

## LECTURE 6: FIBER BUNDLES

In this section we will introduce the interesting class of fibrations given by fiber bundles. Fiber bundles play an important role in many geometric contexts. For example, the Grassmannian varieties and certain fiber bundles associated to Stiefel varieties are central in the classification of vector bundles over (nice) spaces. The fact that fiber bundles are examples of Serre fibrations follows from Theorem ?? which states that being a Serre fibration is a local property.

### 1. FIBER BUNDLES AND PRINCIPAL BUNDLES

**Definition 6.1.** A fiber bundle with fiber  $F$  is a map  $p: E \rightarrow X$  with the following property: every point  $x \in X$  has a neighborhood  $U \subseteq X$  for which there is a homeomorphism  $\phi_U: U \times F \cong p^{-1}(U)$  such that the following diagram commutes in which  $\pi_1: U \times F \rightarrow U$  is the projection on the first factor:

$$\begin{array}{ccc}
 U \times F & \xrightarrow[\cong]{\phi_U} & p^{-1}(U) \\
 \searrow \pi_1 & & \swarrow p \\
 & U &
 \end{array}$$

**Remark 6.2.** The projection  $X \times F \rightarrow X$  is an example of a fiber bundle: it is called the *trivial bundle* over  $X$  with fiber  $F$ . By definition, a fiber bundle is a map which is ‘locally’ homeomorphic to a trivial bundle. The homeomorphism  $\phi_U$  in the definition is a *local trivialization* of the bundle, or a *trivialization over  $U$* .

Let us begin with an interesting subclass. A fiber bundle whose fiber  $F$  is a discrete space is (by definition) a *covering projection* (with fiber  $F$ ). For example, the exponential map  $\mathbb{R} \rightarrow S^1$  is a covering projection with fiber  $\mathbb{Z}$ . Suppose  $X$  is a space which is path-connected and locally simply connected (in fact, the weaker condition of being semi-locally simply connected would be enough for the following construction). Let  $\tilde{X}$  be the space of homotopy classes (relative endpoints) of paths in  $X$  which begin at a given base point  $x_0$ . We can equip  $\tilde{X}$  with the quotient topology with respect to the map  $P(X, x_0) \rightarrow \tilde{X}$ . The evaluation  $\epsilon_1: P(X, x_0) \rightarrow X$  induces a well-defined map  $\tilde{X} \rightarrow X$ . One can show that  $\tilde{X} \rightarrow X$  is a covering projection. (It is called the *universal covering projection*. A later exercise will explain this terminology.)

Let  $p: E \rightarrow X$  be a fiber bundle with fiber  $F$ . If  $f: X' \rightarrow X$  is any map, then the projection

$$f^*(p): X' \times_X E \rightarrow X'$$

is again a fiber bundle with fiber  $F$  (see Exercise 1 of sheet 6).

We will need the following definitions.

**Definition 6.3.** Let  $f: Y \rightarrow X$  be an arbitrary map.

- (i) A *section* of  $f$  over an open set  $U \subseteq X$  is a map  $s: U \rightarrow Y$  such that  $f \circ s = id_U$ .
- (ii) The map  $f: Y \rightarrow X$  has *enough local sections* if every points of  $X$  has an open neighborhood on which some local section of  $f$  exists.

Thus, by the very definition, every fiber bundle has enough local sections. And if you know a bit of differential topology, you'll know that any surjective submersion between smooth manifolds has enough local sections.

Many interesting examples of fiber bundle show up in the context of nice group actions. For this purpose, let us formalize this notion.

**Definition 6.4.** Let  $G$  be a topological group and let  $E$  be a space. A (*right*) *action* of  $G$  on  $E$  is a map

$$\mu: E \times G \rightarrow E: (e, g) \mapsto \mu(e, g) = e \cdot g$$

such that the following identities hold:

$$e \cdot 1 = e \quad \text{and} \quad e \cdot (gh) = (e \cdot g) \cdot h, \quad e \in E, g, h \in G.$$

If  $E$  also comes with a map  $p: E \rightarrow X$  such that  $p(e \cdot g) = p(e)$  for all  $e$  and  $g$ , then the action of  $G$  restricts to an action on each fiber of  $p$ , and one also says that the action is *fiberwise*.

Given a space  $E$  with a right action by  $G$ , then there is an induced equivalence relation  $\sim$  on  $E$  defined by

$$e \sim e' \quad \text{iff} \quad e \cdot g = e' \quad \text{for some } g \in G.$$

The quotient space  $E/\sim$  is called the *orbit space* of the action, and is usually denoted  $E/G$ . The equivalence classes are called the *orbits* of the action. They are the fibers of the quotient map  $\pi: E \rightarrow E/G$ .

**Definition 6.5.** Let  $G$  be a topological group. A *principal  $G$ -bundle* is a map  $p: E \rightarrow B$  together with a fiberwise action of  $G$  on  $E$ , with the property that:

- (i) The map  $\phi: E \times G \rightarrow E \times_B E: (e, g) \mapsto (e, e \cdot g)$  is a homeomorphism.
- (ii) The map  $p: E \rightarrow B$  has enough local sections.

**Proposition 6.6.** *Any principal  $G$ -bundle is a fiber bundle with fiber  $G$ .*

*Proof.* Write

$$\delta = \pi_2 \circ \phi^{-1}: E \times_B E \rightarrow E \times G \rightarrow G$$

for the *difference map*, characterized by the identity

$$e \cdot \delta(e, e') = e'$$

for any  $(e, e') \in E \otimes_B E$ . If  $b \in B$  and  $U \subseteq B$  is a neighborhood on which a local section  $s: U \rightarrow E$  exists, then the map

$$U \times G \rightarrow p^{-1}(U): (x, g) \mapsto s(x) \cdot g$$

is a homeomorphism, with inverse given by  $e \mapsto (p(e), \delta(s(p(e)), e))$ . □

An important source of principal bundles comes from the construction of *homogeneous spaces*. Let  $G$  be a topological group, and suppose that  $G$  is compact and Hausdorff. Let  $H$  be a closed subgroup of  $G$ , and let  $G/H$  be the space of left cosets  $gH$ . Then the projection  $\pi: G \rightarrow G/H$  satisfies the first condition in the definition of principal bundles, because the map

$$\phi: G \times H \rightarrow G \times_{(G/H)} G: (g, h) \mapsto (g, gh)$$

is easily seen to be a continuous bijection, and hence it is a homeomorphism by the compact-Hausdorff assumption. So, we conclude that *if  $G \rightarrow G/H$  has enough local sections, then it is a principal  $H$ -bundle*. (For those who know Lie groups: if  $G$  is a compact Lie group and  $H$  is a

closed subgroup, then  $G/H$  is a manifold and  $G \rightarrow G/H$  is a submersion, hence has enough local sections.)

**Remark 6.7.** In case you have to prove by hand that  $\pi: G \rightarrow G/H$  has enough local sections, it is useful to observe that it suffices to find a local section on a neighborhood  $V$  of  $\pi(1)$  where  $1 \in G$  is the unit, so  $\pi(1) = H \in G/H$ . Because if  $s: V \rightarrow G$  is such a section, then for another coset  $gH$ , the open set  $gV$  is a neighborhood of  $gH$  in  $G/H$ , and  $\tilde{s}: gV \rightarrow G$  defined  $\tilde{s}(\xi) = gs(g^{-1}\xi)$  is a local section on  $gV$ .

**Remark 6.8.** A related construction yields for two closed subgroups  $K \subseteq H \subseteq G$  a map  $G/K \rightarrow G/H$  which gives us a fiber bundle with fiber  $H/K$  under certain assumptions. (See Exercise 5 of sheet 6)

## 2. STIEFEL VARIETIES AND GRASSMANN VARIETIES

We will now consider some classical and important special cases of these general constructions for groups, namely the cases of *Stiefel* and *Grassmann varieties*. We begin by the Stiefel varieties. Consider the vector space  $\mathbb{R}^n$  with its standard basis  $(e_1, \dots, e_n)$ . A *k-frame* in  $\mathbb{R}^n$  (or more explicitly, an orthonormal *k-frame*) is a *k-tuple* of vectors in  $\mathbb{R}^n$ ,

$$(v_1, \dots, v_k)$$

with  $\langle v_i, v_j \rangle = \delta_{ij}$ . Thus,  $v_1, \dots, v_k$  form an orthonormal basis for a *k-dimensional* subspace  $\text{sp}(v_1, \dots, v_k) \subseteq \mathbb{R}^n$ . We can topologize this space of *k-frames* as a subspace of  $\mathbb{R}^n \times \dots \times \mathbb{R}^n$  (*k times*). It is a closed and bounded subspace, hence it is compact. This space is usually denoted

$$V_{n,k}$$

and called the *Stiefel variety*. (It has a well-defined dimension: what is it?) Note that

$$V_{n,1} = S^{n-1}$$

is a sphere. We claim that  $V_{n,k}$  is a homogeneous space, i.e., a space of the form  $G/H$  as just discussed. To see this, take for  $G$  the group  $O(n)$  of orthogonal transformations of  $\mathbb{R}^n$ . We can think of the elements of  $O(n)$  as orthogonal  $n \times n$  matrices, or as *n-tuples* of vectors in  $\mathbb{R}^n$ ,

$$(v_1, \dots, v_n)$$

(the column vectors of the matrix) which form an orthonormal basis in  $\mathbb{R}^n$ . Thus, there is an evident projection

$$\pi: O(n) \rightarrow V_{n,k}$$

which just remembers the first *k* vectors. The group  $O(n-k)$  can be viewed as a closed subgroup of  $O(n)$ , using the group homomorphism

$$O(n-k) \rightarrow O(n): A \mapsto \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$$

where  $I = I_k$  is the  $k \times k$  unit matrix. One easily checks (Exercise!) that the projection induces a homeomorphism

$$O(n)/O(n-k) \xrightarrow{\cong} V_{n,k}.$$

Note that it is again enough to show that we have a continuous bijection since the spaces under consideration are compact Hausdorff. Thus, to see that  $\pi: O(n) \rightarrow V_{n,k}$  is a principal bundle, it suffices to check that there are enough local sections. This can easily be done explicitly, using the Gram-Schmidt algorithm for transforming a basis into an orthonormal one. Indeed, as we said

above, it is enough to find a local section on a neighborhood of  $\pi(1) = \pi(e_1, \dots, e_n) = (e_1, \dots, e_k)$ . Let

$$U = \{(v_1, \dots, v_k) \mid v_1, \dots, v_k, e_{k+1}, \dots, e_n \text{ are linearly independent}\}$$

and let  $s(v_1, \dots, v_k)$  be the result of applying the Gram-Schmidt to the basis  $v_1, \dots, v_k, e_{k+1}, \dots, e_n$ . (This leaves  $v_1, \dots, v_k$  unchanged, changes  $e_{k+1}$  into  $e_{k+1} - \sum \langle v_i, e_{k+1} \rangle v_i$  divided by its length, and so on.)

Let us observe that this construction of the Stiefel varieties also shows that they fit into a tower

$$O(n) = V_{n,n} \rightarrow V_{n,n-1} \rightarrow \dots \rightarrow V_{n,k} \rightarrow V_{n,k-1} \rightarrow \dots \rightarrow V_{n,1} \cong S^{n-1}$$

in which each map  $V_{n,k} \rightarrow V_{n,k-1}$  is a principal bundle with fiber:

$$O(n-k+1)/O(n-k) \cong V_{n-k+1,1} \cong S^{n-k}$$

From these Stiefel varieties we can now construct the Grassmann varieties. In fact, the group  $O(k)$  obviously acts on the Stiefel variety  $V_{n,k}$  of  $k$ -frames in  $\mathbb{R}^n$ . The orbit space of this action is called the *Grassmann variety*, and denoted

$$G_{n,k} = V_{n,k}/O(k).$$

The orbit of a  $k$ -frame  $(v_1, \dots, v_k)$  only remembers the subspace  $W$  spanned by  $(v_1, \dots, v_k)$ , because any two orthogonal bases for  $W$  can be related by acting by an element of  $O(k)$ . Thus,  $G_{n,k}$  is the *space of  $k$ -dimensional subspaces of  $\mathbb{R}^n$* . Since  $V_{n,k} = O(n)/O(n-k)$ , the Grassmann variety is itself a homogeneous space

$$G_{n,k} \cong O(n)/(O(k) \times O(n-k))$$

where  $O(k)$  and  $O(k) \times O(n-k)$  are viewed as the subgroups of matrices of the forms

$$\begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$$

respectively. The quotient map

$$q: O(n) \rightarrow G_{n,k}$$

is again a principal bundle (with fiber  $O(k) \times O(n-k)$ ), because  $q$  again has enough local sections. Indeed, it suffices to construct a local section on a neighborhood of  $q(I)$ . As a  $k$ -dimensional subspace of  $\mathbb{R}^n$  this is  $\mathbb{R}^k \times \{0\}$ . Let

$$U = \{W \subseteq \mathbb{R}^n \mid W \oplus \mathbb{R}^{n-k} = \mathbb{R}^n\}$$

be the subspace of complements of the subspace  $\mathbb{R}^{n-k} \subseteq \mathbb{R}^n$  spanned by  $e_{k+1}, \dots, e_n$ , and define a section  $s$  on  $U$  as follows: write  $w_i$  for the projection of  $e_i$  on  $W$ ,  $1 \leq i \leq k$ , i.e.,

$$e_i = w_i + \sum_{j>k} \lambda_j e_j.$$

Then  $(w_1, \dots, w_k, e_{k+1}, \dots, e_n)$  still span all of  $\mathbb{R}^n$ , and we can transform this into an orthonormal basis by Gram-Schmidt, the result of which defines  $s(W)$ .

It follows that  $V_{n,k} \rightarrow G_{n,k}$  also has enough local sections (why?), so this is a principal bundle too (for the group  $O(k)$ ). Summarizing, we have a diagram of three principal bundles

$$\begin{array}{ccc} & O(n) & \\ \swarrow & & \searrow \\ V_{n,k} & \longrightarrow & G_{n,k} \end{array}$$

constructed as

$$O(n) \longrightarrow O(n)/O(n-k) \longrightarrow O(n)/(O(k) \times O(n-k)).$$

### 3. FIBER BUNDLES ARE FIBRATIONS

The relation of the previous considerations to the previous lecture is given by the following result.

**Theorem 6.9.** *A fiber bundle is a Serre fibration.*

Before proving this theorem, we draw some immediate consequences by applying the long exact sequence of homotopy groups associated to a Serre fibration to our examples of fiber bundles. More applications of this kind can be found in the exercises.

#### Application 6.10.

- (i) Let  $p: E \rightarrow B$  be a covering projection, let  $e_0 \in E$  and let  $b_0 = p(e_0)$ . If we denote the fiber by  $F$  (a discrete space), then we have pointed maps

$$(F, e_0) \rightarrow (E, e_0) \rightarrow (B, b_0).$$

Then  $p_*: \pi_i(E, e_0) \rightarrow \pi_i(B, b_0)$  is an isomorphism for all  $i > 1$ . Moreover, if  $E$  is connected then there is short exact sequence

$$0 \rightarrow \pi_1(E) \rightarrow \pi_1(B) \rightarrow F \rightarrow 0$$

where we have omitted base points from notation, and where we view  $F$  as a pointed set  $(F, e_0)$ . Thus, for the covering  $\mathbb{R} \rightarrow S^1$  this gives us  $\pi_i(S^1) \cong 0$  for  $i > 1$ , since  $\mathbb{R}$  is contractible.

More generally, for the universal covering projection  $\tilde{X} \rightarrow X$  with fiber  $\pi_1(X, x_0)$  we have  $\pi_i(\tilde{X}) \xrightarrow{\cong} \pi_i(X)$  for  $i > 1$  and  $\pi_1(\tilde{X}) \cong 0$ . These statements all follow by applying the long exact sequence of a Serre fibration.

- (ii) In the second lecture we stated that  $\pi_i(S^n) \cong 0$  for  $i < n$  (a fact that can easily be proved using a bit of differential topology, but which we haven't given an independent proof yet). Using this, we can analyze the long exact sequence associated to the fiber bundle

$$O(n) \rightarrow V_{n,1} \cong S^{n-1}$$

with fiber  $O(n-1)$ , to conclude that the map

$$\pi_i(O(n-1)) \rightarrow \pi_i(O(n))$$

induced by the inclusion (always with the unit of the group as the base point) is an *isomorphism* for  $i+1 < n-1$  and a *surjection* for  $i < n-1$ . Writing  $O(n-k) \rightarrow O(n)$  as a composition

$$O(n-k) \rightarrow O(n-k+1) \rightarrow \dots \rightarrow O(n-1) \rightarrow O(n),$$

we find that

$$\pi_i(O(n-k)) \rightarrow \pi_i(O(n))$$

is an isomorphism if  $i+1 < n-k$  and is surjective if  $i < n-k$ . Feeding this back in the long exact sequence for the fiber bundle

$$O(n) \rightarrow O(n)/O(n-k) \cong V_{n,k},$$

we conclude that

$$\pi_i(V_{n,k}) \cong 0, \quad i < n-k.$$

Now back to the proof of Theorem ???. Instead of proving this theorem, we will prove a slightly more general result (Theorem ???), which can informally be phrased by saying that ‘being a Serre fibration is a local property’. Theorem ??? immediately follows from this result and the fact that trivial fibrations are Serre fibrations.

**Theorem 6.11.** *Let  $p: E \rightarrow B$  be a map with the property that every point  $b \in B$  has a neighborhood  $U \subseteq B$  such that the restriction  $p|_U: p^{-1}(U) \rightarrow U$  is a Serre fibration. Then  $p: E \rightarrow B$  is itself a Serre fibration.*

The proof of this theorem is relatively straightforward if we assume the following lemma. Recall the following notation from a previous lecture. Let  $F = \{F_a \mid a \in A\}$  be a family of faces of the cube  $I^n$ , and let

$$J_{(F)}^n = (I^n \times \{0\}) \cup \left( \bigcup_a F_a \times I \right) \rightarrow I^n \times I$$

be the inclusion.

**Lemma 6.12.** *A map  $p: E \rightarrow B$  is a Serre fibration if and only if it has the RLP with respect to all maps of the form  $J_{(F)}^n \rightarrow I^n \times I$ .*

Note that the ‘if’-part is clear because the case  $F = \emptyset$  gives the definition of a Serre fibration. Earlier on, we have also used the case where  $F$  is the family of *all* the faces, when  $J_{(F)}^n \rightarrow I^n \times I$  is homeomorphic to  $I^n \times \{0\} \rightarrow I^n \times I$ . The same is actually true for an arbitrary family  $F$ , but one can also use an inductive argument to reduce the general case to the two cases where  $F = \emptyset$  or  $F$  consists of all faces. We will do this after the proof of Theorem ???.

*Proof of Theorem ???.* Let  $p: E \rightarrow B$  be as in the statement of the theorem, and consider a diagram of solid arrows of the form

$$\begin{array}{ccc} I^{n-1} \times \{0\} & \xrightarrow{g} & E \\ \downarrow & \nearrow h & \downarrow p \\ I^n & \xrightarrow{f} & B \end{array}$$

in which we wish to find a diagonal  $h$  as indicated. By assumption on  $p$  and compactness of  $I^n$ , we can find a natural number  $k$  large enough so that for any sequence  $(i_1, \dots, i_n)$  of numbers  $0 \leq i_1, \dots, i_n \leq k-1$ , the small cube

$$[i_1/k, (i_1+1)/k] \times \dots \times [i_n/k, (i_n+1)/k]$$

is mapped by  $f$  into an open set  $U \subseteq B$  over which  $p$  is a Serre fibration (use the Lebesgue lemma!). Now order all these tuples

$$(i_1, \dots, i_n)$$

lexicographically, and list them as  $C_1, \dots, C_{k^n}$ . We will define a lift  $h$  by consecutively finding lifts  $h_r$  on  $C_1 \cup \dots \cup C_r \subseteq I^n$  making the diagram

$$\begin{array}{ccc} I^{n-1} \times \{0\} & \xrightarrow{g} & E \\ \downarrow & \nearrow h_r & \downarrow p \\ C_1 \cup \dots \cup C_r & \xrightarrow{f} & B \end{array}$$

commute. We can find  $h_1$  because  $(I^{n-1} \times \{0\}) \cap C_1$  is a small copy of  $I^{n-1} \times \{0\} \rightarrow I^n$ . And given  $h_r$ , we can extend it to  $h_{r+1}$  by defining  $h_{r+1}|_{C_{r+1}}$  as a lift in

$$\begin{array}{ccc} C_{r+1} \cap ((I^{n-1} \times \{0\}) \cup (C_1 \cup \dots \cup C_r)) & \xrightarrow{(g, h_r)} & E \\ \downarrow & \nearrow h_{r+1}| & \downarrow p \\ C_{r+1} & \xrightarrow{f} & B. \end{array}$$

Such a lift exists, because  $C_{r+1} \cap ((I^{n-1} \times \{0\}) \cup (C_1 \cup \dots \cup C_r)) \rightarrow C_{r+1}$  is (essentially) a small copy of an inclusion  $J_{(F)}^n \rightarrow I^n \times \{0\}$ . (You should draw some pictures for yourself in the cases  $n = 2, 3$  to see what is going on.)  $\square$

*Proof of Lemma ??.* As we already said it only remains to establish the ‘only if’-direction which we already know in the cases of  $F = \emptyset$  or the collection of all faces. We will reduce the intermediate cases to the case of all the faces by induction on  $n$ . For  $n = 0$ ,  $I^0$  has no faces so only the case  $A = \emptyset$  applies and there is nothing to prove. For  $n = 1$ , there are four cases,

$$A = \emptyset, \quad A = \{0\}, \quad A = \{1\}, \quad \text{and} \quad A = \{0, 1\},$$

of which the first and the last have already been dealt with. For the intermediate case  $A = \{0\}$ , for example, consider a diagram of the form

$$\begin{array}{ccc} I \times \{0\} \cup \{0\} \times I & \xrightarrow{f} & E \\ \downarrow & & \downarrow p \\ I^2 & \longrightarrow & B \end{array}$$

where  $p: E \rightarrow B$  is a Serre fibration. Now one can first find a lift for

$$\begin{array}{ccc} \{1\} \times \{0\} & \longrightarrow & E \\ \downarrow & \nearrow g & \downarrow p \\ \{1\} \times I & \longrightarrow & B \end{array}$$

by the case  $n = 0$ . Then next fill the following diagram

$$\begin{array}{ccc} (I \times \{0\} \cup \{0\} \times I) \cup \{1\} \times I & \xrightarrow{(f, g)} & E \\ \downarrow & \nearrow & \downarrow p \\ I^2 & \longrightarrow & B \end{array}$$

by the case where  $F$  consists of all the faces (the fourth case).

The induction from  $n$  to  $n + 1$  proceeds in exactly the same way: suppose  $F = \{F_a \mid a \in A\}$  is a family of faces of  $I^{n+1}$  for which we wish to find a lift in a diagram of the form

$$\begin{array}{ccc} J_{(F)}^{n+1} & \xrightarrow{f} & E \\ \downarrow & \nearrow & \downarrow p \\ I^{n+1} \times I & \longrightarrow & B \end{array}$$

If  $F$  does not consist of all the faces, we add a face  $G$  and extend  $f$  to  $J_F^{n+1} \cup (G \times I) = J_{F \cup \{G\}}^{n+1}$  by lifting in

$$\begin{array}{ccc} G \times \{0\} \cup \bigcup_a (G \cap F_a) \times I & \xrightarrow{\quad} & E \\ \downarrow & \nearrow \text{---} & \downarrow p \\ G \times I & \xrightarrow{\quad} & B \end{array}$$

which is possible by the earlier case of the induction, because  $\bigcup_a (G \cap F_a)$  is a family of faces of a cube of lower dimension. After having done this for all the faces not in  $F$ , we arrive at the case where  $A$  is the set of all faces which was already settled.  $\square$