LECTURE 13: REPRESENTABLE FUNCTORS AND THE BROWN REPRESENTABILITY THEOREM

1. Representable functors

Let $\mathcal{C}$ be a category. A functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Sets}$ is called *representable* if there exists an object $B = B_F$ in $\mathcal{C}$ with the property that there is a *natural* isomorphism of functors

$$\varphi\colon \mathcal{C}(-, B_F) \longrightarrow F.$$

Thus, for every object X in $\mathcal{C}$, there is an isomorphism φ_X from the set of arrows $\mathcal{C}(X, B_F)$ to the value $F(X)$ of the functor. The naturality condition states that for any map $f\colon Y \to X$ in $\mathcal{C}$, the identity

$$F(f)(\varphi_X(\alpha)) = \varphi_Y(\alpha \circ f)$$

holds, for any $\alpha\colon X \to B$. One usually expresses this in terms of a commutative diagram

$$\begin{array}{ccc} \mathcal{C}(X,B) & \xrightarrow{\varphi_X} & F(X) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathcal{C}(Y,B) & \xrightarrow[\varphi_Y]{} & F(Y), \end{array}$$

where f^* denotes the contravariant functoriality in f; that is, f^* is composition with f on the left of the diagram, and $f^* = F(f)$ on the right. By applying φ_B to the identity map $B \to B$ we obtain a special element $\gamma = \varphi_B(\mathrm{id}) \in F(B)$, which is *generic* in the sense that any element $\xi \in F(X)$ can be obtained as $\xi = f^*(\gamma)$, for a suitable $f\colon X \to B$. Indeed, one can take $f = \varphi_X^{-1}(\xi)$ and apply naturality to check that $\xi = f^*(\gamma)$.

Clearly, if the functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Sets}$ is representable, then it "respects" all colimits that exist in $\mathcal{C}$. Since F is contravariant, these colimits are limits in $\mathcal{C}^{\mathrm{op}}$ and are turned into limits in Sets by F. For example, a pushout diagram in $\mathcal{C}$ as below on the left is turned into a pullback diagram on the right

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ B & \xrightarrow[h]{} & D \end{array} \qquad\qquad \begin{array}{ccc} F(A) & \xleftarrow{g^*} & F(C) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle k^*} \\ F(B) & \xleftarrow[h^*]{} & F(D), \end{array}$$

and a coproduct $X = \coprod_{i\in I} X_i$ in $\mathcal{C}$ is turned into a product $F(X) = \prod_{i \in I} F(X_i)$.

Thus, for a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Sets}$ to be representable it is necessary that F turns colimits that exist in $\mathcal{C}$ into limits in Sets. It is not necessary to check this condition for all types of existing colimits, because some can be obtained from others. For example, *coequalizers* below on the left can be obtained from pushouts and (binary) coproducts, as indicated on the right:

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \longrightarrow C \qquad\qquad \begin{array}{ccc} A \coprod A & \xrightarrow{(f,g)} & B \\ {\scriptstyle \nabla}\downarrow & & \downarrow \\ A & \longrightarrow & C, \end{array} \qquad (1.1)$$

where ∇ denotes the "codiagonal". Also, the colimit $X = \varinjlim X_n$ of a sequence

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots$$

can be constructed from coequalizers and coproducts, as

$$\coprod X_n \underset{f}{\overset{\mathrm{id}}{\rightrightarrows}} \coprod X_n \longrightarrow \varinjlim X_n, \tag{1.2}$$

where f sends the summand X_n to the summand to the summand X_{n+1} via f_n.

Many algebraic structures can be expressed in terms of finite products and commutative diagrams, hence if $F\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Sets}$ is representable, it sends such an algebraic structure to a similar structure in Sets. For example, a group G in $\mathcal{C}^{op}$, that is, a "cogroup" in $\mathcal{C}$, given by comultiplication and counit

$$* \xleftarrow{\varepsilon} G \xrightarrow{\nabla} G \coprod G,$$

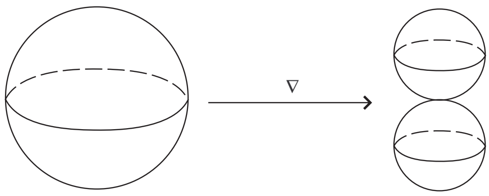
($*$ denotes the initial object in $\mathcal{C}$) is turned into a group $F(G)$ with multiplication

$$\nabla^*\colon F(G) \times F(G) \cong F(G \coprod G) \longrightarrow F(G)$$

and unit ε^*. Such coalgebraic structures are quite familiar in topology. As a basic example, recall that the group structure on $\pi_n(X, x_0) = [(S^n, *), (X, x_0)]$ comes from a cogroup structure on the sphere

$$* \longleftarrow S^n \xrightarrow{\nabla} S^n \vee S^n$$

given by the "pinch map" ∇



(1.3)

So, if F is a contravariant functor from the homotopy category of pointed spaces $\mathsf{Ho}(\mathsf{Top}_*)$ to Sets, then $F(S^n, *)$ is a group for each $n \geq 1$ (abelian for $n \geq 2$).

In $\mathsf{Ho}(\mathsf{Top}_*)$ and other cases we wish to study, the category $\mathcal{C}$ is a *pointed* category: it has an object, usually denoted by $*$ or pt, which is both initial and terminal. So for any two objects X and Y there is a canonical arrow $X \to * \to Y$, and any representable functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Sets}$, from a pointed category naturally takes values in the category Sets_* of *pointed* sets.

Moreover, as we will see in our example, $\mathcal{C}$ will have coproducts, but not very many other types of colimits. Instead, $\mathcal{C}$ will have some "weak" colimits though: a *weak colimit* A of a diagram $\{A_i\}_{i \in I}$ has the existence property of a colimit, but not the uniqueness property. In other words, for a compatible system of maps $\{A_i \to X\}_{i \in I}$ (a "cocone") there is *some* $A \to X$ making the appropriate diagram commute, but it need not be unique. For example, if a square

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ B & \xrightarrow[h]{} & D \end{array}$$

is a weak pushout, then for any $u\colon B \to X$ and $v\colon C \to X$ with $u \circ f = v \circ g$, there is at least one $w\colon D \to X$ with $w \circ h = u$ and $w \circ k = v$, but there can be more such w.

Note that, exactly as for ordinary colimits, one can construct weak coequalizers and weak colimits of sequences from weak pushouts and coproducts (cf. (1.1) and (1.2)). Also note that *a representable functor necessarily sends weak colimits in* $\mathcal{C}$ *to weak limits in* Sets (or in the category Sets_* of pointed sets, if $\mathcal{C}$ is pointed). Of course, weak limits are defined exactly like weak colimits, by dropping the uniqueness condition in the definition of ordinary limit.

The category $\mathcal{C}$ that we are primarily interested in is the category $\mathsf{Ho}(\mathsf{Top}_*)$ of *pointed spaces* and *homotopy classes of pointed maps.* This category has coproducts, the coproduct of a family of pointed spaces $\{X_i\}_{i\in I}$ being their wedge product $\vee_{i\in I} X_i$, obtained from the disjoint union by identifying all the base points. The wedge product of the empty family also exists, and is the zero object, that is, a single point. However, most other types of colimits do no exist. On the other hand, if

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pushout of pointed spaces (a pushout in the category Top_*) and $A \to X$ is a cofibration, then the *homotopy extension property for cofibrations* (see Lecture 8) at least says that this square is a *weak pushout* in $\mathsf{Ho}(\mathsf{Top}_*)$. Thus, in $\mathsf{Ho}(\mathsf{Top}_*)$, *weak pushouts along cofibrations exist.* From this fact, we can deduce the following:

Proposition 13.1. *Let F be a contravariant functor from $\mathsf{Ho}(\mathsf{Top}_*)$ into the category Sets_* of pointed sets. Suppose that F maps coproducts to products and pushouts along cofibrations in Top_* to weak pullbacks. Then*

(i) *If $A \rightrightarrows B \to C$ is a coequalizer in Top_* and the map $A \vee A \to B$ is a cofibration, then $FC \to FB \rightrightarrows FA$ is a weak coequalizer of pointed sets.*

(ii) *If $X_0 \overset{f_0}{\to} X_1 \overset{f_1}{\to} X_2 \overset{f_2}{\to} \cdots$ is a sequence of cofibrations with colimit X in Top_*, then $F(X)$ is a weak limit of the inverse sequence $F(X_0) \leftarrow F(X_1) \leftarrow F(X_2) \leftarrow \cdots$, that is, the map $F(X) \to \varprojlim F(X_n)$ is a surjection of pointed sets.*

Proof. Part (i) is clear from the description of coequalizers in terms of pushouts and coproducts as in diagram (1.1).

For part (ii), we need to do a bit more work. Recall that we can construct $\varinjlim X_n$ as the coequalizer of the two maps

$$\bigvee X_n \underset{f}{\overset{i}{\rightrightarrows}} \bigvee X_n,$$

where the wedge is the coproduct in the category of pointed spaces, and where i is the identity while f sends the summand X_n to X_{n+1} by the given cofibration f_n. Thus, this colimit is the pushout in the diagram

$$\begin{array}{ccc} (\bigvee X_n) \vee (\bigvee X_n) & \xrightarrow{(i,f)} & \bigvee X_n \\ \nabla \downarrow & & \downarrow q \\ \bigvee X_n & \xrightarrow[q]{} & \varinjlim X_n \end{array} \tag{1.4}$$

as discussed before. We have written q for the canonical map $q\colon \bigvee X_n \to \varinjlim X_n$, with components $q_n\colon X_n \to \varinjlim X_n$. The problem is that (i, f) is not (necessarily) a cofibration. To resolve this, we are going to "thicken" the colimit and construct a "telescope" T, which decomposes the pushout (1.4) above into a composition of two pushouts:

$$\begin{array}{ccccc} (\bigvee X_n) \vee (\bigvee X_n) & \overset{(i',f')}{\rightarrowtail} & \bigvee (X_n \wedge I^+) & \overset{\mathrm{pr}_1}{\longrightarrow} & \bigvee X_n \\ \Big\downarrow{\scriptstyle \nabla} & & \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle q} \\ \bigvee X_n & \underset{v}{\rightarrowtail} & T & \underset{\pi}{\longrightarrow} & \varinjlim X_n. \end{array} \tag{1.5}$$

To see what these maps are, write points of $\bigvee X_n$ as pairs (n, x), where $x \in X_n$, and points of $\bigvee(X \wedge I^+)$ as triples (n, x, t), where $x \in X_n$ and $t \in I$. Then

$$i'(n, x) = (n, x, 1) \quad \text{ and } \quad f'(n, x) = (n + 1, f_n(x), 0).$$

So points of T are equivalence classes of triples (n, x, t), with identifications $(n, x_0, t) \sim (n, x_0, t')$ for the base point x_0 coming from the definition of $X \wedge I^+$, and identifications

$$(n, x, 1) \sim (n + 1, f_n(x), 0)$$

coming from the definition of the pushout. Let us write $[n, x, t] = p(n, x, t)$ for the equivalence class. Then $v\colon \bigvee X_n \to T$ is the map given on a summand X_n by

$$v_n\colon X_n \longrightarrow T, \qquad v_n(x) = [n, x, 1]$$

and $\pi\colon T \to \varinjlim X_n$ is the obvious projection, $\pi[n, x, t] = q_n(x)$.

Now observe that (i', f') is a cofibration. Indeed, it is a wedge (coproduct) of cofibrations $i'_0\colon X_0 \to X_0 \wedge I^+$ (sending $x \mapsto (x, 1)$) and for $n \geq 0$

$$X_n \vee X_{n+1} \overset{f_n \vee \mathrm{id}}{\longrightarrow} X_{n+1} \vee X_{n+1} \longrightarrow X_{n+1} \wedge I^+,$$

where the second map is the standard cofibration mapping the two copies of X_{n+1} to $X_{n+1} \times \{0\}$ and $X_{n+1} \times \{1\}$, respectively.

We are now ready to prove that the map $F(\varinjlim X_n) \to \varprojlim F(X_n)$ is surjective. Choose a sequence $\xi_n \in F(X_n)$ with $(f_n)^*(\xi_{n+1}) = \xi_n$ $(n \geq 0)$. These ξ_n together make up an element ξ in $\prod F(X_n) \cong F(\bigvee X_n)$. Let $\overline{\xi}_n \in F(X \wedge I^+)$ be obtained from ξ_n by applying F to the projection $\mathrm{pr}_1\colon X_n \wedge I^+ \to X_n$, $\overline{\xi}_n = \mathrm{pr}_1^*(\xi_n)$. Then the $\overline{\xi}_n$ together define an element $\overline{\xi} \in F(\bigvee(X_n \wedge I^+))$. The assumption that $(f_n)^*(\xi_{n+1}) = \xi_n$ means precisely that $(i')^*(\overline{\xi}) = f'(\overline{\xi}) = \xi$. So by applying (i) to the pushout square on the left of (1.5) we find a $\zeta \in F(T)$ with $v^*(\zeta) = \xi$ and $p^*(\zeta) = \overline{\xi}$. In particular, $(v_n)^*(\zeta) = \xi_n$.

We now wish to "push down" ζ to an element $\eta \in F(\varinjlim X_n)$. To this end, we construct a map $w\colon \varinjlim_n \to T$. Consider the maps $v_n\colon X_n \to T$, and observe that each triangle

$$\begin{array}{ccc} X_n & \overset{f_n}{\rightarrowtail} & X_{n+1} \\ \Big\downarrow{\scriptstyle v_n} & \swarrow{\scriptstyle v_{n+1}} & \\ T & & \end{array}$$

commutes up to homotopy. Indeed, $v_n(x) = [n, x, 1] = [n + 1, f_n(x), 0]$ and $v_{n+1}(f_n(x)) = [n+1, f_n(x), 1]$, which are connected by the homotopy sending x to $[n+1, f_n(x), t]$ for $0 \leq t \leq 1$. We can now successively apply the homotopy extension property to the cofibrations $f_0, f_1, f_2, \ldots$ and replace the v_i by homotopic maps $w_i \simeq v_i$ so that $w_{n+1} \circ f_n = w_n$. This gives a map

$$w\colon \varinjlim X_n \to T \quad \text{ with } \quad w \circ q_n = w_n \simeq v_n.$$

Let $\eta = w^*(\zeta)$. Then η is the element in $F(\varinjlim X_n)$ we are looking for, because

$$(q_n)^*(\eta) = (q_n)^*(w^*(\zeta)) = (w \circ q_n)^*(\zeta) = (w_n)^*(\zeta) = (v_n)^*(\zeta) = \xi_n,$$

proving that $F(X) \to \varprojlim F(X_n)$ is a surjection. □

Exercise 13.2. Prove the result in the last part of the proof stating that, by applying the homotopy extension property to the cofibrations f_i, we can replace the maps v_i by homotopic maps w_i so that $w_{n+1} \circ f_n = w_n$,

The kinds of colimits mentioned in the proposition have a special property, viz. they are "homotopy invariant". For coproducts this is clear: a family of pointed homotopy equivalences $X_i \xrightarrow{\simeq} Y_i$ (where i ranges over some index set I) induces a pointed homotopy equivalence $\bigvee X_i \xrightarrow{\simeq} \bigvee_i Y_i$. For pushouts along cofibrations, it is a bit more complicated. The next statement can be proved using the properties of cofibrations stated at the end of Lecture 8.

Proposition 13.3. *Let $F\colon \mathsf{Ho}(\mathsf{Top}_*)^{\mathrm{op}} \to \mathsf{Sets}_*$ be a contravariant functor, from the homotopy category of pointed spaces to pointed sets. Suppose that F sends each pushout*

$$\begin{array}{ccc} A & \rightarrowtail & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \cup_A C \end{array}$$

of two cofibrations to a weak pullback in Sets_*. Then F sends each pushout along a cofibration to a weak pullback.

Proof. If we have a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \cup_A X \end{array}$$

of a map $f\colon A \to X$ along a cofibration $A \to B$, we can factor $A \to X$ as a cofibration followed by a homotopy equivalence, and construct the pushout in two steps:

$$\begin{array}{ccccc} A & \rightarrowtail & M_f & \xrightarrow{\simeq} & X \\ \downarrow & & \downarrow & & \downarrow \\ B & \rightarrowtail & B \cup_A M_f & \xrightarrow[\simeq]{} & B \cup_A X. \end{array}$$

Then the lower map on the right is a homotopy equivalence by Corollary 8.15 from Lecture 8. If we apply F to this diagram, we obtain a diagram

$$\begin{array}{ccccc} F(A) & \longleftarrow & F(M_f) & \xleftarrow{\cong} & F(X) \\ \uparrow & & \uparrow & & \uparrow \\ F(B) & \longleftarrow & F(B \cup_A M_f) & \xleftarrow[\cong]{} & F(B \cup_A X) \end{array}$$

in which the square on the left is a weak pullback by hypothesis, while in the one on the right, the horizontal maps are isomorphisms. It follows that the large rectangle is also a weak pullback. □

2. Brown representability theorem

Let us summarize the discussion so far. Suppose that

$$F\colon \mathsf{Ho}(\mathsf{Top}_*)^{\mathrm{op}} \longrightarrow \mathsf{Sets}_*$$

is a functor having the following two properties:

(i) $F(\bigvee_{i\in I} X_i) \to \prod_{i\in I} F(X_i)$ is an isomorphism, for any family of pointed spaces $\{X_i\}_{i\in I}$.

(ii) $F(B\cup_A C) \to F(B)\times_{F(A)} F(C)$ is a surjection for any two cofibrations $A\to B$ and $A\to C$.

Then we also have

(iii) $F(*) = *$

(iv) If we have a pushout of pointed spaces

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

in which i is a cofibration, then $F(Y)\to F(B)\times_{F(A)} F(X)$ is a surjection.

(v) If $A \rightrightarrows B \to C$ is a coequalizer of pointed spaces in which the two maps form a cofibration $A\vee A\to B$, then $F(C)\to F(B)$ maps surjectively to the equalizer of $F(B)\rightrightarrows F(A)$.

(vi) If $X_0\to X_1\to X_2\to\cdots$ is a sequence of cofibrations, then $F(X)\to \varprojlim F(X_n)$ is a surjection.

Our aim is now to show that conditions (i) and (ii) are in fact enough to show that F is representable, at least when we restrict ourselves to connected CW-complexes.

Theorem 13.4 (Brown representability theorem)**.** *Let F be a contravariant functor from the homotopy category of pointed connected CW-complexes to pointed sets. If F satisfies conditions* (i) *and* (ii) *above (for any pointed connected CW-complexes X_i, A, B, C), then F is representable.*

Remark 13.5.
(i) Recall that this means that there is a space $B = B_F$ (itself a pointed CW-complex) for which there is a natural isomorphism

$$\varphi_X\colon [X, B_F]_* \xrightarrow{\cong} F(X),$$

for any pointed connected CW-complex X. This space B_F is called a *classifying space* for F. Recall also that when such a φ exists, it is completely determined by a *generic* element $\gamma\in F(B_F)$.

(ii) Suppose that (B_1,γ_1) and (B_2,γ_2) are two classifying spaces for F, with generic elements γ_1 and γ_2, respectively. Then there exists a homotopy equivalence $f\colon B_1\to B_2$ with $f^*(\gamma_2)=\gamma_1$. In other words, the pair (B,γ) of a classifying space and its generic element is unique up to homotopy. Indeed, writing $\mathcal{C}$ for the category of pointed connected CW-complexes and homotopy classes of maps, there are natural isomorphism

$$\varphi^1_X\colon \mathcal{C}(X,B_1) \xrightarrow{\cong} F(X) \text{ and } \varphi^2_X\colon \mathcal{C}(X,B_2) \xrightarrow{\cong} F(X)$$

defined by $\varphi^i_X(f) = f^*(\gamma_i)$, for $i=1,2$. Then

$$(\varphi^1_{B_2})^{-1}(\gamma_1)\colon B_2 \longrightarrow B_1 \text{ and } (\varphi^2_{B_1})^{-1}(\gamma_2)\colon B_1 \longrightarrow B_2$$

are mutually inverse maps in the category $\mathcal{C}$.

(iii) We can use the Whitehead theorem to make a slightly different statement. Let us say that (B,γ) is a *spherical classifying space* for F is $\gamma\in F(B)$ is an element inducing an isomorphism

$$\gamma_*\colon [S^n, B]_* \longrightarrow F(S^n), \qquad \gamma_*(f) = f^*(\gamma)$$

for any sphere S^n ($n > 0$; S^0 is not connected). If (B_1, γ_1) and (B_2, γ_2) are two such spherical classifying spaces in the category $\mathcal{C}$, and $f\colon B_1 \to B_2$ is a map with $f^*(\gamma_2) = \gamma_1$, then f induces isomorphisms

$$\pi_n(B_1) \longrightarrow \pi_n(B_2)$$

between the homotopy groups for each $n > 0$ (we dropped the basepoints of B_1 and B_2 from the notation). Since B_1 and B_2 are pointed connected CW-complexes, this means that f is a weak homotopy equivalence, hence a homotopy equivalence by the Whitehead theorem.

Let us now turn to the proof of Brown representability theorem. It is based on the following lemmas.

Lemma 13.6. *Let X be a pointed CW-complex and let $\xi \in F(X)$. Then there exists a spherical classifying space (B, γ) for F with a cofibration $f\colon X \to B$ with $f^*(\gamma) = \xi$.*

Lemma 13.7. *Any spherical classifying space (B, γ) for F is a classifying space. (Thus, $\gamma_*\colon [X, B]_* \to F(X)$ is an isomorphism for any pointed connected CW-complex X, nor just for spheres.)*

Indeed, Brown's theorem follows by taking X to be a point in Lemma 13.6, and then applying Lemma 13.7 to the spherical classifying space provided by Lemma 13.6. We will now first show that Lemma 13.7 follows from Lemma 13.6, and then prove Lemma 13.6.

Proof of Lemma 13.7 (using Lemma 13.6). Let X be a pointed connected CW-complex, and let (B, γ) be a spherical classifying space for F.

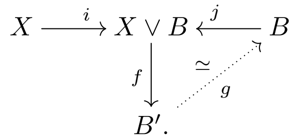
We first prove that $\gamma_*\colon [X, B]_* \to F(X)$ is a surjection. Let $\xi \in F(X)$. Form the wedge

$$X \xrightarrow{\;i\;} X \vee B \xleftarrow{\;j\;} B.$$

Since $F(X \vee B) \cong F(X) \times F(B)$ (by an isomorphism identifying i^* and j^* with the projections), we find an element $(\xi, \gamma) \in F(X \vee B)$ with $i^*(\xi, \gamma) = \xi$ and $j^*(\xi, \gamma) = \gamma$. By Lemma 13.6, there is a spherical classifying space (B', γ') and a cofibration

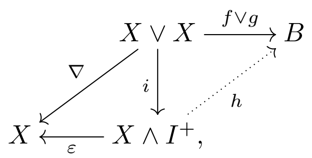
$$f\colon X \vee B \longrightarrow B'$$

with $f^*(\gamma') = (\xi, \gamma)$. Thus $(f \circ i)^*(\gamma') = \xi$ and $(f \circ j)^*(\gamma') = \gamma$. But then $f \circ j\colon B \to B'$ is a homotopy equivalence by Remark 13.5(iii). If $g\colon B' \to B$ is a homotopy inverse for $f \circ j$, then $g \circ f \circ i\colon X \to B$ is a map with $(g \circ f \circ i)^*(\gamma) = \xi$



This proves that $\gamma_*\colon [X, B]_* \to F(X)$ is a surjection.

Next, we prove that $\gamma_*\colon [X, B]_* \to F(X)$ is injective. Suppose that f and g are two maps $X \rightrightarrows B$ with $f^*(\gamma) = g^*(\gamma) \in F(X)$. Consider the diagram



where $X \wedge I^+$ is the reduced cylinder $(X \times I)/(\{x_0\} \times I)$. We wish to find a map $h\colon X \wedge I^+ \to B$ with $h \circ i = f \vee g$ because this would show that $f \simeq g$. To this end, form the pushout

$$\begin{array}{ccc} X \vee X & \xrightarrow{f \vee g} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle u} \\ X \wedge I^+ & \xrightarrow[v]{} & W. \end{array}$$

Now $F(X \vee X) = F(X) \times F(X)$ and $(f \vee g)^*(\gamma) = (f^*(\gamma), g^*(\gamma))$ under this identification. Let $\zeta = \varepsilon^* \circ f^* \circ \gamma = \varepsilon^* \circ g^* \circ \gamma \in F(X \wedge I^+)$. Then $i^*(\zeta) = (f \vee g)^*(\gamma)$, so since F transforms the pushout into a weak pullback, there exists an $\eta \in F(W)$ with $v^*(\eta) = \zeta$ and $u^*(\eta) = \gamma$. By Lemma 13.6, there exists a spherical classifying space (B', γ') and a cofibration $w\colon W \to B'$ with $w^*(\gamma') = \eta$. Then $(w \circ u)^*(\gamma') = \gamma$, so $w\colon B \to B'$ is a homotopy equivalence.

In particular, there is a map $p\colon B' \to B$ with $p^*(\gamma) = \gamma'$ and $p \circ w \circ u \simeq \mathrm{id}$. Then, $p \circ w \circ v \circ i = p \circ w \circ u(f \vee g) \simeq f \vee g$, so by the homotopy extension property applied to the cofibration $X \vee X \to X \wedge I^+$ we find a map $q\colon X \wedge I^+ \to B$ with $q \simeq p \circ w \circ v$ and $q \circ i = f \vee g$. In particular, q is a homotopy between f and g. □

Proof of Lemma 13.6. Let X be a pointed connected CW-complex and $\xi \in F(X)$. We are going to construct a sequence of cofibrations

$$X \subseteq B^1 \subseteq B^2 \subseteq B^3 \subseteq \cdots$$

together with elements $\gamma^n \in F(B^n)$ (for $n > 0$), such that the map

$$(\gamma^n)_*\colon [S^q, B^n]_* \longrightarrow F(S^q)$$

which sends f to $f^*(\gamma^n)$, is a surjection for $q = n$ and a bijection for $0 < q < n$. Moreover, the γ^i will be compatible with each other and with ξ in the obvious sense that the image of $X \to B^n \to B^{n+1}$ under F sends γ^{n+1} to γ^n and then to ξ. These B^n will be constructed in quite a straightforward way, by attaching cells, much as in the proof of the CW-approximation theorem. For $n = 1$, let

$$B^1 = X \vee \bigvee_\zeta S^1_\zeta,$$

where ζ ranges over all elements of $F(S^1)$ and S^1_ζ is a copy of S^1. Then, by property (i) on page 6, $F(B^1) \cong F(X) \times \prod_\zeta F(S^1_\zeta)$, and we let γ^1 be the element with coordinate ξ on $F(X)$ and coordinate ζ on the factor $F(S^1_\zeta)$. Then, for the inclusion $i_\zeta\colon S^1_\zeta \to B^1$ we have $i^*_\zeta(\gamma^1) = \zeta$. In particular, $[S^1, B^1]_* \to F(S^1)$ is surjective.

Suppose that (B^n, γ^n) has been constructed with the desired properties. In particular, $(\gamma^n)_*\colon [S^n, B^n]_* \to F(S^n)$ is a surjection of pointed sets. In fact, it is a surjection of groups, because S^n is an H-cogroup, cf. (1.3). Let K be the kernel of $(\gamma^n)_*$. Let $B^{n\frac{1}{2}}$ be the space obtained form B^n by attaching an $(n+1)$-cell along the attaching map $k\colon S^n \to B^n$, one k for each homotopy class $[k]$ in this kernel K. Thus, we have a pushout:

$$\begin{array}{ccccc} \coprod_k S^n & \twoheadrightarrow & \bigvee_k S^n & \longrightarrow & B^n \\ \downarrow & & \downarrow & & \downarrow \\ \coprod_k e^{n+1} & \twoheadrightarrow & \bigvee_k e^{n+1} & \longrightarrow & B^{n\frac{1}{2}}. \end{array}$$

Since e^{n+1} is contractible, $F(e^{n+1})$ is a point, so the pullback of $F(B^n) \to \prod_k F(S^n)$ along $\prod F(e^{n+1}) \to \prod F(S^n)$ is the kernel of the map $F(B^n) \to \prod_k F(S^n)$, sending γ^n to the element with coordinate $k^*(\gamma^n) = (\gamma^n)_*(k)$ on the factor k. The map $F(B^{n\frac{1}{2}}) \to F(B^n)$ surjects onto this kernel (by property (iv) on page 6), so there is an element $\gamma^{n\frac{1}{2}} \in B^{n\frac{1}{2}}$ with $j^*(\gamma^{n\frac{1}{2}}) = \gamma^n$.

For each $q \leq n$ we now have a diagram

$$\begin{array}{ccc} [S^q, B^{n\frac{1}{2}}]_* & \xrightarrow{(\gamma^{n\frac{1}{2}})_*} & F(S^q) \\ {\scriptstyle j_*}\uparrow & \nearrow{\scriptstyle (\gamma^n)_*} & \\ [S^q, B^n]_* & & \end{array}$$

By cellular approximation, j_* is an isomorphism for $q < n$, and hence $(\gamma^{n\frac{1}{2}})_*$ is because $(\gamma^n)_*$ is by induction hypothesis. Moreover, $(\gamma^{n\frac{1}{2}})_*$ is surjective for $q = n$ because $(\gamma^n)_*$ is. It is also a surjection for $q = n$, because if $k\colon S^n \to B^{n\frac{1}{2}}$ is (or represents) a homotopy class with $(\gamma^{n\frac{1}{2}})_*(k) = 0$, then by cellular approximation k is homotopic to $j \circ k'$ for a map $k'\colon S^n \to B^n$, and $(\gamma^n)_*(k') = 0$ so k' (or more precisely its homotopy class) lies in K. Then $j_*(k') = k = 0$ in $[S^n, B^{n\frac{1}{2}}]_*$ by construction of $B^{n\frac{1}{2}}$.

Finally, we construct B^{n+1} from $B^{n\frac{1}{2}}$ much as we constructed B^1 from X, as

$$B^{n+1} = B^{n\frac{1}{2}} \vee \bigvee_{\zeta} S^{n+1}_{\zeta},$$

where ζ ranges over all elements of $F(S^{n+1})$ and each S^{n+1}_{ζ} is a copy of S^{n+1}. Then

$$F(B^{n+1}) \cong F(B^{n\frac{1}{2}}) \times \prod_{\zeta} F(S^{n+1}_{\zeta})$$

has a canonical element γ^{n+1} with coordinates $\gamma^{n\frac{1}{2}}$ and ζ. Moreover, we have that the map $(\gamma^{n+1})_*\colon [S^q, B^{n+1}]_* \to F(S^q)$ is an isomorphism for $q \leq n$ (as before, for $B^{n\frac{1}{2}}$) and a surjection for $q = n+1$ (by construction).

To conclude the proof, let $B = \varinjlim B^n$ and use property (vi) on page 6 to find an element $\gamma \in F(B)$ such that for every n, the element γ is mapped to γ^n by $F(B^n \to B)$. Then (B, γ) is a spherical classifying space.

This completes the proof of the Brown representability theorem. $\square$