wedge K, Y] \cong [X, Y^K]$$

Exercise 2.13.

(i) Give a proof of Proposition 2.11 using the corresponding result about ('unpointed') spaces. Hint: compare the proof of the special case of $K = S^1$ and realize that the smash product is designed so that this proposition becomes true.
(ii) Give a proof of Corollary 2.12. There are some hints on how to attack this on the exercise sheet.

We will now recall the notion of path components which allows us to establish a relation between loop spaces and the fundamental group.

Let X be a space and let $x_0, x_1 \in X$. We say that x_0 and x_1 are equivalent, notation $x_0 \simeq x_1$, if and only if there there is a path in X connecting them. It is easy to check that this defines an equivalence relation on X with equivalence classes the *path components* of X. We write $\pi_0(X)$ for the set of path components of X and denote the path component of $x \in X$ by $[x]$. A point in X can be identified with a map $x\colon * \to X$ sending the unique point $*$ to x. Under this identification, the above equivalence relation becomes the homotopy relation on maps $* \to X$. Thus, we have a bijection:

$$\pi_0(X) \cong [*, X]$$

In the case of a pointed space the set of path components has a naturally distinguished element given by the path component of the base point. Thus, the set $\pi_0(X, x_0)$ of path components of a pointed space (X, x_0) is a *pointed set*. Now, a point $x \in (X, x_0)$ can be identified with a *pointed* map $S^0 \to X$. In fact, a bijection is obtained by evaluating such a map on $1 \in S^0$ (which is not the base-point!). Moreover, it is easy to see that we have an isomorphism of pointed sets

$$\pi_0(X, x_0) \cong [(S^0, -1), (X, x_0)].$$

Exercise 2.14. Verify the details of the above discussion.

As a consequence of our work so far we obtain the following corollary.

Corollary 2.15. *Let (X, x_0) be a pointed space. Then there is a canonical bijection of pointed sets:*

$$\pi_1(X, x_0) \cong \pi_0(\Omega(X, x_0))$$

Proof. It suffices to assemble our results from above. Since the sphere S^1 is a compact, Hausdorff space we can apply Corollary 2.12 in order to obtain:

$$\begin{aligned} \pi_0(\Omega(X, x_0)) &\cong [(S^0, -1), \Omega(X, x_0)] \\ &\cong [(S^0, -1) \wedge (S^1, *), (X, x_0)] \\ &\cong [(S^1, *), (X, x_0)] \\ &\cong \pi_1(X, x_0) \end{aligned}$$

The third identification is a special case of Example 2.10(ii). □

Thus, at least for theoretical purposes, in order to calculate the fundamental group of a given space it is enough to calculate the path components of the associated loop space. However, this does not really simplify the task since the loop space is, in general, less tractable than the original space. This corollary also suggests a definition of higher homotopy groups. Namely, given a pointed space (X, x_0) we could simply set

$$\pi_n(X, x_0) = \pi_0((X, x_0)^{(S^n, *)}), \quad n \geq 2.$$

This will be pursued further in the next section where we will see that we actually obtain *abelian* groups this way.

The final aim of this section is to introduce the notion of a functor and to remark that many of the constructions introduced so far are in fact functorial. Here is the key definition.

Definition 2.16. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F \colon \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ is given by:

(i) An object function which assigns to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$.
(ii) For each pair of objects $X, Y \in \mathcal{C}$ a morphism function $\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$.

This data is compatible with the composition and the identity morphisms in the sense that:

- For morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$ we have $F(g \circ f) = F(g) \circ F(f) \colon F(X) \to F(Z)$ in $\mathcal{D}$.
- For every object $X \in \mathcal{C}$ we have $F(id_X) = id_{F(X)} \colon F(X) \to F(X)$ in $\mathcal{D}$.

In order to give some examples of functors, we introduce notation for some prominent categories.

Notation 2.17.

(i) The category of sets and maps of sets will be denoted by Set, the one of pointed sets and maps preserving the chosen elements by Set_*.
(ii) We will write Grp for the category of groups and group homomorphisms.
(iii) The category of abelian groups and group homomorphisms is denoted by Ab.
(iv) Given a ring R, we write R-Mod for the category of (left) R-modules and R-linear maps.

You know already many examples of functorial constructions. Let us only give a few examples.

Example 2.18.

(i) The formation of free abelian groups generated by a set defines a functor $\mathsf{Set} \to \mathsf{Ab}$. More generally, given a ring then there is a free R-module functor $\mathsf{Set} \to R$-Mod.
(ii) Given a group G then we obtain the abelianization of G by dividing out the subgroup generated by the commutators $aba^{-1}b^{-1}$, $a, b \in G$. This quotient is an abelian group and gives us a functor $(-)^{ab} \colon \mathsf{Grp} \to \mathsf{Ab} \colon G \mapsto G^{ab}$.

(iii) There are many functors which forget structures or properties. For example there is the following chain of forgetful functors where R is an arbitrary ring:

$$R\text{-}\mathsf{Mod} \to \mathsf{Ab} \to \mathsf{Grp} \to \mathsf{Set}_* \to \mathsf{Set}$$

We first forget the action by scalars $r \in R$ and only keep the abelian group. We then forget the fact that our group is abelian. Next, we drop the group structure and only keep the neutral element. Finally, we also forget the base point.

In this course, many of the constructions on spaces or unpointed spaces turn out to be instances of suitable functors. Let us mention some of the constructions which were already implicit in this course. More examples will be considered in the exercises.

Example 2.19.

(i) The fundamental group construction defines a functor $\pi_1\colon \mathsf{Top}_* \to \mathsf{Grp}$.

(ii) The formation of path components defines functors

$$\pi_0\colon \mathsf{Top} \to \mathsf{Set} \qquad \text{and} \qquad \pi_0\colon \mathsf{Top}_* \to \mathsf{Set}_*.$$

(iii) The first two examples are special cases of the following more general construction. Let K be a space, then we can consider the assignment

$$[K,-]\colon \mathsf{Top} \to \mathsf{Set}\colon X \mapsto [K,X].$$

Given a map $f\colon X \to Y$ of spaces then we obtain an induced map $[K,X] \to [K,Y]$ by sending a homotopy class $[g]\colon K \to X$ to $[f]\circ[g]\colon K \to X \to Y$. It is easy to check that this defines a functor $\mathsf{Top} \to \mathsf{Set}$.

Similarly, if (K,k_0) is a pointed space, then the assignment $(X,x_0) \mapsto [(K,k_0),(X,x_0)]$ is functorial. Note that the set $[(K,k_0),(X,x_0)]$ has a natural base point given by the homotopy class of the constant map $\kappa_{x_0}\colon (K,k_0) \to (X,x_0)$. Given a *pointed* map

$$f\colon (X,x_0) \to (Y,y_0)$$

then the induced map $[(K,k_0),(X,x_0)] \to [(K,k_0),(Y,y_0)]\colon [g] \mapsto [f]\circ[g]$ preserves the base point. Thus, we obtain a functor

$$[(K,k_0),-]\colon \mathsf{Top}_* \to \mathsf{Set}_*.$$

The first two examples are obtained by considering the special cases of $K = * \in \mathsf{Top}$ and $K = S^0, S^1 \in \mathsf{Top}_*$ respectively.

(iv) The construction of the reduced suspension is functorial and similarly for the loop space. Thus, we have two functors:

$$\Sigma\colon \mathsf{Top}_* \to \mathsf{Top}_* \qquad \text{and} \qquad \Omega\colon \mathsf{Top}_* \to \mathsf{Top}_*$$

Let us give some details about the functoriality of Σ (the case of Ω will be treated in the exercises). By definition, $\Sigma(X,x_0)$ is the following quotient space:

$$X \times S^1/(\{x_0\} \times S^1 \cup X \times \{*\})$$

If we have a pointed map $f\colon (X,x_0) \to (Y,y_0)$, we can form the product with the identity to obtain a map

$$f \times id_{S^1}\colon X \times S^1 \to Y \times S^1$$

In order to obtain a well-defined map $\Sigma(f)\colon \Sigma(X,x_0) \to \Sigma(Y,y_0)$ we want to apply the universal property of the construction of quotient spaces. Thus, it suffices to check that:

$$(f \times id_{S^1})\big(\{x_0\} \times S^1 \cup X \times \{*\}\big) \quad \subseteq \quad \{y_0\} \times S^1 \cup Y \times \{*\}$$

But this is true since f is a pointed map. Thus, we deduce the existence of a *unique* map $\Sigma(f)\colon \Sigma(X,x_0)\to\Sigma(Y,y_0)$ such that the following square commutes:

$$\begin{array}{ccc} X\times S^1 & \longrightarrow & Y\times S^1 \\ \downarrow & & \downarrow \\ \Sigma(X,x_0) & \underset{\exists!}{\dashrightarrow} & \Sigma(Y,y_0) \end{array}$$

We leave it as an exercise to check that the uniqueness implies that we indeed get a functor $\Sigma\colon \mathsf{Top}_*\to\mathsf{Top}_*$. For example, you might want to consider the following diagram to prove the compatibility with respect to compositions:

$$\begin{array}{ccccc} X\times S^1 & \longrightarrow & Y\times S^1 & \longrightarrow & Z\times S^1 \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma(X,x_0) & \underset{\exists!}{\dashrightarrow} & \Sigma(Y,y_0) & \underset{\exists!}{\dashrightarrow} & \Sigma(Z,z_0) \end{array}$$

(v) By adding disjoint base points to spaces we obtain a functor $(-)_+\colon \mathsf{Top}\to\mathsf{Top}_*$. There is also a forgetful functor $\mathsf{Top}_*\to\mathsf{Top}$ in the opposite direction which forgets the base points.