LECTURE 3: HIGHER HOMOTOPY GROUPS

In this section we will introduce the main objects of study of this course, the *homotopy groups*

$$\pi_n(X, x_0)$$

of a pointed space (X, x_0), for each natural number $n \geq 2$. (Recall that the (pointed) set of components $\pi_0(X, x_0)$ and the fundamental group $\pi_1(X, x_0)$ have already been defined.) One goal of this course is to develop some techniques which will allow us to calculate these homotopy groups in interesting examples.

1. Higher homotopy groups

We begin by introducing some notation for important spaces. Let us denote by

$$I^n = [0,1] \times \ldots \times [0,1] \subseteq \mathbb{R}^n$$

the n-cube and $\partial I^n \subseteq I^n$ for its boundary. Thus,

$$\partial I^n = \{(t_1, \ldots, t_n) \in I^n \mid \text{ at least one of the } t_i \in \{0,1\}\}.$$

Let us agree on the convention that $\partial I^0 = \emptyset$ is empty. Note that the boundary satisfies (and is completely determined by ∂I and) the Leibniz formula

$$\partial(I^n \times I^m) = (\partial I^n) \times I^m \cup I^n \times (\partial I^m).$$

The n-sphere is denoted by

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}.$$

Note that there are homeomorphisms $I^n/\partial I^n \cong S^n$; we will write $[t_1, \ldots, t_n] \in S^n$ for the image of $(t_1, \ldots, t_n) \in I^n$ under the composition $I^n \to I^n/\partial I^n \cong S^n$.

The definition of the underlying (pointed) set of $\pi_n(X, x_0)$ is simple enough:

$$\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$$

Thus, an element $[\alpha]$ of $\pi_n(X, x_0)$ is represented by a map $\alpha\colon I^n \to X$ sending the entire boundary ∂I^n to the base point x_0; and two such α and α' represent the same element of $\pi_n(X, x_0)$ if and only if there is a homotopy $H\colon I^n \times I \to X$ such that

$$H(\partial I^n \times I) = x_0, \qquad H(-,0) = \alpha, \qquad \text{and} \qquad H(-,1) = \alpha'.$$

Obviously, a map $f\colon (X, x_0) \to (Y, y_0)$ induces a function

$$f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

which in fact only depends on the homotopy class of f. Just like for the fundamental group, we have

Proposition 3.1.

(i) *For each pointed space (X, x_0) and $n \geq 1$, the set $\pi_n(X, x_0)$ is a group, the n-th homotopy group of (X, x_0).*

(ii) *For each map* $(X, x_0) \to (Y, y_0)$*, the induced operation* $\pi_n(X, x_0) \to \pi_n(Y, y_0)$ *is a group homomorphism, defining a functor* $\pi_n\colon \mathsf{Top}_* \to \mathsf{Grp}$*. The functor* π_n *is* homotopy invariant*, i.e.,* $\pi_n(f) = \pi_n(g)$ *for homotopic maps* $f \simeq g$*.*

Proof. For two elements $[\alpha]$ and $[\beta]$ in $\pi_n(X, x_0)$, the product $[\beta] \circ [\alpha]$ is represented by the map $\beta * \alpha\colon I^n \to X$ defined by:

$$(\beta * \alpha)(t_1, \ldots, t_n) = \begin{cases} \alpha(2t_1, t_2, \ldots, t_n) & , \quad 0 \leq t_1 \leq 1/2 \\ \beta(2t_1 - 1, t_2, \ldots, t_n) & , \quad 1/2 \leq t_1 \leq 1 \end{cases}$$

Notice that the definition agrees with the known group structure on the fundamental group for $n = 1$. The proof that $\circ$ is well-defined and associative, that the constant map $\kappa_{x_0}\colon I^n \to X$ represents a neutral element, and that each element $[\alpha]$ has an inverse represented by

$$\alpha^{-1}(t_1, \ldots, t_n) = \alpha(1 - t_1, t_2, \ldots, t_n)$$

is exactly the same as for π_1, and we leave the details as an exercise. Also the functoriality is an exercise. □

Remark 3.2. Let X be a space and let $x_0, x_1 \in X$. In general, $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ can be very different. In fact, homotopy groups only 'see the path-component of the base point'. More precisely, let (X, x_0) be a pointed space and let $X' = [x_0]$ be the path-component of x_0. Then the inclusion $i\colon (X', x_0) \to (X, x_0)$ induces an isomorphism $i_*\colon \pi_n(X', x_0) \to \pi_n(X, x_0)$ for all $n \geq 1$. This follows immediately from the fact that S^n is path-connected for $n \geq 1$. We will see later that any path between two points $x_0, x_1 \in X$ induces an isomorphism $\pi_n(X, x_0) \cong \pi_n(X, x_1)$.

Remark 3.3.

(i) Of course it is not only the validity of the proposition which is important, but also the explicit description of the product. However, it can be shown that this group structure is unique for $n \geq 2$.

(ii) We said that the proof of the group structure is analogous to the argument for π_1. In fact, there is a more formal way to see this, as we will see below.

One may object that the definition of the group structure is a bit unnatural, because the first coordinate t_1 is given a preferred rôle in the definition of the group structure. We could also define a product as follows:

$$(\beta *_i \alpha)(t_1, \ldots, t_n) = \begin{cases} \alpha(t_1, \ldots, 2t_i, \ldots, t_n) & , \quad 0 \leq t_i \leq 1/2 \\ \beta(t_1, \ldots, 2t_i - 1, \ldots, t_n) & , \quad 1/2 \leq t_i \leq 1 \end{cases}$$

The explanation is that these two products induce the *same* operation on homotopy classes. The proof of this fact is given by the following observation (Lemma 3.4) together with the so-called Eckmann-Hilton argument (Proposition 3.5).

Lemma 3.4. *The operation* $*$ *distributes over the operation* $*_i$ *in the sense that*

$$(\alpha *_i \beta) * (\gamma *_i \delta) = (\alpha * \gamma) *_i (\beta * \delta)$$

for all maps α, β, γ, $\delta\colon (I^n, \partial I^n) \to (X, x_0)$.

Proof. We only have to look at the case $n = 2$, $i = 2$. Then the expressions on the left and right correspond to the same subdivisions of the square so define identical maps (draw the picture!). □

Proposition 3.5. ***('Eckmann-Hilton trick')***
Let S be a set with two operations $\bullet, \circ \colon S \times S \to S$ having a common unit $e \in S$. Suppose $\bullet$ and $\circ$ distribute over each other, in the sense that

$$(\alpha \bullet \beta) \circ (\gamma \bullet \delta) = (\alpha \circ \gamma) \bullet (\beta \circ \delta)$$

Then $\bullet$ and $\circ$ coincide, and define a commutative and associative operation on S.

Proof. Taking $\beta = e = \gamma$ in the distributive law yields $\alpha \circ \delta = \alpha \bullet \delta$, while taking $\alpha = e = \delta$ yields $\beta \circ \gamma = \gamma \bullet \beta$. The associativity is obtained by taking $\beta = e$ in the distributive law. □

Applying this proposition to $*$ and $*_i$ shows that these define the same operation on $\pi_n(X, x_0)$ for $n \geq 2$. The proposition also shows:

Corollary 3.6. *The groups $\pi_n(X, x_0)$ are abelian for $n \geq 2$.*

Remark 3.7. For this reason, one often employs *additive* notation for the group structure on $\pi_n(X, x_0)$, writing:

$$\begin{aligned} [\beta] + [\alpha] &= [\beta * \alpha] \\ 0 &= [\kappa_{x_0}] \\ -[\alpha] &= [\alpha^{-1}] \end{aligned}$$

There is yet another way of describing $\pi_n(X, x_0)$.

Proposition 3.8.
(i) *There is a bijection of sets natural in the pointed space (X, x_0):*

$$[(S^n, *), (X, x_0)] \xrightarrow{\cong} \pi_n(X, x_0)$$

(ii) *The group structure on $[(S^n, *), (X, x_0)]$ induced by this bijection coincides with the one obtained by composition with the 'pinch map'*

$$\nabla \colon S^n \to S^n \vee S^n$$

defined by collapsing the equator in S^n to a single point.

Proof. Part (i) follows immediately from the isomorphism

$$(I^n / \partial I^n, *) \to (S^n, *).$$

For part (ii), recall from the exercises that the wedge $\vee$ defines a coproduct in the category of pointed spaces, so that two maps $\alpha, \beta \colon (S^n, *) \to (X, x_0)$ together uniquely define a map

$$\alpha \vee \beta \colon (S^n \vee S^n, *) \to (X, x_0).$$

Thus we get an induced operation on $[(S^n, *), (X, x_0)]$ defined by

$$\beta * \alpha = (\alpha \vee \beta) \circ \nabla$$

It is easy to check that this corresponds to the operation $*$ on maps from $(I^n, \partial I^n)$, once one takes the equator in S^n to be the image of $\{t_1 = 1/2\} \subseteq I^n$ under the map $I^n \to I^n / \partial I^n \cong S^n$. □

Example 3.9. For the one-point space $* \in \mathsf{Top}_*$ there is precisely one pointed map $S^n \to *$ for each $n \geq 0$. Thus we have

$$\pi_0(*) \cong *, \qquad \pi_1(*) \cong 1, \qquad \text{and} \qquad \pi_n(*) \cong 0, \quad n \geq 2.$$

We will refer to this by saying that $\pi_n(*)$ is trivial for all $n \geq 0$.

This example together with the homotopy invariance immediately gives us the following.

Corollary 3.10. *Let $f\colon (X,x_0)\to(Y,y_0)$ be a pointed homotopy-equivalence. Then the induced map*

$$f_*\colon \pi_n(X,x_0)\to\pi_n(Y,y_0)$$

is an isomorphism. In particular, $\pi_n(X,x_0)$ is trivial for all choices of base points in a contractible space X and all $n\geq 0$.

Proof. Let $g\colon (Y,y_0)\to(X,x_0)$ be an inverse pointed homotopy equivalence so that we have:

$$g\circ f\simeq id_X \quad \text{rel } x_0 \qquad \text{and} \qquad f\circ g\simeq id_Y \quad \text{rel } y_0$$

Homotopy invariance gives $g_*\circ f_*=(g\circ f)_*=id$, and similarly $f_*\circ g_*=id$. For the second claim, given a contractible space X and $x_0\in X$, it suffices to consider the pointed homotopy equivalence $(X,x_0)\to *$. □

We will later see that these groups $\pi_n(X,x_0)$ are non-trivial and highly informative, but we need to develop (or know) a little more theory before we can make this precise. However, assuming a bit of background knowledge, we observe the following.

Example 3.11. (Preview of examples)

(i) The identity map $id\colon S^n\to S^n$ defines an element of $\pi_n(X,x_0)$. If you know something about degrees, you know that the constant map has degree zero while the identity has degree 1. It is a fact that the degrees of homotopic maps coincide, we conclude that $0\neq[id]$ in $\pi_n(S^n,*)$. In fact, we will show that there is an isomorphism $\pi_n(S^n,*)\cong\mathbb{Z}$ and that $[id]$ is a generator. This could be proved, e.g., using singular homology but we will obtain this calculation as a consequence of the *homotopy excision theorem*.

(ii) If you know a bit of differential topology then you know that any map $S^k\to S^n$ is homotopic to a smooth map, and that a smooth map $f\colon S^k\to S^n$ cannot be surjective if $k<n$. So such a map f factors as a composition

$$S^k\to S^n-\{x\}\cong\mathbb{R}^n\to S^n$$

for some point $x\in S^n$ not in the image of f. The contractibility of $\mathbb{R}^n$ implies that f is homotopic to a constant map. Thus,

$$\pi_k(S^n,*)\cong 0, \qquad k<n.$$

We will later deduce this result from the *cellular approximation theorem*.

(iii) Consider the scalar multiplication on the complex vector space of dimension 2,

$$\mu\colon \mathbb{C}\times\mathbb{C}^2\to\mathbb{C}^2.$$

When we restrict to the complex numbers, respectively vectors of norm 1, we obtain a map

$$\mu\colon S^1\times S^3\to S^3,$$

an *action* of the circle-group on the 3-sphere. It can be shown that the orbit space of this action is S^2. The quotient map $S^3\to S^2$ is the famous *Hopf fibration*, and defines a non-zero element in $\pi_3(S^2,*)$.

2. H-spaces and H-groups

Let us now examine loop spaces in some more detail. Recall the construction of the loop space $\Omega(X, x_0)$ associated to a pointed space (X, x_0), and the isomorphism

$$\pi_1(X, x_0) \cong \pi_0(\Omega(X, x_0)).$$

The group structure on $\pi_1(X, x_0)$ comes from a 'group structure up to homotopy' on $\Omega(X, x_0)$. Explicitly, writing

$$H = \Omega(X, x_0)$$

and $e = \kappa_{x_0} \in H$ for the constant loop at x_0, there is a multiplication map on H given by the concatenation:

$$\mu \colon H \times H \to H \colon \quad (\beta, \alpha) \mapsto \beta * \alpha$$

This multiplication is *associative up to homotopy* in the sense that the following two maps are homotopic:

$$\begin{array}{ccc} H \times H \times H & \xrightarrow{id \times \mu} & H \times H \\ {\scriptstyle \mu \times id}\downarrow & & \downarrow{\scriptstyle \mu} \\ H \times H & \xrightarrow[\mu]{} & H \end{array} \qquad \begin{array}{ccc} (\gamma, \beta, \alpha) & \longmapsto & (\gamma, \beta * \alpha) \\ & & \downarrow \\ & & \gamma * (\beta * \alpha) \end{array} \qquad \begin{array}{ccc} (\gamma, \beta, \alpha) & & \\ \downarrow & & \\ (\gamma * \beta, \alpha) & \longmapsto & (\gamma * \beta) * \alpha \end{array}$$

Moreover, this multiplication is *unital up to homotopy*, i.e., we have homotopies from id to $\mu \circ (e \times id)$ and $\mu \circ (id \times e)$:

$$\begin{array}{ccccc} H & \xrightarrow{e \times id} & H \times H & \xleftarrow{id \times e} & H \\ & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle id} & \\ & & H & & \end{array} \qquad \begin{array}{ccc} \alpha & \longmapsto & (e, \alpha) \\ & & \downarrow \\ & & e * \alpha \end{array} \qquad \begin{array}{ccc} (\alpha, e) & \longleftarrow\!\!| & \alpha \\ \downarrow & & \\ \alpha * e & & \end{array}$$

Indeed, these homotopies are given by the usual reparametrization homotopies. A pointed space (H, e) with such an additional structure is called an *(associative) H-space* (or *Hopf space*).

Given any such H-space (H, e), composition with μ defines an associative multiplication on the set of homotopy classes of maps

$$[(Y, y_0), (H, e)]$$

for an arbitrary pointed space (Y, y_0). Moreover, this 'multiplicative structure' is *natural* in (Y, y_0). (If you do not know what we mean by this naturality, then see the exercise sheet.)

Moreover, the associative multiplication defines a group structure on this set if H has a *homotopy inverse*, i.e., if there is a map $i \colon H \to H$ such that the following diagram commutes up to homotopies:

$$\begin{array}{ccccc} H & \xrightarrow{i \times id} & H \times H & \xleftarrow{id \times i} & H \\ & {\scriptstyle e}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle e} & \\ & & H & & \end{array} \qquad \begin{array}{ccc} \alpha & \longmapsto & (i(\alpha), \alpha) \\ & & \downarrow \\ & & i(\alpha) * \alpha \end{array} \qquad \begin{array}{ccc} (\alpha, i(\alpha)) & \longleftarrow\!\!| & \alpha \\ \downarrow & & \\ \alpha * i(\alpha) & & \end{array}$$

In this case (H, e) is called an *H-group*. Thus, $\Omega(X, x_0)$ is an H-group, and for each pointed space (Y, y_0) the set $[(Y, y_0), \Omega(X, x_0)]$ carries a natural group structure. For the one-point space $* \in \mathsf{Top}_*$, this defines the usual group structure on:

$$[*, \Omega(X, x_0)] \cong \pi_0(\Omega(X, x_0)) \cong \pi_1(X, x_0)$$

Exercise 3.12. Define the notion of a *commutative H-space* and a *commutative H-group.* Is the loop space of a pointed space with the concatenation pairing a commutative H-group?

Theorem 3.13. *For every $n \geq 1$, there is a natural isomorphism of groups:*

$$\pi_n(X, x_0) \cong \pi_{n-1}(\Omega(X, x_0))$$

Let us begin with a lemma. Recall our notation $[x, y] \in X \wedge Y$ for points in a smash product. Moreover, let us use a similar notation for elements in a quotient space, i.e., we will write $[x] \in X/A$. For convenience, we will drop base points from notation in the next lemma and we use $I^n/\partial I^n$ as our model for the n-sphere.

Lemma 3.14. *The following map is a pointed homeomorphism:*

$$S^n \wedge S^m \to S^{n+m}\colon \quad \big[[t_1, \ldots, t_n], [t'_1, \ldots, t'_m]\big] \mapsto [t_1, \ldots, t_n, t'_1, \ldots, t'_m]$$

Proof. One checks directly that this is a well-defined continuous bijection between compact Hausdorff spaces and hence a homeomorphism. □

This map can be described as:

$$\begin{aligned} S^n \wedge S^m &= (I^n/\partial I^n) \wedge (I^m/\partial I^m) \\ &= (I^n/\partial I^n \times I^m/\partial I^m)/(* \times I^m/\partial I^m \cup I^n/\partial I^n \times *) \\ &\cong I^n \times I^m/(\partial I^n \times I^m \cup I^n \times \partial I^m) \\ &\cong I^{n+m}/\partial I^{n+m} \\ &= S^{n+m} \end{aligned}$$

Proof. (of Theorem 3.13). The multiplication on $\pi_{n-1}(\Omega(X, x_0))$ is the 'loop multiplication'. We already know from an earlier lecture that there is a natural isomorphism:

$$[(S^{n-1}, *), \Omega(X, x_0)] \cong [(S^{n-1} \wedge S^1, *), (X, x_0)]$$

Using the homeomorphism of the above lemma, it is immediate that the pairing on $[(S^n, *), (X, x_0)]$ induced by the loop multiplication is $*_n$, the 'concatenation with respect to the last coordinate'. But we already know that this is identical to the group structure on $\pi_n(X, x_0)$. □

If we look at the theorem for $n \geq 2$, we see that $\pi_{n-1}(\Omega(X, x_0))$ has *two* group structures: one is the group structure on π_{n-1} for any pointed space, and the other is an instance of the group structure on $[(Y, y_0), (H, e)]$ for any H-group H, in this case for $Y = S^{n-1}$ and $H = \Omega(X, x_0)$. Moreover, these group structures distribute over each other. Indeed, the multiplication $\mu\colon H \times H \to H$ induces a group homomorphism

$$\mu_*\colon \pi_{n-1}(H, e) \times \pi_{n-1}(H, e) \to \pi_{n-1}(H, e)$$

and this precisely means that the multiplication coming from the H-group distributes over the one coming from π_{n-1}. So, by Eckmann-Hilton (Proposition 3.5), the two multiplications coincide and are commutative.

Corollary 3.15. *The fundamental group $\pi_1(H, e)$ of an H-group (H, e) is abelian.*

We now come to a different model for the loop space. We have seen that $\Omega(X, x_0)$ has a multiplication which is associative and unital 'up to homotopy'. One may wonder whether there is a way to make this multiplication strictly associative and unital. For a general H-space this need not be possible. But in this special case there is an easy way to do this. Let $M(X, x_0)$ be the space of *Moore loops* (named after J. C. Moore). Its points are pairs (t, α) where $t \in \mathbb{R}$, $t \geq 0$, and $\alpha\colon [0, t] \to X$ is a loop at x_0 of length t (i.e., $\alpha(0) = x_0 = \alpha(t)$). We can topologize this set as a subspace of $\mathbb{R} \times X^{[0,\infty)}$, identifying a path $\alpha\colon [0, t] \to X$ with the map $[0, \infty) \to X$ which is constant on $[t, \infty)$, and the resulting space is the *Moore loop space*. Then there is a continuous and *strictly* associative multiplication on $M(X, x_0)$, given by

$$(t, \beta) \cdot (s, \alpha) \quad = \quad (t + s, \beta *_M \alpha)$$

where:

$$(\beta *_M \alpha)(r) = \begin{cases} \alpha(r) & , \quad 0 \leq r \leq s \\ \beta(r - s) & , \quad s \leq r \leq t + s \end{cases}$$

A *strict* unit for this multiplication is $(0, \kappa_{x_0})$.

The space $M(X, x_0)$ is homotopy equivalent to $\Omega(X, x_0)$. Indeed there are maps

$$\Omega(X, x_0) \underset{\phi}{\overset{\psi}{\rightleftarrows}} M(X, x_0),$$

ψ is simply the inclusion, while ϕ is defined by

$$\phi(t, \alpha)(r) \quad = \quad \alpha(t \cdot r), \qquad 0 \leq r \leq 1.$$

Then obviously $\phi \circ \psi$ is the identity, while

$$H_s(t, \alpha) \quad = \quad \left(\frac{t}{(1 - s) + st}, \alpha(((1 - s) + st)) \cdot -) \right)$$

defines a homotopy from $H_1 = \psi \circ \phi$ to the identity H_0.

Remark 3.16. We just observed that the loop space and the Moore loop space are homotopy equivalent spaces. Note that the respective H-group structures correspond to each other under these homotopy equivalences. However the multiplications have different formal properties: the Moore loop space is *strictly associative* while the loop space is only *associative up to homotopy*. Thus we see that a space X homotopy equivalent to a space with a strictly associative multiplication does not necessarily inherit the same structure. But it is easy to see that X can be turned into an H-space that way. To put it as a slogan:

'strictly associative multiplications do not live in homotopy theory'

As we already mentioned not all H-spaces can be *rectified* in the sense that they would be homotopy equivalent to spaces with a strictly associative multiplication. One might wonder what additional structure would be needed for this to become true. There is an answer to this question lying beyond the scope of these lectures. Nevertheless, these questions and the more general search for *homotopy invariant algebraic structures* initiated the development of a good deal of mathematics.

Let us formalize the notion of a homotopy invariant functor. Let $\mathcal{C}$ be an arbitrary category. Then a functor $F\colon \mathsf{Top}_* \to \mathcal{C}$ is *homotopy invariant* if pointed maps which are homotopic relative to the base point always have the same image under F:

$$f \simeq g \qquad \text{implies} \qquad F(f) = F(g)$$

Now, note that there is a canonical functor

$$\gamma\colon \mathsf{Top}_* \to \mathsf{Ho}(\mathsf{Top}_*)$$

which is the identity on objects and which sends a pointed map to its pointed homotopy class.

Exercise 3.17.

(i) The above assignments, in fact, define a functor $\gamma\colon \mathsf{Top}_* \to \mathsf{Ho}(\mathsf{Top}_*)$ and this functor is homotopy invariant.

(ii) Let $\mathcal{C}$ be a category. A functor $F\colon \mathsf{Top}_* \to \mathcal{C}$ is homotopy invariant if and only if there is a functor $F'\colon \mathsf{Ho}(\mathsf{Top}_*) \to \mathcal{C}$ such that $F = F' \circ \gamma\colon \mathsf{Top}_* \to \mathcal{C}$. In this case the functor F' is unique.

(iii) Redo a similar reasoning for the categories Top and Top^2.

Thus a homotopy invariant functor 'is the same thing' as a functor defined on the homotopy category of (pointed or pairs of) spaces. In particular, we have:

$$\pi_0\colon \mathsf{Ho}(\mathsf{Top}_*) \to \mathsf{Set}_*, \qquad \pi_1\colon \mathsf{Ho}(\mathsf{Top}_*) \to \mathsf{Grp}, \qquad \text{and} \qquad \pi_n\colon \mathsf{Ho}(\mathsf{Top}_*) \to \mathsf{Ab}$$