LECTURE 4: RELATIVE HOMOTOPY GROUPS AND THE ACTION OF THE FUNDAMENTAL GROUP

In this section we will introduce *relative homotopy groups* of a (pointed) pair of spaces. Associated to such a pair we obtain a long exact sequence in homotopy relating the absolute and the relative groups. This and related long exact sequences are useful in calculations as we will see later. Moreover, we want to clarify the rôle played by the choice of base points. Expressed in a fancy way, we will show that the assignment $x_0 \mapsto \pi_n(X, x_0)$ defines a functor on the fundamental groupoid $\pi(X)$ of X. This encodes, in particular, an action of the fundamental group on higher homotopy groups.

1. Relative homotopy groups

To begin with let us consider a pointed space (X, x_0) and a subspace $A \subseteq X$ containing the base point x_0. Thus we have an inclusion of pointed spaces $i\colon (A, x_0) \to (X, x_0)$ and we refer to (X, A, x_0) as a *pointed pair of spaces.* The inclusion induces a map at the level of homotopy groups (or sets)

$$i_*\colon \pi_n(A, x_0) \to \pi_n(X, x_0), \qquad n \geq 0.$$

which, in general, is *not* injective. A homotopy class $\alpha \in \pi_n(A, x_0)$ lies in the kernel of i_* if for any map $f\colon (I^n, \partial I^n) \to (A, x_0)$ representing it the induced map $i \circ f\colon (I^n, \partial I^n) \to (X, x_0)$ is homotopic to the constant map κ_{x_0}. Such a homotopy is a map $H\colon I^n \times I \to X$ satisfying the following relations:

$$H(-,1) = f, \qquad H(-,0) = \kappa_{x_0}, \qquad \text{and} \qquad H\,|_{\partial I^n \times I} = \kappa_{x_0}$$

Thus, if we denote by J^n the subspace of the boundary $\partial I^{n+1} = I^n \times \partial I \cup \partial I^n \times I$ given by

$$J^n = I^n \times \{0\} \cup \partial I^n \times I$$

then such a homotopy is a map of *triples of spaces* (in the obvious sense):

$$H\colon (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0)$$

There is also an adapted notion of homotopies of maps of triples which we want to introduce in full generality. Let $X_2 \subseteq X_1 \subseteq X_0$ and $Y_2 \subseteq Y_1 \subseteq Y_0$ be triples of spaces and let

$$f, g\colon (X_0, X_1, X_2) \to (Y_0, Y_1, Y_2)$$

be maps of triples. Then a homotopy $H\colon f \simeq g$ is a map of triples

$$H\colon (X_0, X_1, X_2) \times I = (X_0 \times I, X_1 \times I, X_2 \times I) \to (Y_0, Y_1, Y_2)$$

which satisfies $H(-,0) = f$ and $H(-,1) = g$. Thus, we are asking for a homotopy $H\colon X_0 \times I \to Y_0$ which has the property that each map $H(-,t)$ respects the subspace inclusions, i.e., is a map of triples $H(-,t)\colon (X_0, X_1, X_2) \to (Y_0, Y_1, Y_2)$. In the special case that X_2 and Y_2 are just base points, this gives us the notion of homotopies of maps of pointed pairs.

Exercise 4.1. This homotopy relation is an equivalence relation which is well-behaved with respect to maps of triples. Similarly, we get such a result for pointed pairs of spaces. There are homotopy categories of triples of spaces and pointed pairs of spaces.

Maybe you should not carry out this exercise in detail but only play a bit with the notions in order to convince yourself that they behave as expected.

Now, back to our pointed pair (X, A, x_0). The above discussion motivates the following definition:

$$\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)], \qquad n \geq 1$$

(Note that in the case of $A = \{x_0\}$ we have $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$.) A priori, the $\pi_n(X, A, x_0)$ are only pointed sets, the base point being given by the homotopy class of the constant map κ_{x_0}. However, it turns out that we get groups for $n \geq 2$ which are abelian for $n \geq 3$. To this end let us consider maps

$$\alpha, \beta\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0), \qquad n \geq 2$$

Then we can define the concatenation $\beta * \alpha\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ by the 'usual formula':

$$(\beta * \alpha)(t_1, \ldots, t_n) = \begin{cases} \alpha(2t_1, t_2, \ldots, t_n) & , \quad 0 \leq t_1 \leq 1/2 \\ \beta(2t_1 - 1, t_2, \ldots, t_n) & , \quad 1/2 \leq t_1 \leq 1 \end{cases}$$

It follows immediately that $\beta * \alpha$ again is a map of triples. As in earlier lectures one checks that this concatenation is well-defined on homotopy classes and defines a group structure on $\pi_n(X, A, x_0)$ with neutral element given by the homotopy class of the constant map.

Definition 4.2. Let (X, A, x_0) be a pointed pair of spaces. Then the group

$$\pi_n(X, A, x_0) = [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)], \qquad n \geq 2,$$

is the n-th *relative homotopy group* of (X, A, x_0). The pointed set

$$\pi_1(X, A, x_0) = [(I^1, \partial I^1, 0), (X, A, x_0)]$$

is the *first relative homotopy set* of (X, A, x_0).

To avoid awkward notation we will simply write $\pi_n(X, A)$ instead of $\pi_n(X, A, x_0)$ unless there is a risk of ambiguity. Now, if $n \geq 3$ one could again object that the above definition for the concatenation is not very natural. In fact, one could also define pairings $*_i$, where $1 \leq i \leq n-1$, given by the formula:

$$(\beta *_i \alpha)(t_1, \ldots, t_n) = \begin{cases} \alpha(t_1, \ldots, 2t_i, \ldots, t_n) & , \quad 0 \leq t_i \leq 1/2 \\ \beta(t_1, \ldots, 2t_i - 1, \ldots, t_n) & , \quad 1/2 \leq t_i \leq 1 \end{cases}$$

(Note that there is no $*_n$ unless $A = \{x_0\}$ and this is why $\pi_1(X, A)$ is only a pointed set in general.) Following the lines of the last lecture ('Eckmann-Hilton trick') one checks that these different pairings induce the same group structure and that $\pi_n(X, A)$ is abelian for $n \geq 3$. If we denote by Top^2_* the category of pointed pairs of spaces, then our discussion gives us the following:

Corollary 4.3. *The assignments $(X, A, x_0) \mapsto \pi_n(X, A)$ can be extended to define functors:*

$$\pi_1\colon \mathsf{Top}^2_* \to \mathsf{Set}_*, \qquad \pi_2\colon \mathsf{Top}^2_* \to \mathsf{Grp}, \qquad \textit{and} \qquad \pi_n\colon \mathsf{Top}^2_* \to \mathsf{Ab}, \quad n \geq 3$$

Exercise 4.4. Convince yourself that $(X, A, x_0) \mapsto \pi_2(X, A)$ really defines a functor taking values in groups by drawing some diagrams. If you are ambitious, then do similarly in order to see that $\pi_3(X, A)$ always is an *abelian* group.

A different way of proving this corollary is sketched in the exercises. There, you will show that $\pi_{n+1}(X, A)$ is naturally isomorphic to the n-th homotopy group of a certain space $P(X; x_0, A)$.

2. The long exact sequence for relative homotopy groups

The motivation for this discussion was the observation that an inclusion $i\colon (A,x_0)\to(X,x_0)$ induces a morphism of homotopy groups which is not necessarily injective. The relative homotopy groups are designed to measure the deviation from this. In fact, if j denotes the inclusion $j\colon (X,x_0)\to(X,A)$ then there is the following result.

Proposition 4.5. *Given a pointed pair of spaces* (X,A,x_0)*, there are* connecting homomorphisms $\partial\colon \pi_n(X,A)\to\pi_{n-1}(A,x_0)$, $n\geq 1$, *such that the following sequence is exact:*

$$\ldots\to\pi_{n+1}(X,A)\stackrel{\partial}{\to}\pi_n(A,x_0)\stackrel{i_*}{\to}\pi_n(X,x_0)\stackrel{j_*}{\to}\pi_n(X,A)\stackrel{\partial}{\to}\ldots\stackrel{\partial}{\to}\pi_0(A,x_0)\stackrel{i_*}{\to}\pi_0(X,x_0)$$

This is the long exact homotopy sequence *of the pointed pair* (X,A,x_0)*. Moreover, this sequence is natural in the pointed pair.*

Before we attack the proof let us be a bit more precise about the statement. Recall that a diagram of groups and group homomorphisms $G_1\stackrel{f}{\to}G_2\stackrel{g}{\to}G_3$ is *exact* at G_2 if we have the equality $\mathrm{im}(f)=\ker(g)$ of subgroups of G_2. In particular, the composition $g\circ f$ sends everything to the neutral element of G_3, but we also have a converse inclusion. Namely, if $x_2\in G_2$ lies in $\ker(g)$, then it already comes from G_1, i.e., there is an element $x_1\in G_1$ such that $f(x_1)=x_2$.

More generally, a diagram of groups and group homomorphisms

$$G_1\to G_2\to\ldots\to G_{n-1}\to G_n$$

is *exact* if it is exact at G_i for all $2\leq i\leq n-1$. A special case is a *short exact sequence* which is an exact diagram of the form:

$$1\to G_1\to G_2\to G_3\to 1$$

Example 4.6. Let G and H be groups.

(i) A homomorphism $G\to H$ is injective if and only if $1\to G\to H$ is exact.
(ii) A homomorphism $G\to H$ is surjective if and only if $G\to H\to 1$ is exact.
(iii) A homomorphism $G\to H$ is an isomorphism if and only if $1\to G\to H\to 1$ is exact.
(iv) A group G is trivial if and only if $1\to G\to 1$ is exact.

In particular, a short exact sequence basically encodes a surjective homomorphism $G_2\to G_3$ together with the inclusion of the kernel $N=G_1\subseteq G_2$.

Now, in the diagram we consider in the above proposition not all maps are homomorphisms of groups. In fact, the last three entries $\pi_1(X,A)$, $\pi_0(A,x_0)$, and $\pi_0(X,x_0)$ are only pointed sets. The notion of exactness is extended to the context of maps of pointed sets by defining the kernel of such a map to be the preimage of the base point.

Finally, let us make precise the meaning of the naturality in the above proposition. If we have a map of pointed pairs $f\colon(X,A,x_0)\to(Y,B,y_0)$, then we have a connecting homomorphism for each of the pointed pairs. The naturality means that the following square commutes:

$$\begin{array}{ccc}\pi_{n+1}(X,A) & \xrightarrow{\partial} & \pi_n(A,x_0)\\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*}\\ \pi_{n+1}(Y,B) & \xrightarrow[\partial]{} & \pi_n(B,y_0)\end{array}$$

It is easy to check that from this we actually get a commutative ladder of the form:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \pi_1(X,x_0) & \longrightarrow & \pi_1(X,A) & \longrightarrow & \pi_0(A,x_0) & \longrightarrow & \pi_0(X,x_0) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\dots & \longrightarrow & \pi_1(Y,y_0) & \longrightarrow & \pi_1(Y,B) & \longrightarrow & \pi_0(B,y_0) & \longrightarrow & \pi_0(Y,y_0)
\end{array}$$

For the purpose of the following lemma let us introduce some notation. Recall that J^n is obtained from I^{n+1} by removing the 'interior of the cube and the interior of the top face'. From a different perspective J^n is obtained from $I^n = I^n \times \{0\}$ by gluing a further copy of I^n on each face of ∂I^n. Now, if F is a collection of faces of I^n, then let $J^n_F \subseteq J^n$ be obtained from I^n by gluing only those copies of I^n which correspond to faces in F. More formally, the cube $I^n \subseteq \mathbb{R}^n$ has $2n$ faces. These can be parametrized by the set $\{1,\ldots,n\} \times \{0,1\}$ in a way that the first component of such a pair (j,i_j) tells us which coordinate is constant while the second coordinate is the value of that coordinate. Thus the face $I^{n-1}_f \subseteq I^n$ corresponding to an index $f=(j,i_j)$ is given by:

$$I^{n-1}_f = \{(t_1,\ldots,t_n) \in I^n \mid t_j = i_j\}$$

With this notation the space $J^n_F \subseteq J^n \subseteq \partial I^{n+1}$ associated to a set F of faces is given by:

$$J^n_F = I^n \times \{0\} \cup \big(\bigcup_{f\in F} I^{n-1}_f \times I\big)$$

Lemma 4.7.

(i) *The map $i\colon J^{n-1} \to I^n$ is the inclusion of a strong deformation retract, i.e., there is a map $r\colon I^n \to J^{n-1}$ which satisfies $r\circ i = id_{J^{n-1}}$ and $i\circ r \simeq id_{I^n}$ (rel J^{n-1}).*

(ii) *Given a set F of faces of I^{n-1} then $J^{n-1}_F \subseteq I^n$ is the inclusion of a strong deformation retract.*

Proof. We will only give the proof of the first claim, the second one is an exercise. If we consider the space $I^n \subseteq \mathbb{R}^n$ as the unit cube of length one, then let s be the point $s = (1/2,\ldots,1/2,2)$ sitting 'above the center of the cube'. For each point $x \in I^n$ let $l(x)$ be the unique line in $\mathbb{R}^n$ passing through s and x. This line $l(x)$ intersects J^{n-1} in a unique point which we take as the definition of $r(x)$. It is easy to see that the resulting map $r\colon I^n \to J^{n-1}$ is continuous and that we have $r\circ i = id$. The homotopy $i \circ r \simeq id$ (rel J^{n-1}) is obtained by 'collapsing the line segments between x and $r(x)$' to $r(x)$. We leave it to the reader to write down an explicit formula for this and to check that this gives us the intended relative homotopy. □

With this preparation we can now turn to the proof of the proposition.

Proof. (of Proposition 4.5) Let us begin by defining the connecting homomorphism. Given a class ω in $\pi_n(X,A)$ it can be represented by a map of triples $H\colon (I^n,\partial I^n, J^{n-1}) \to (X,A,x_0)$. This map can be restricted to the top face $I^{n-1}\times\{1\}$ to give a map $h = H|\colon (I^{n-1},\partial I^{n-1}) \to (A,x_0)$. We set:

$$\partial\colon \pi_n(X,A) \to \pi_{n-1}(A,x_0)\colon \qquad [H] \mapsto [h] = [H|]$$

We leave it to the reader to check that this defines a group homomorphism or a map of pointed sets depending on the value of n. The naturality of ∂ follows immediately from the definition.

Let us prove that the sequence is exact. Thus, we have to establish exactness at three different positions, one of which we will leave as an exercise. So, we will content ourselves showing exactness at $\pi_n(A,x_0)$ and at $\pi_n(X,A)$. So, we have to show that there are four inclusions:

(i) $\mathrm{im}(\partial) \subseteq \ker(i_*)$: This inclusion is immediate; given the homotopy class of a map

$$H\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0),$$

we have to show that the map $i \circ h\colon (I^{n-1}, \partial I^{n-1}) \to (X, x_0)$ is homotopic to the constant map (relative to the boundary). But such a homotopy is given by H itself.

(ii) $\ker(i_*) \subseteq \mathrm{im}(\partial)$: This follows by definition of the relative homotopy groups and the connecting homomorphism (see the motivational discussion!).

(iii) $\mathrm{im}(j_*) \subseteq \ker(\partial)$: Given an arbitrary $\alpha \in \pi_n(X, x_0)$ it is easy to see that $\partial \circ j_*$ is by definition represented by the constant map $\kappa_{x_0}\colon I^{n-1} \to X$.

(iv) $\ker(\partial) \subseteq \mathrm{im}(j_*)$: Let us consider a map $H\colon (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ which lies in the kernel of ∂. By definition this means that the restriction

$$h = H|\colon (I^{n-1} \times \{1\}, \partial I^{n-1} \times \{1\}) \to (A, x_0)$$

is homotopic to the constant map κ_{x_0} relative to the boundary. Choose an arbitrary such homotopy $H'\colon h \simeq \kappa_{x_0}$ (rel x_0). Then we obtain a map

$$H''\colon J^n = I^n \times \{0\} \cup \partial I^n \times I \to X$$

which is H on $I^n \times \{0\}$, the homotopy H' on $I^{n-1} \times \{1\} \times I$ and which takes the constant value x_0 on the rest of $\partial I^n \times I$.[1] An application of Lemma 4.7 gives us a map $K\colon I^{n+1} \to X$ which restricts to H'' along $J^n \subseteq I^{n+1}$. By construction, K is a homotopy of maps of triples $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ from H to $K(-,1)\colon (I^n, \partial I^n) \to (X, x_0)$. Thus, we have $[H] = j_*([K(-,1)])$ as intended.

□

Exercise 4.8. Conclude the proof of Proposition 4.5 by showing that the sequence is exact at $\pi_n(X, x_0)$.

Corollary 4.9.

(i) *Given a pointed pair of spaces (X, A, x_0) such that there is a pointed homotopy equivalence $X \simeq *$ then there are isomorphisms $\pi_n(X, A) \cong \pi_{n-1}(A)$, $n \geq 1$.*

(ii) *Let $i\colon (A, x_0) \to (X, x_0)$ be the inclusion of a retract, i.e., we have $r \circ i = id$ for some pointed map $r\colon (X, x_0) \to (A, x_0)$. Then there are split short exact sequences*

$$1 \to \pi_n(A, x_0) \to \pi_n(X, x_0) \to \pi_n(X, A) \to 1, \qquad n \geq 1,$$

i.e., short exact sequences such that $\pi_n(A, x_0) \to \pi_n(X, x_0)$ admits a retraction.

We can apply the first part to the special case of the reduced cone CA of a pointed space $(A, *)$. The reduced cone comes naturally with an inclusion $(A, *) \to (CA, *)$ so that we have a pointed pair $(CA, A, *)$. By the corollary, the connecting homomorphism $\partial\colon \pi_{n+1}(CA, A) \to \pi_n(A, *)$ is an isomorphism. We can combine this with the map induced by the quotient map $q\colon (CA, A) \to (\Sigma A, *)$ in order to obtain the *suspension homomorphism*:

$$S\colon \pi_n(A, *) \overset{\delta^{-1}}{\to} \pi_{n+1}(CA, A) \overset{q_*}{\to} \pi_{n+1}(\Sigma A, *)$$

As opposed to the context of singular homology, this suspension homomorphism is not an isomorphism (even not for nice spaces as –say– CW-complexes). However, this map can be iterated

[1] In the notation introduced before Lemma 4.7 we thus put the homotopy H on $I^{n-1}_{(n,1)} \times I$ and constant maps κ_{x_0} on $I^{n-1}_f \times I \subseteq \partial I^n \times I$, $f \neq (n,1)$.

and we will later show that the suspension homomorphisms $S\colon \pi_{n+k}(\Sigma^k A,*)\to\pi_{n+k+1}(\Sigma^{k+1}A,*)$ *eventually* are isomorphisms. Thus, the groups $\pi_{n+k}(\Sigma^k A,*)$ *stabilize* for large values of k.

3. The action of the fundamental group

We will now turn to the action of the fundamental group on higher homotopy groups. This will also allow us to understand more precisely the difference between $\pi_n(X,x_0)$ and $[S^n,X]$. To begin with let us collect some basic facts about free homotopies. Given a space X and a homotopy $H\colon S^n\times I\to X$ we obtain a path u in X by setting

$$u = H(*,-)\colon I\to X$$

where $*$ is the base point of S^n. If H is a homotopy from f to g and if u is the path of the base point, then this will be denoted by:

$$H\colon f\simeq_u g$$

The fact that the homotopy relation is an equivalence relation takes the following form if we keep track of the paths of the base point.

Lemma 4.10. (1) *For every map $f\colon S^n\to X$ we have $f\simeq_{\kappa_{f(*)}} f$.*
(2) *If for two maps $f,g\colon S^n\to X$ there is a homotopy $f\simeq_u g$ then we also have $g\simeq_{u^{-1}} f$.*
(3) *Let $f,g,h\colon S^n\to X$ be maps such that $f\simeq_u g$ and $g\simeq_v h$. Then there is a homotopy $f\simeq_{v*u} h$.*

Lemma 4.11. *For every map $f\colon S^n\to X$ and every path $u\colon I\to X$ such that $u(0)=f(*)$ there is a map $g\colon S^n\to X$ such that $f\simeq_u g$.*

Proof. Let $q\colon I^n\to I^n/\partial I^n\cong S^n$ be the quotient maps. The maps $f\circ q\colon I^n\times\{0\}\to X$ and $u\circ pr\colon \partial I^n\times I\to I\to X$ together define a map as follows:

$$\begin{array}{ccc} (f\circ q,u\circ pr)\colon J^n=I^n\times\{0\}\cup\partial I^n\times I & \longrightarrow & X \\ \downarrow & \nearrow_{\exists H} & \\ I^{n+1} & & \end{array}$$

It follows from Lemma 4.7 that we can find an extension $H\colon I^{n+1}\to X$ as indicated in the diagram. By construction, $H(-,t)\colon I^n\to X$ takes the constant value $u(t)$ on the boundary ∂I^n and hence factors as $I^n\times I\to S^n\times I\to X$. The induced map $S^n\times I\to X$ defines a homotopy $f\simeq_u g$. □

Thus g is obtained from f by 'stacking a copy of the path on top of each point of ∂I^n' and then choosing a certain reparametrization. In the special case of $n=1$ it is easy to see that this way we obtain $g=u*f*u^{-1}$. In the notation of the lemma, we want to show that the assignment

$$([u],[f])\mapsto[g]$$

is well-defined.

Lemma 4.12. *Let $f,f_0,f_1,g,g_0,g_1\colon S^n\to X$ be maps and let $u,v\colon I\to X$ be paths in X.*
(i) *If $f\simeq_u g$ and $u\simeq v$ $(\mathrm{rel}\ \partial I)$ then also $f\simeq_v g$.*
(ii) *Let us assume that $f_0(*)=f_1(*)=x_0$ and $g_0(*)=g_1(*)=x_1$. If $f_0\simeq f_1$ $(\mathrm{rel}\ x_0)$, $g_0\simeq g_1$ $(\mathrm{rel}\ x_1)$ and $f_0\simeq_u g_0$ then also $f_1\simeq_u g_1$.*

Proof. Let us begin with a proof of the first claim. We recommend that you draw a picture in the case of $n = 1$ to see what is happening. Now, let $H\colon I^n \times I \to X$ be a homotopy $f \simeq_u g$ and similarly $G\colon I \times I \to X$ a homotopy $u \simeq v$ (rel ∂I) which both exist by assumption. From this we construct a new map $K\colon J^{n+1} \to X$ as follows. Note that $J^{n+1} \subseteq \partial I^{n+2}$ can be written as a union of three subspaces (use the Leibniz rule!):

$$J^{n+1} = I^n \times I \times \{0\} \cup \partial I^n \times I \times I \cup I^n \times \partial I \times I$$

On the first subspace we take the homotopy H, on the second subspace $\partial I^n \times I \times I \overset{pr}{\to} I \times I \overset{G}{\to} X$, and on the remaining one the constant homotopies of f and g, i.e., we take:

$$I^n \times \partial I \times I \overset{pr}{\to} I^n \times \partial I \cong I^n \sqcup I^n \overset{(f,g)}{\to} X$$

We leave it to the reader to check that these maps fit together in the sense that they define a map $K\colon J^{n+1} \to X$. Now, an application of Lemma 4.7 shows that K can be extended to a map $L = K \circ r\colon I^{n+2} \to J^{n+1} \to X$. By construction it follows that the restriction of L to $I^n \times I \times \{1\}$ gives us the desired homotopy $f \simeq_v g$.

The second claim is now easy. By assumption we have a chain of homotopies:

$$f_1 \simeq_{\kappa_{x_0}} f_0 \simeq_u g_0 \simeq_{\kappa_{x_1}} \simeq g_1$$

But since $\kappa_{x_1} * u * \kappa_{x_0} \simeq u$ (rel ∂I) we can conclude $f_1 \simeq_u g_1$ (by the first part of this lemma). □

Recall that given a space X we denote its fundamental groupoid by $\pi(X)$. The objects in $\pi(X)$ are the points in X while morphisms are given by homotopy classes of paths relative to the boundary.

Corollary 4.13. *Let $f\colon (S^n, *) \to (X, x_0)$, let $u\colon I \to X$ be a path from x_0 to x_1, and let $f \simeq_u g$ for some $g\colon (S^n, *) \to (X, x_1)$. Then the homotopy class $[g] \in \pi_n(X, x_1)$ only depends on the homotopy classes $[f] \in \pi_n(X, x_0)$ and $[u] \in \pi(X)(x_0, x_1)$.*

Proof. Let us assume we were also given $f \simeq_{\kappa_{x_0}} f'$, $u \simeq v$ (rel ∂I), and $f' \simeq_v g'$. Then in order to show that $g \simeq_{\kappa_{x_1}} g'$ we observe that:

$$g \simeq_{u^{-1}} f \simeq_{\kappa_{x_0}} f' \simeq_v g'$$

But since $v * \kappa_{x_0} * u^{-1} \simeq \kappa_{x_1}$ (rel ∂I) we can conclude by Lemma 4.10 and Lemma 4.12. □

Thus, we obtain a well-defined pairing

$$\pi(X)(x_0, x_1) \times \pi_n(X, x_0) \to \pi_n(X, x_1)\colon \quad ([u], [f]) \mapsto {}^{[u]}[f] = [g]$$

for $f \simeq_u g$ as in the notation of Lemma 4.11.

Proposition 4.14. *Given a space X then we have a functor $\pi_n(X, -)\colon \pi(X) \to \mathsf{Grp}$ which sends an object $x_0 \in \pi(X)$ to $\pi_n(X, x_0)$ and a map $[u] \in \pi(X)(x_0, x_1)$ to ${}^{[u]}(-)\colon \pi_n(X, x_0) \to \pi_n(X, x_1)$.*

Proof. We know already that $\pi_n(X, x_0)$ is a group for all $x_0 \in X$ and that we have a well-defined map of sets ${}^{[u]}(-)\colon \pi_n(X, x_0) \to \pi_n(X, x_1)$. To check that the assignment $[u] \mapsto {}^{[u]}(-)$ is compatible with compositions and identities it suffices to recall the definition of this action. In fact, since it was obtained from 'stacking copies of u on top of ∂I^n' it is easy to see that this is true. It remains to show that the maps ${}^{[u]}(-)\colon \pi_n(X, x_0) \to \pi_n(X, x_1)$ are group homomorphisms. But this is left as an exercise. □

Exercise 4.15. Given a path $u\colon I \to X$ with $u(0) = x_0$ and $u(1) = x_1$ show that $[f] \mapsto {}^{[u]}[f]$ defines a group homomorphism

$${}^{[u]}(-)\colon \pi_n(X, x_0) \to \pi_n(X, x_1).$$

Thus, we have isomorphisms $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ whenever $x_0, x_1 \in X$ lie in the same path-component. Note that such an isomorphism is, in general, not canonical, since it depends on the choice of a homotopy class of paths from x_0 to x_1. However, if $\pi_1(X, x_0) \cong 1$ then there is only a unique such homotopy class so that the identification $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ is canonical.

Corollary 4.16. *Given a pointed space (X, x_0) then there is an action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$. For $n = 1$ this specializes to the conjugation action, i.e., we have:*

$${}^{[u]}[f] = [u][f][u]^{-1}, \quad [u], [f] \in \pi_1(X, x_0)$$

Proof. Since we have a functor $\pi_n(X, -)\colon \pi(X) \to \mathsf{Grp}$, it is completely formal that we get an action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$. In the context of Lemma 4.11, we already observed that our construction sends (u, f) to $u * f * u^{-1}$. Thus, at the level of homotopy classes we obtain the conjugation. □

Instead of using the actual construction of Lemma 4.11 to deduce this corollary, we can also argue using the essential uniqueness of the construction (Corollary 4.13): we just have to observe that there is a homotopy:

$$f \simeq_u u * f * u^{-1}$$

Whenever we have a group acting on a set we can pass to the set of orbits. In the case of the action of the fundamental group on higher homotopy groups we obtain the following convenient result.

Corollary 4.17. *Let X be a path-connected space and let $x_0 \in X$. Then the forgetful map*

$$\pi_n(X, x_0) = [(S^n, *), (X, x_0)] \to [S^n, X]$$

exhibits $[S^n, X]$ as the set of orbits of the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

Exercise 4.18. Give a proof of this corollary, i.e., show that the forgetful map is surjective and that two elements $[f]$ and $[g]$ have the same image if and only if there is a loop u at x_0 such that ${}^{[u]}[f] = [g]$.