ALGEBRAIC TOPOLOGY

MASTERMATH (FALL 2014)

Written exam, 21/01/2015, 3 hours

Outline of solutions

Exercise 1.

(i) There are various definitions in the literature. Based on the discussion on p. 5 of Lecture 3, as well as the fact that all spaces and maps should be taken *pointed*, the correct answer should at least be an example of the following definition:

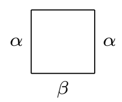
Definition. An H-space is a space X equipped with a multiplication map $\mu\colon X \times X \to X$ and a unit element $e\colon * \to X$ such that the maps

$$\mu(e,-)\colon X \longrightarrow X \qquad \mu(-,e)\colon X \longrightarrow X$$

are homotopic to the identity maps on X, via a homotopy that fixes the unit e (so that $\mu(e,e) = e$).

There are variants asking for associativity up to homotopy or the existence of strict units. All these variants are counted as a correct answer. Note that exercise (iii) gives an extra hint for taking (X, e) as a pointed space, so that all homotopies should fix the unit e.

(ii) For $[\alpha] \in \pi_1(X)$ and $[\beta] \in \pi_n(X)$, take representatives $\alpha\colon I \to X$ and $\beta\colon I^n \to X$ sending ∂I and ∂I^n to the basepoint. Consider the map $I^n \times \{0\} \cup \partial I^n \times I \to X$ given by β on $I^n \times \{0\}$ and by $\alpha \circ \pi_2$ on $\partial I^n \times I$. It has an extension to a map $H\colon I^n \times I \to X$, whose restriction to $I^n \times \{1\}$ presents the element $[\alpha] \cdot [\beta]$. For $n = 1$, the map H can be pictured as



This clearly shows that the remaining face represents $[\alpha][\beta][\alpha]^{-1}$. See p. 6-7 of Lecture 4.

(iii) $\pi_1(X)$ is abelian by the Eckman-Hilton trick (p. 3 of Lecture 3). It also follows from the argument for generic n, by taking $n = 1$.

If the unit is a strict unit, one can take the extension H from part (ii) to be given by

$$H(x,t) = \mu(\beta(x), \alpha(t))$$

Since $\alpha(1) = e$, one sees that $H(x, 1) = \beta(x)$, showing that the action is trivial. If the e is only the unit up to *pointed* homotopy, observe that $H(x, 1) = \mu(\beta(x), e)$ is homotopic to $\beta(x)$ via the homotopy witnessing unitality. But the restriction of $H(x, 1)$ to ∂I^n is constant with value e and the unitality homotopy fixes e. It follows that $H(-, 1) \simeq \beta$ via a homotopy that fixes the boundary ∂I^n.

Exercise 2.

(i) False, $\pi_3(S^2) \neq 0$ (see p. 4 of Lecture 3 or Exercise 6 from sheet 6).

(ii) True, by cellular approximation (see p. 3 of Lecture 9).

(iii) True, to check the defining right lifting property one produces a lifting in two steps.

(iv) True, to check the defining right lifting property one produces a lifting in two steps.

(v) False in general, f should land in $X^{(n-1)}$.

(vi) True, f necessarily lands in $X^{(n-1)}$.

(vii) False. As was mentioned in the lectures, there are many maps $K(A,n) \to K(B,m)$ which are not nullhomotopic, even when $n \neq m$. Such maps always induce the zero map on homotopy groups when $n \neq m$. However, this remark does not appear in the lecture notes and therefore *exercise 2(vii) has not been marked.*

A concrete counterexample to (vii) is given by the quotient map

$$S^1 \times S^1 \to S^1 \times S^1/S^1 \vee S^1 \simeq S^2.$$

However, to prove that this map is indeed not nullhomotopic requires material beyond the scope of this course (the map induces an isomorphism between the second homology groups).

Exercise 3.

(i) In general, if

$$\begin{array}{ccccc} q^{-1}(c_0) & \dashrightarrow & D & \longrightarrow & B \\ \downarrow & & \downarrow q & & \downarrow p \\ * & \xrightarrow{c_0} & C & \xrightarrow{\phi} & A \end{array}$$

realizes q as the pullback of a Serre fibration p, then q is a Serre fibration as well. Furthermore, the fiber $q^{-1}(c_0)$ can be realized by the dotted pullback square. By the 'pasting lemma' for pullbacks, the total square is a pullback as well, which show that $q^{-1}(c_0)$ is homeomorphic to $p^{-1}(\phi(c_0))$.

Since we know that the map $(\epsilon_0, \epsilon_1)\colon X^I \to X \times X$ is a Serre fibration (see p. 2 of Lecture 5), this gives (i). Alternatively, one can construct an explicit homeomorphism between the fiber of $Y \times^h_X Z \to Y \times Z$ and $\Omega(X, x_0)$.

(ii) The long exact sequence of the fibration $Y \times^h_X Z \to Y \times Z$ gives an exact sequence

$$\cdots \longrightarrow \pi_n(\Omega(X,x_0)) \longrightarrow \pi_n(Y \times^h_X Z) \longrightarrow \pi_n(Y \times Z) \longrightarrow \pi_{n-1}(\Omega(X,x_0)) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_0(\Omega(X,x_0)) \longrightarrow \pi_0(Y \times^h_X Z) \longrightarrow \pi_0(Y \times Z)$$

Using that $\pi_n(X, x_0) \simeq \pi_{n-1}(\Omega(X, x_0))$ for $n \geq 1$ and that $\pi_n(Y \times Z) \simeq \pi_n(Y) \times \pi_n(Z)$, one obtains a long exact sequence of the desired form.

The map $\pi_1(Y) \times \pi_1(Z) \to \pi_1(X)$ corresponds to the connecting homomorphism of the long exact sequence for $Y \times^h_X Z \to Y \times Z$. It is constructed as follows: let $\alpha\colon I \to Y$ and $\beta\colon I \to Z$ present an element in $\pi_1(Y) \times \pi_1(Z)$. Pick a lift

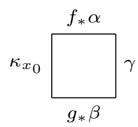
$$\begin{array}{ccccc} \{0\} & \xrightarrow{(\kappa_{x_0}, y_0, z_0)} & Y \times^h_X Z & \longrightarrow & X^I \\ \downarrow & \overset{h}{\nearrow} & \downarrow & & \downarrow (\epsilon_0, \epsilon_1) \\ I & \xrightarrow[(\alpha, \beta)]{} & Y \times Z & \xrightarrow[f \times g]{} & X \times X. \end{array}$$

Then $h(1)$ is a point in the fiber over (y_0, z_0) – which was $\Omega(X, x_0)$ – whose homotopy class provides the desired element in $\pi_0(\Omega(X, x_0))$.

Now the composite $I \to Y \times^h_X Z \to X^I$ determines a map $H\colon I \times I \to X$ with the property that

$$H(-,0) = f \circ \alpha \qquad H(-,1) = g \circ \beta \qquad H(0,-) = \kappa_{x_0}$$

In other words, H determines a square of the form



for some $\gamma\colon I \to X$ with $\gamma(0) = \gamma(1) = x_0$. It follows that $[\gamma] \cdot f_*[\alpha] = g_*[\beta] \cdot [\kappa_{x_0}]$ in $\pi_1(X)$, so that $[\gamma] = g_*[\beta] \cdot f_*[\alpha]^{-1}$.

By construction, the $h(1)$ is given by (γ, y_0, z_0). It follows that the connecting homomorphism sends (α, β) to $[\gamma] = g_*[\beta] \cdot f_*[\alpha]^{-1}$ in $\pi_0(\Omega(X, x_0)) \simeq \pi_1(X)$.

Remark: the map $\pi_n(Y) \times \pi_n(Z) \to \pi_n(X)$ sends (α, β) to $g_*(\beta) - f_*(\alpha)$ for all $n \geq 2$. This either follows from a similar argument as the one for $n = 1$, or one can reduce to the case $n = 1$ as follows: applying Ω^n to pullback square (**??**), we find a pullback square

$$\begin{array}{ccc} \Omega^n(Y \times^h_X Z) & \longrightarrow & (\Omega^n X)^I \\ \downarrow & & \downarrow \\ \Omega^n Y \times \Omega^n Z & \longrightarrow & \Omega^n X \times \Omega^n X \end{array}$$

in which the vertical maps are the n-fold loopings of the original Serre fibrations.

If $p\colon E \to X$ is a Serre fibration with fiber F, then $\Omega^n p\colon \Omega^n E \to \Omega^n X$ is a fiber sequence with fiber $\Omega^n F$. Under the isomorphism $\pi_0(\Omega^n X) \simeq \pi_n(X)$, the long exact sequence of $\Omega^n p$ corresponds to the part of the long exact sequence of p that sits in dimensions $\geq n$. The exercise then shows that the map

$$\pi_{n+1}(Y) \times \pi_{n+1}(Z) \simeq \pi_1(\Omega^n Y \times \Omega^n Z) \longrightarrow \pi_1(\Omega^n X) \simeq \pi_{n+1}(X)$$

sends (α, β) to $g_*\beta \cdot f_*(\alpha)^{-1}$.

Exercise 4.

(i) See Lecture 11. A correct answer should include: iteratively attaching cells along maps $\partial e^n \to X$ presenting nontrivial elements in $\pi_{n-1}(X)$ and replacing the resulting sequence of relative CW-complexes by a homotopy equivalent sequence of fibrations.

(ii) See Lecture 11. The fibration ψ_{n-1} induces isomorphisms of homotopy groups in dimensions $\neq n$: in dimensions $> n$ the homotopy groups of $P_n(X)$ and $P_{n-1}(X)$ are both trivial and in dimensions $k < n$ there is a commuting diagram

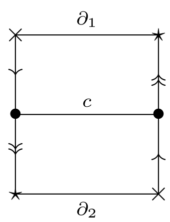
$$\begin{array}{ccc} & & \pi_k\big(P_n(X)\big) \\ & \overset{\simeq}{\nearrow} & \downarrow \psi_{n-1} \\ \pi_k(X) & \xrightarrow[f_{n-1}]{\simeq} & \pi_k\big(P_{n-1}(X)\big) \end{array}$$

so that ψ_{n-1} induces isomorphisms of homotopy groups in dimensions $k < n$.

Furthermore, the map ψ_{n-1} induces the zero map on the n-th homotopy group. Inspection of the long exact sequence of the fibration ψ_{n-1} now shows that the fiber of ψ_{n-1} is a $K(\pi, n)$, where $\pi = \pi_n(X)$.

Exercise 5.

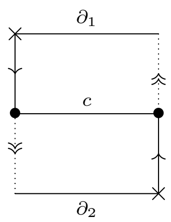
(i) One possible CW-decomposition is given by



This has three 0-cells $\times, \bullet, \star$, five 1-cells and two 2-cells. The inclusion of the central cirlce is the inclusion of a subcomplex (we take $\bullet$ to be the unique 0-cell of the central circle).

If we give the boundary circle the CW-decomposition with a unique 0-cell $\times$ and one 1-cell, then the inclusion of the boundary becomes cellular (since it sends the 0- , resp. 1-skeleton of the circle to the 0- , resp. 1-skeleton of M).

For exercise (iii), it is useful to pick a different CW-structure for the Möbius strip, for which both the central circle and the boundary circle are inclusions of subcomplexes:



This decomposition has two 0-cells, three 1-cells (the central circle, the boundary circle $\partial_2 \circ \partial_1$ and the segment between $\times$ and $\bullet$) and one 2-cell.

(ii) The most obvious retraction is given by

$$p\colon M \longrightarrow S^1; \qquad p[s,t] = [s,0]$$

It is immediate that $p \circ c$ is the identity on $S^1 \simeq I/\partial I$. Furthermore, $c \circ p$ is homotopic to the identity on M via

$$H\colon M \times I \longrightarrow M; \qquad H\big([s,t],\tau\big) = [s, t\cdot\tau].$$

The main point is that the composite $S^1 \xrightarrow{\partial} M \xrightarrow{p} S^1$ wraps the boundary circle twice around the central circle, so it induces multiplication by two on $\pi_1(S^1) = \mathbb{Z}$. The map p is a homotopy equivalence by construction, while the map ∂ is a cofibration since it is the inclusion of a subcomplex.

(iii) Using the second CW-structure from (i), we have that c and ∂ are both inclusions of subcomplexes. In general, if $A \to B$ and $A \to C$ are inclusions of subcomplexes (giving the same CW-structure on A!), then the pushout $B \to B \coprod_A C$ is the inclusion of a subcomplex. In particular, $B \coprod_A C$ is itself a CW-complex.

Indeed, define $\big(B \coprod_A C\big)^{(n)}$ as the pushout

$$\begin{array}{ccc} A^{(n)} & \longrightarrow & C^{(n)} \\ \downarrow & & \downarrow \\ B^{(n)} & \longrightarrow & \big(B \coprod_A C\big)^{(n)}. \end{array}$$

Then $\big(B \coprod_A C\big)^{(n+1)}$ is obtained from $\big(B \coprod_A C\big)^{(n)}$ by adding the $(n+1)$-cells of B and the $(n+1)$-cells of C (identifying the $(n+1)$-cells of A).

Applying this inductively to the sequence of $M_{(n)}$ shows that $M_{(n)}$ is a CW-complex and that both $M_{(n-1)} \to M_{(n)}$ and the inclusion of the central circle $c_{(n)}\colon S^1 \to M_{(n)}$ are inclusions of subcomplexes; the latter allows one to proceed inductively.

Phrased differently, one obtains a CW-structure on $M_{(n)}$ by
(1) taking the CW-structure on $M_{(n-1)}$
(2) adding a 0-cell (the 0-cell of the new central circle)
(3) adding two 1-cells: add the new central circle to the added point, and add a 1-cell between the newly added point and the 0-cell in the old central circle of $M_{(n-1)}$.

(4) finally, adding a 2-cell according to the CW-structure of the Möbius strip.

The result is a CW-complex since we only attach n-cells to the $(n-1)$-skeleton. Furthermore, it is immediate that $M_{(n-1)}$ is a CW subcomplex.

For the case $n = \infty$, the CW-structure is given by taking the images of the 0-, 1- and 2-cells of each $M_{(n)}$.

(iv) Any finite subcomplex of $M_{(\infty)}$ is contained in some $M_{(n)}$ (this holds in general for the colimit of a sequence of subcomplex inclusions). This implies that any map $S^k \to M_{(\infty)}$ takes values in some $M_{(n)}$.

But it is given that $\pi_k(M_{(n)})$ is zero when $k \neq 1$, so any map $S^k \to M_{(\infty)}$ is (pointed) homotopic to the constant map when $k \neq 1$. This shows that $M_{(\infty)}$ is a $K(G, 1)$. In particular, it is path-connected, so we just have to look at the fundamental group at some basepoint that lies in $M_{(1)} \subseteq M_{(\infty)}$.

For any sequence of subcomplex inclusions $X_0 \subseteq X_1 \subseteq ...$, one has that

$$\pi_1(\operatorname{colim} X_n) = \operatorname{colim}_n \pi_1(X_n)$$

It this case, we know that each $\pi_1(M_{(n)}) \simeq \mathbb{Z}$ and that each map $\pi_1(M_{(n)}) \to \pi_1(M_{(n+1)}$ is given by 'multiplication by 2'. We thus have that $\pi_1(M_{(\infty)})$ is the colimit of the sequence

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \cdots$$

The colimit of this sequence is $\mathbb{Z}[\frac{1}{2}]$, i.e. the (additive) group of fractions $\frac{p}{q}$ where q is a power of 2.

Alternatively, one can explicitly construct an isomorphism between $\mathbb{Z}[\frac{1}{2}]$ and $\pi_1(M_{(\infty)})$ as follows: send the element $1 \in \mathbb{Z}[\frac{1}{2}]$ to the image of the generating element $1 \in \pi_1(M_{(1)})$ in $\pi_1(M_{(\infty)})$. Similarly, send $\frac{1}{2^n}$ to the image in $\pi_1(M_{(\infty)})$ of the generating element of $\pi_1(M_{(n+1)})$. This gives a well-defined group homomorphism because the generator of $\pi_1(M_{(n)})$ is exactly identified with twice the generator of $\pi_1(M_{(n+1)})$.

The resulting group homomorphism is surjective: this follows from the fact that any map $S^1 \to M_{(\infty)}$ takes values in some $M_{(n+1)}$ and therefore is given by $\frac{p}{2^n}$ for some p and n.

The resulting group homomorphism is injective: if $\frac{p}{2^n}$ and $\frac{q}{2^m}$ are sent to the same element in $\pi_1(M_{(\infty)})$, then the representing loops inside $M_{(n)}$, resp. $M_{(m)}$, become homotopic in some $M_{(N)}$ for N large enough. But this means precisely that

$$\frac{p}{2^n} = \frac{2^{N-n} \cdot p}{2^N} = \frac{2^{N-m} \cdot q}{2^N} = \frac{q}{2^m}.$$