Generalized Ohkawa's theorem

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Introduction

The *homological Bousfield* class $\langle E\rangle$ of a spectrum E consists of all E_*-acyclic spectra, where E_* is the reduced homology theory associated with E

$$\langle E\rangle = \{X \mid E \wedge X \simeq *\}.$$

- **Ohkawa's theorem:** There is a set of Bousfield classes [Ohkawa, 1989].
- Simpler proof and relationship with the Bousfield lattice [Dwyer–Palmieri, 2001].
- Bousfield classes in the derived category of modules over some non-noetherian rings [Dwyer–Palmieri, 2008].
- Generalization to well-generated tensor triangulated categories [Iyengar–Krause, 2011].

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Theorem (Iyengar–Krause, 2011)

Let $\mathcal{T}$ be a α-well generated triangulated category and consider the collection $\mathcal{H}$ of functors $H : \mathcal{T} \to \mathcal{A}$ such that

(i) *$\mathcal{A}$ is abelian and has coproducts and exact α-filtered colimits.*

(ii) *H is cohomological and preserves coproducts.*

Then the localizing subcategories of the form $\ker H$ for some $H \in \mathcal{H}$ form a set of cardinality at most $2^{2^{|\mathcal{T}^\alpha|}}$.

Corollary

For any α-well generated tensor triangulated category $\mathcal{T}$, the collection of Bousfield classes forms a set of cardinality at most $2^{2^{|\mathcal{T}^\alpha|}}$.

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Accessible categories

Let $\mathcal{C}$ be any category.

- An object X of $\mathcal{C}$ is called *λ-presentable* if the functor $\mathcal{C}(X, -)$ preserves λ-filtered colimits.
- A category $\mathcal{C}$ is *λ-accessible* if all λ-filtered colimits exist in $\mathcal{C}$ and there is a set $\mathcal{S}$ of λ-presentable objects such that every object of $\mathcal{C}$ is a λ-filtered colimit of objects from $\mathcal{S}$. It is called *accessible* if it is λ-accessible for some λ.
- A cocomplete accessible category is called *locally presentable*.
- If $\mathcal{C}$ and $\mathcal{C}'$ are λ-accessible categories, then a functor $H\colon \mathcal{C} \to \mathcal{C}'$ is called *λ-accessible* if it preserves λ-filtered colimits.

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Combinatorial model categories

A model category $\mathcal{M}$ is *λ-combinatorial* if it is locally λ-presentable and cofibrantly λ-generated. $\mathcal{M}$ is *combinatorial* if it is λ-combinatorial for some λ.

For a model category $\mathcal{M}$, the composition

$$\mathcal{M} \xrightarrow{R} \mathcal{M}_{cf} \xrightarrow{Q} Ho(\mathcal{M}),$$

is the canonical functor to its homotopy category, where Q is the quotient functor.

Definition

A functor $H\colon \mathcal{M} \to \mathcal{M}'$ between model categories is called a *homotopy functor* if it sends weak equivalences between fibrant and cofibrant objects to weak equivalences.

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Generalized Bousfield classes

Every homotopy functor $H\colon \mathcal{M} \to \mathcal{M}'$ restricts to a functor

$$\overline{H}\colon Ho(\mathcal{M}) \longrightarrow Ho(\mathcal{M}').$$

Let H be a homotopy endofunctor on $\mathcal{M}$. An object X in $Ho(\mathcal{M})$ is called *H-acyclic* if $\overline{H}X$ is isomorphic to the terminal object $Ho(\mathcal{M})$.

We denote by $\mathcal{A}(H)$ the full subcategory of $Ho(\mathcal{M})$ consisting of all H-acyclic objects.

Definition

Let $\mathcal{M}$ be a model category. A *generalized Bousfield class* is a full subcategory of $Ho(\mathcal{M})$ of the form $\mathcal{A}(H)$ for some homotopy endofunctor H on $\mathcal{M}$. If H is a λ-accessible homotopy functor, then $\mathcal{A}(H)$ will be called a *generalized λ-Bousfield class*.

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Main result

Theorem (CGR)

Let $\mathcal{M}$ be a combinatorial pointed model category and λ a regular cardinal. Then there is only a set of generalized λ-Bousfield classes in $Ho(\mathcal{M})$.

Proof

$Ho(\mathcal{M})$ has a set $\mathcal{G}$ of weak generators. By the Uniformization Theorem we can choose a regular cardinal $\mu \geq \lambda$ such that

(i) $\mathcal{M}$ is μ-combinatorial.

(ii) Each $G \in \mathcal{G}$ is μ-presentable.

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Proof (cont.)

Since every λ-accessible functor $H\colon \mathcal{M} \to \mathcal{M}$ is μ-accessible, it suffices to prove that there is only a set of generalized μ-Bousfield classes in $Ho(\mathcal{M})$.

Given a μ-accessible homotopy functor $H\colon \mathcal{M} \to \mathcal{M}$, let

$$\mathcal{J}(H) = \{f\colon A \to B \text{ in } \mathcal{M}_\mu \cap \mathcal{M}_{cf} \text{ such that } \overline{H}Q(f) = 0\}.$$

Then, one shows that $\mathcal{A}(H_1) = \mathcal{A}(H_2)$ whenever $\mathcal{J}(H_1) = \mathcal{J}(H_2)$. Since $\mathcal{M}_\mu$ is small, this finishes the proof. □

Remark

The argument in the proof can be adapted to *semipointed* model categories, i.e., if for every X the morphism $X \to *$ is an epimorphism.

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A set of left Quillen functors

A functor $F : \mathcal{M} \to \mathcal{M}'$ is *left Quillen* if it is a left adjoint and preserves cofibrations and trivial cofibrations.

Every left Quillen functor preserves weak equivalences between cofibrant objects, hence they are homotopy functors and they are λ-accessible for any λ (since they preserve all colimits).

Let $\mathcal{M}$ be a combinatorial (semi)pointed model category and let $\mathcal{F}$ be the class of all left Quillen functors $F : \mathcal{M} \to \mathcal{M}$. Given two functors F_1 and F_2 in $\mathcal{F}$, we say that $F_1 \sim F_2$ if $\mathcal{A}(F_1) = \mathcal{A}(F_2)$.

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Corollary 1

Let $\mathcal{M}$ be a (semi)pointed combinatorial model category. Then there is a set of equivalence classes in $\mathcal{F}/\sim$.

If $\mathcal{M}$ is a monoidal model category, then the homological Bousfield class of an object E in $\mathcal{M}$ is the class

$$\langle E\rangle = \{X \in \mathcal{M} \mid E \otimes X \simeq *\}.$$

Corollary 2

Let $\mathcal{M}$ be a (semi)pointed combinatorial monoidal model category. Then there is a set of homological Bousfield classes.

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