Coreflective semilocalizing subcategories in triangulated categories with models

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Reflection and coreflections

- A full subcategory $\mathcal{L}$ of a category $\mathcal{T}$ is ***reflective*** if the inclusion $\mathcal{L} \hookrightarrow \mathcal{T}$ has a left adjoint $\mathcal{T} \to \mathcal{L}$.
- The composite $L\colon \mathcal{T} \to \mathcal{T}$ is called a ***reflection*** or a ***localization*** onto $\mathcal{L}$.
- There is a natural transformation $l\colon \mathrm{Id} \to L$ such that $Ll\colon L \to LL$ is an isomorphism, lL is equal to Ll, and, for each X, the morphism $l_X\colon X \to LX$ is initial in $\mathcal{T}$ among morphisms from X to objects in $\mathcal{L}$.
- Similarly, a full subcategory $\mathcal{C}$ of $\mathcal{T}$ is ***coreflective*** if the inclusion $\mathcal{C} \hookrightarrow \mathcal{T}$ has a right adjoint.
- The composite $C\colon \mathcal{T} \to \mathcal{T}$ is called a ***coreflection*** or a ***colocalization*** onto $\mathcal{C}$.
- There is a natural transformation $c\colon C \to \mathrm{Id}$ such that Cc is an isomorphism, cC is equal to Cc, and for each X, the morphism $c_X\colon CX \to X$ is terminal among morphisms from objects in $\mathcal{C}$.

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Weak reflections

- A full subcategory $\mathcal{L}$ of a category $\mathcal{T}$ is called ***weakly reflective*** if for every object X of $\mathcal{T}$ there is a morphism $l_X\colon X \to X^*$ with X^* in $\mathcal{L}$ and such that the function

$$\mathcal{T}(l_X, Y)\colon \mathcal{T}(X^*, Y) \longrightarrow \mathcal{T}(X, Y)$$

  is surjective for all objects Y of $\mathcal{L}$.

- Every morphism from X to an object of $\mathcal{L}$ factors through l_X, not necessarily in a unique way.

- If such a factorization is unique for all objects X, then the morphisms $l_X\colon X \to X^*$ for all X define together a reflection, so $\mathcal{L}$ is then reflective.

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Closure properties for triangulated categories

Let $\mathcal{T}$ be a triangulated category with products and coproducts. Let Σ be the shift or suspension operator. Triangles in $\mathcal{T}$ will be denoted by

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X,$$

Let $\mathcal{S}$ be a full subcategory of $\mathcal{T}$. We define the following closure properties in $\mathcal{S}$:

(i) ***Fibres***: If Y and Z are in $\mathcal{S}$ then X is in $\mathcal{S}$.

(ii) ***Cofibres***: If X and Y are in $\mathcal{S}$ then Z is in $\mathcal{S}$.

(iii) ***Extensions***: If X and Z are in $\mathcal{S}$ then Y is in $\mathcal{S}$.

Definition

A full subcategory of $\mathcal{T}$ is called ***semilocalizing*** if it is closed under coproducts, (ii) and (iii). It is called ***localizing*** if it is closed under coproducts, (i), (ii) and (iii). Dually, a full subcategory is called ***semicolocalizing*** if it is closed under products, (i) and (iii). It is called ***colocalizing*** if it is closed under products, (i), (ii) and (iii).

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Closure properties for triangulated categories

Let $\mathcal{T}$ be a *tensor triangulated category*, i.e., $\mathcal{T}$ has a closed symmetric monoidal structure with a unit object S, tensor product $\wedge$ and internal hom $F(-,-)$, compatible with the triangulated structure and such that $\mathcal{T}(X, F(Y,Z)) \cong \mathcal{T}(X \wedge Y, Z)$ naturally in all variables. In this case we have in addition the following closure properties in $\mathcal{S}$:

(iv) ***Tensor***: If X is in $\mathcal{S}$, then so is $X \wedge W$ for all W in $\mathcal{T}$.

(v) ***Internal hom***: If X is in $\mathcal{S}$, then so is $F(W,X)$ for all W in $\mathcal{T}$.

Definition

A full subcategory of $\mathcal{T}$ is called a ***localizing ideal*** if it is a localizing subcategory and closed under (iv). It is called a ***colocalizing coideal*** if it is a colocalizing subcategory and it is closed under (v).

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Closure properties for triangulated categories

- A reflection L on $\mathcal{T}$ will be called ***semiexact*** if the subcategory of L-local objects is semicolocalizing, and ***exact*** if it is colocalizing.
- Dually, a coreflection C will be called ***semiexact*** if the subcategory of C-colocal objects is semilocalizing and ***exact*** if it is localizing.
- A (co)reflection is exact if and only if it preserves all triangles (or, equivalently, commutes with the suspension operator)

Examples

- Bousfield–Farjoun ***localizations*** are reflections.
- ***Cellular approximations*** are coreflections.
- ***Nullifications*** are semiexact reflections.
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Orthogonality and semiorthogonality

For a class of objects $\mathcal{D}$ in $\mathcal{T}$ we introduce the following notation:

$$^{\lrcorner}\mathcal{D} = \{X \mid \mathcal{T}(X, \Sigma^k D) = 0 \text{ for all } D \in \mathcal{D} \text{ and } k \leq 0\}$$

$$\mathcal{D}^{\llcorner} = \{Y \mid \mathcal{T}(\Sigma^k D, Y) = 0 \text{ for all } D \in \mathcal{D} \text{ and } k \geq 0\}$$

$$^{\perp}\mathcal{D} = \{X \mid \mathcal{T}(X, \Sigma^k D) = 0 \text{ for all } D \in \mathcal{D} \text{ and } k \in \mathbb{Z}\}$$

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Theorem

In every triangulated category $\mathcal{T}$ there is a bijective correspondence between semiexact reflections and semiexact coreflections such that, if a reflection L is paired with a coreflection C then:

(i) *For every X, the morphisms $l_X \colon X \to LX$ and $c_X \colon CX \to X$ fit into a triangle*

$$CX \longrightarrow X \longrightarrow LX \longrightarrow \Sigma CX.$$

(ii) *The class $\mathcal{L}$ of L-local objects coincides with the class of C-acyclics, and the class $\mathcal{C}$ of C-colocal objects coincides with the class of L-acyclics.*

(iii) *The class $\mathcal{C}$ is equal to ${}^{\lrcorner}\mathcal{L}$, and $\mathcal{L}$ is equal to $\mathcal{C}^{\llcorner}$.*

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(ii) *The class $\mathcal{L}$ of L-local objects coincides with the class of C-acyclics, and the class $\mathcal{C}$ of C-colocal objects coincides with the class of L-acyclics.*

(iii) *The class $\mathcal{C}$ is equal to ${}^{\lrcorner}\mathcal{L}$, and $\mathcal{L}$ is equal to $\mathcal{C}^{\llcorner}$.*

(iv) *L is exact if and only if C is exact. In this case, $\mathcal{C} = {}^{\perp}\mathcal{L}$ and $\mathcal{L} = \mathcal{C}^{\perp}$.*

Orthogonality and semiorthogonality

Theorem

In every triangulated category $\mathcal{T}$ there is a bijective correspondence between semiexact reflections and semiexact coreflections such that, if a reflection L is paired with a coreflection C then:

(i) *For every X, the morphisms $l_X \colon X \to LX$ and $c_X \colon CX \to X$ fit into a triangle*
$$CX \longrightarrow X \longrightarrow LX \longrightarrow \Sigma CX.$$

(ii) *The class $\mathcal{L}$ of L-local objects coincides with the class of C-acyclics, and the class $\mathcal{C}$ of C-colocal objects coincides with the class of L-acyclics.*

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Orthogonality and semiorthogonality

Proposition

Let $\mathcal{D}$ be any class of objects in a triangulated category with products and coproducts.

(i) *If* $\mathrm{scoloc}(\mathcal{D})$ *is reflective, then* $\mathrm{scoloc}(\mathcal{D}) = ({}^{\lrcorner}\mathcal{D})^{\llcorner}$, *and if* $\mathrm{sloc}(\mathcal{D})$ *is coreflective, then* $\mathrm{sloc}(\mathcal{D}) = {}^{\lrcorner}(\mathcal{D}^{\llcorner})$.

(ii) *If* $\mathrm{coloc}(\mathcal{D})$ *is reflective, then* $\mathrm{coloc}(\mathcal{D}) = ({}^{\perp}\mathcal{D})^{\perp}$, *and if* $\mathrm{loc}(\mathcal{D})$ *is coreflective, then* $\mathrm{loc}(\mathcal{D}) = {}^{\perp}(\mathcal{D}^{\perp})$.

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Torsion theories
in abelian categories
(e.g., abelian groups)

Torsion
classes

Torsion-free
classes

Torsion theories in abelian categories
(e.g., abelian groups)

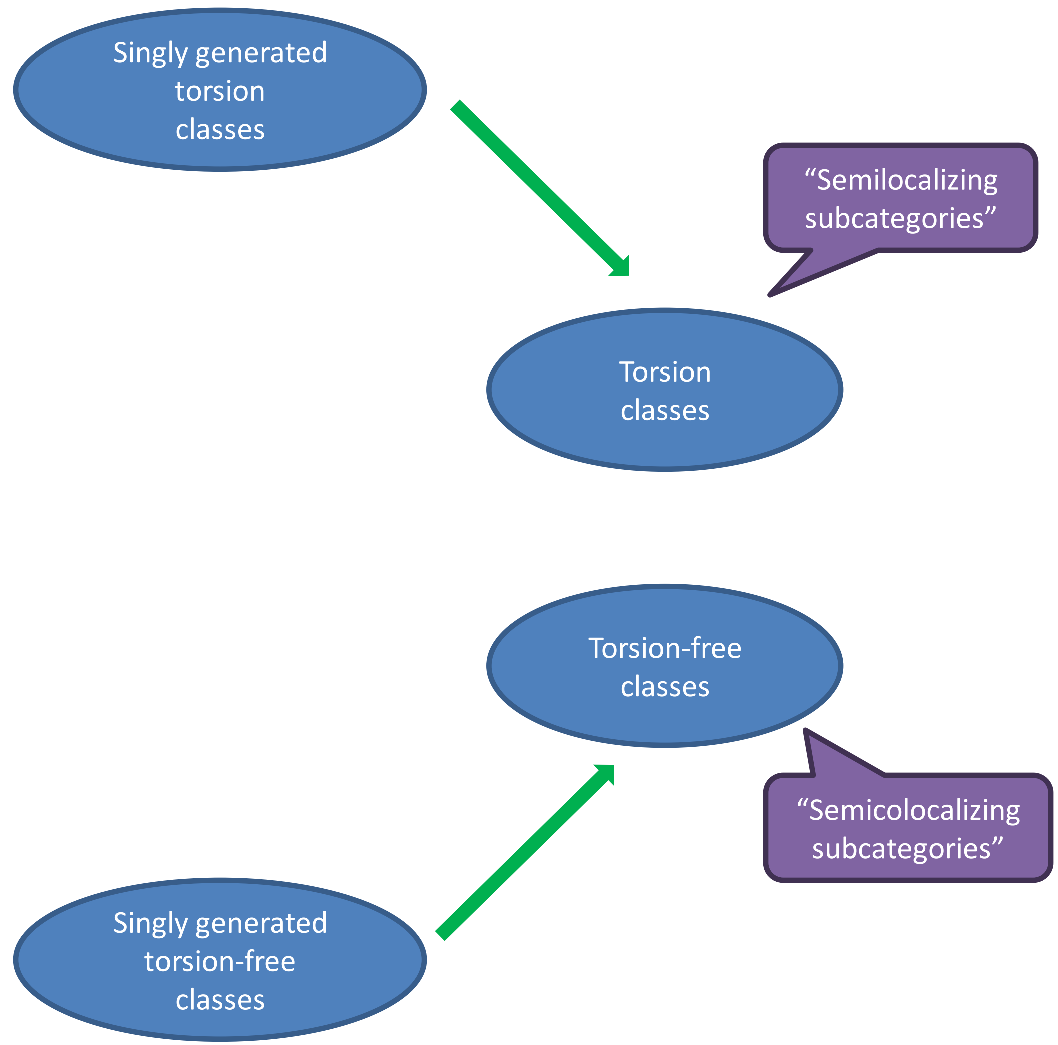
"Semilocalizing subcategories"

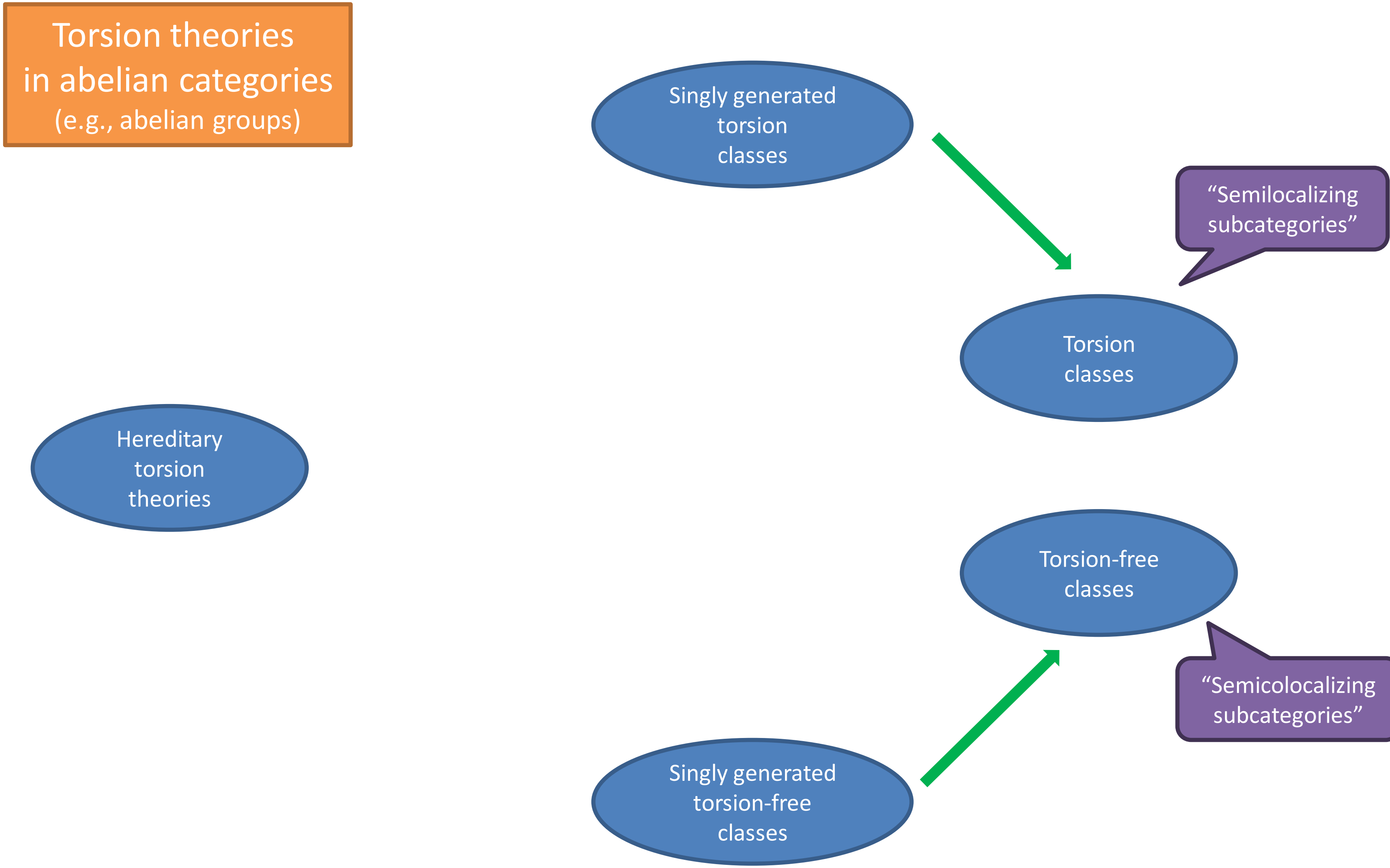
Torsion classes

Torsion-free classes

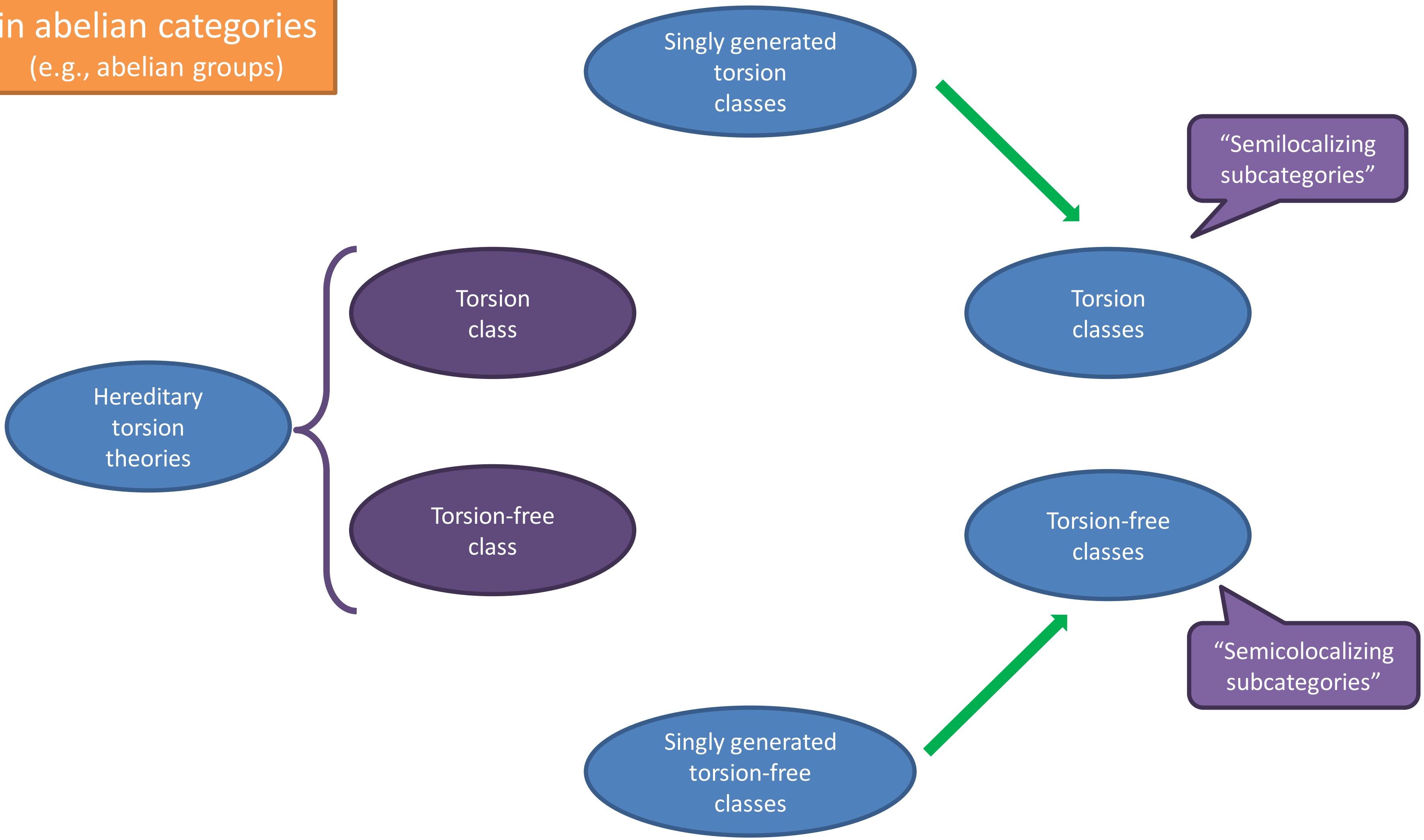
"Semicolocalizing subcategories"

Torsion theories in abelian categories
(e.g., abelian groups)

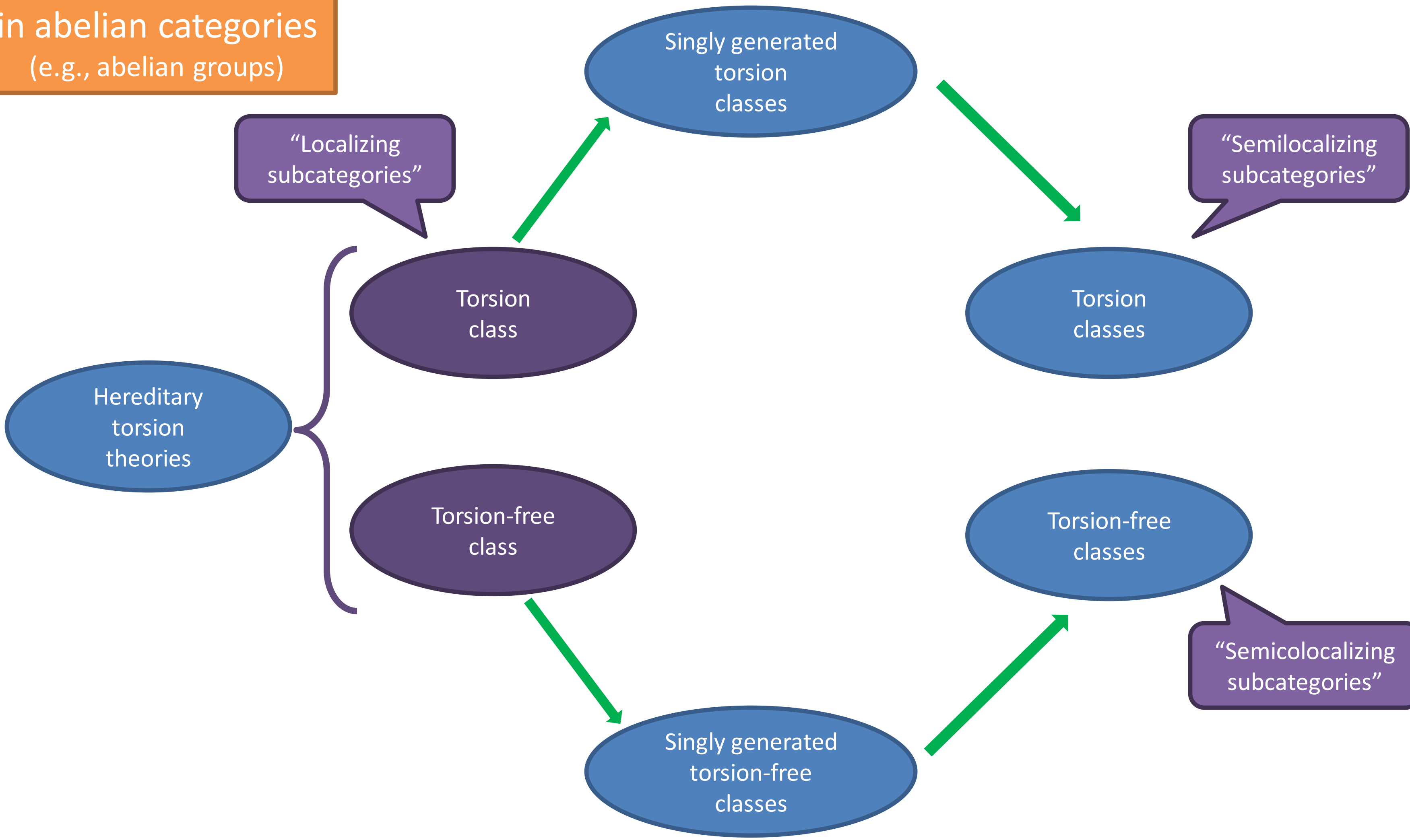


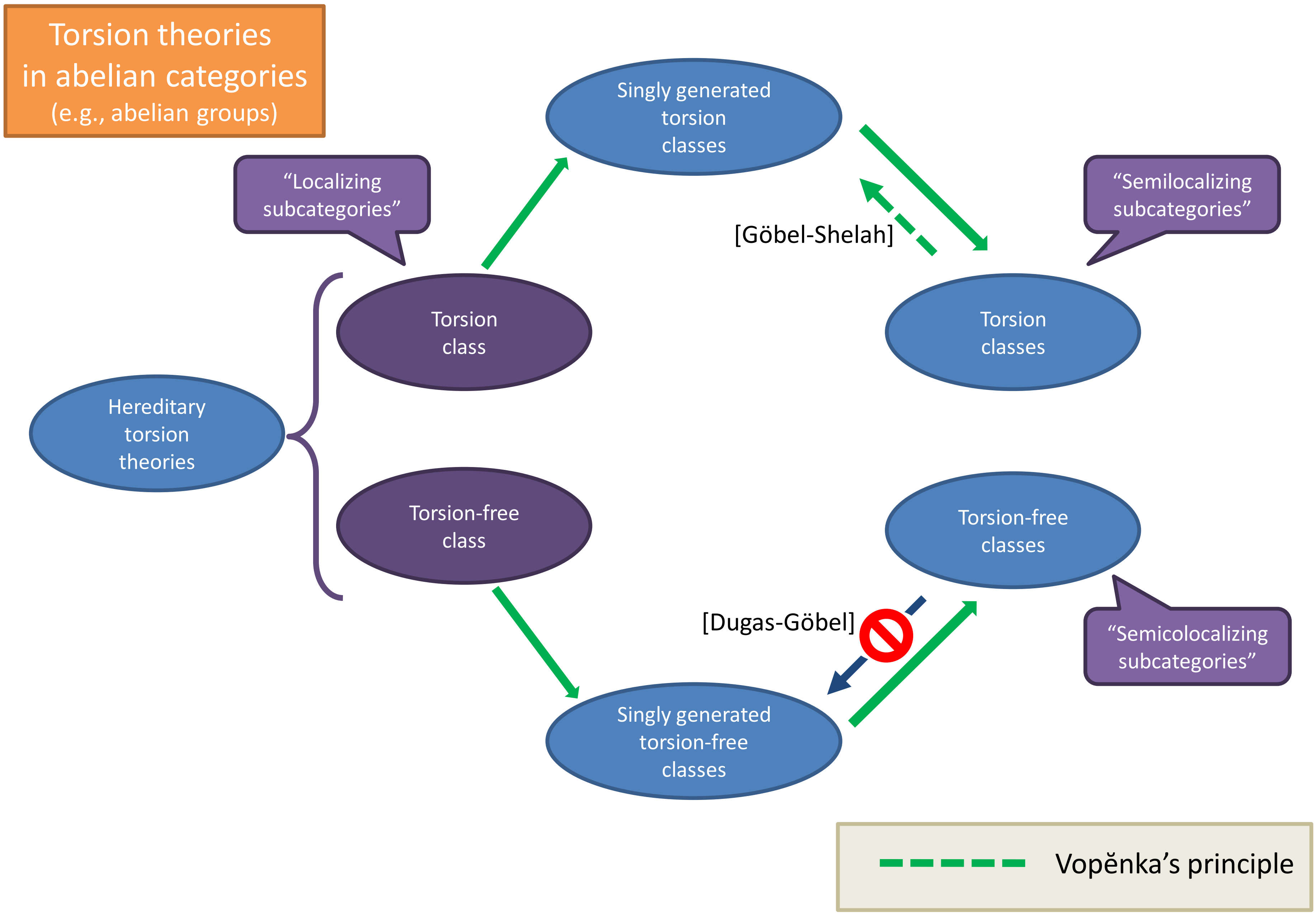

Torsion theories
in abelian categories
(e.g., abelian groups)
Singly generated torsion classes
Torsion classes
“Semilocalizing subcategories”
Hereditary torsion theories
Torsion-free classes
“Semicolocalizing subcategories”
Singly generated torsion-free classes

Torsion theories in abelian categories
(e.g., abelian groups)



Torsion theories in abelian categories
(e.g., abelian groups)




Torsion theories
in abelian categories
(e.g., abelian groups)
"Localizing subcategories"
Singly generated torsion classes
Torsion class
Torsion classes
[Göbel-Shelah]
"Semilocalizing subcategories"
Hereditary torsion theories
Torsion-free class
Torsion-free classes
[Dugas-Göbel]
"Semicolocalizing subcategories"
Singly generated torsion-free classes
Vopěnka's principle

Triangulated categories
(with combinatorial models)

Semilocalizing subcategories

Semicolocalizing subcategories

Triangulated categories
(with combinatorial models)

Closed semilocalizing subcategories

Semilocalizing subcategories

Closed semicolocalizing subcategories

Semicolocalizing subcategories

Triangulated categories
(with combinatorial models)

Coreflective semilocalizing subcategories

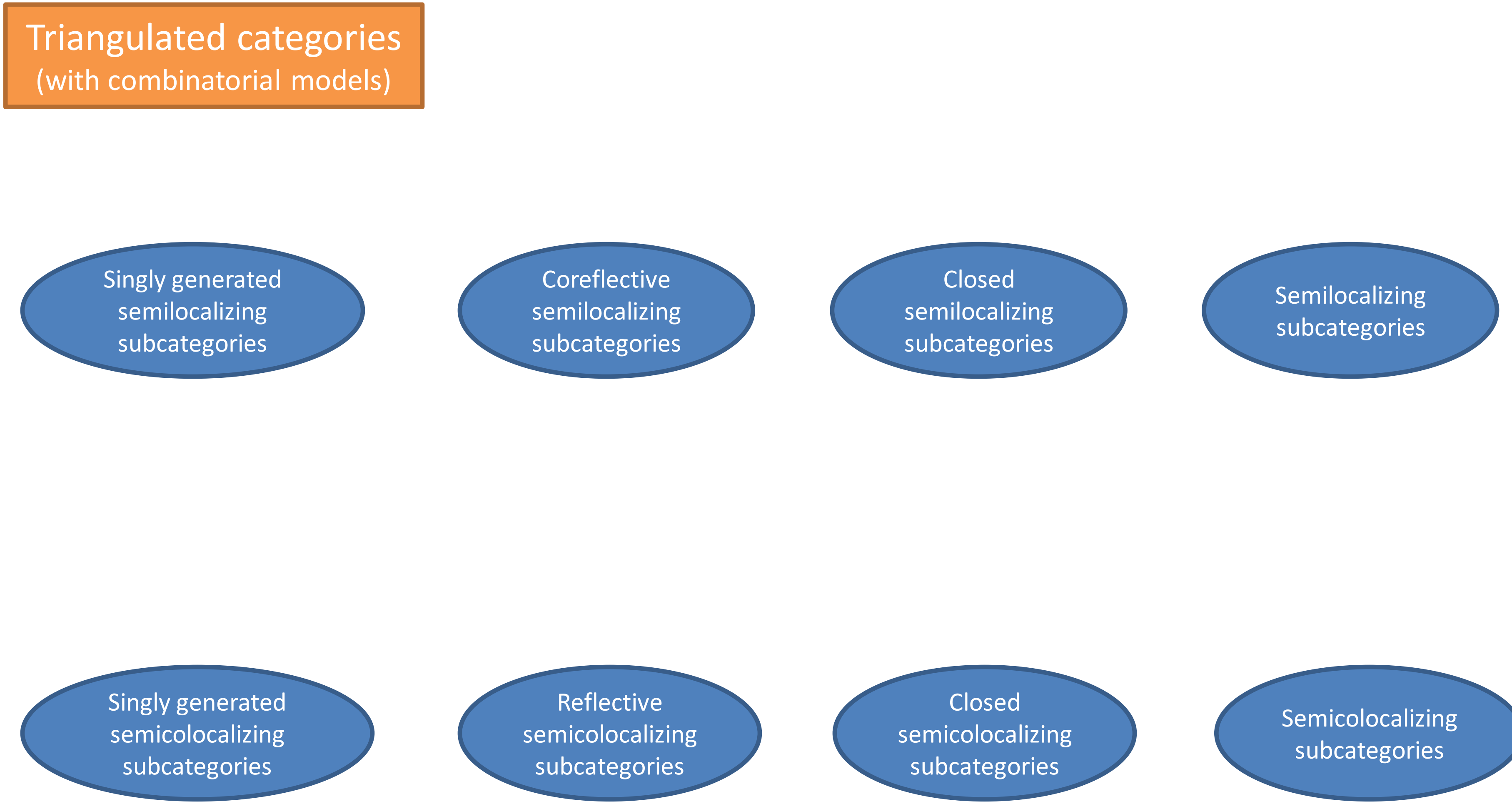
Closed semilocalizing subcategories

Semilocalizing subcategories

Reflective semicolocalizing subcategories

Closed semicolocalizing subcategories

Semicolocalizing subcategories


Triangulated categories
(with combinatorial models)
Singly generated semilocalizing subcategories
Coreflective semilocalizing subcategories
Closed semilocalizing subcategories
Semilocalizing subcategories
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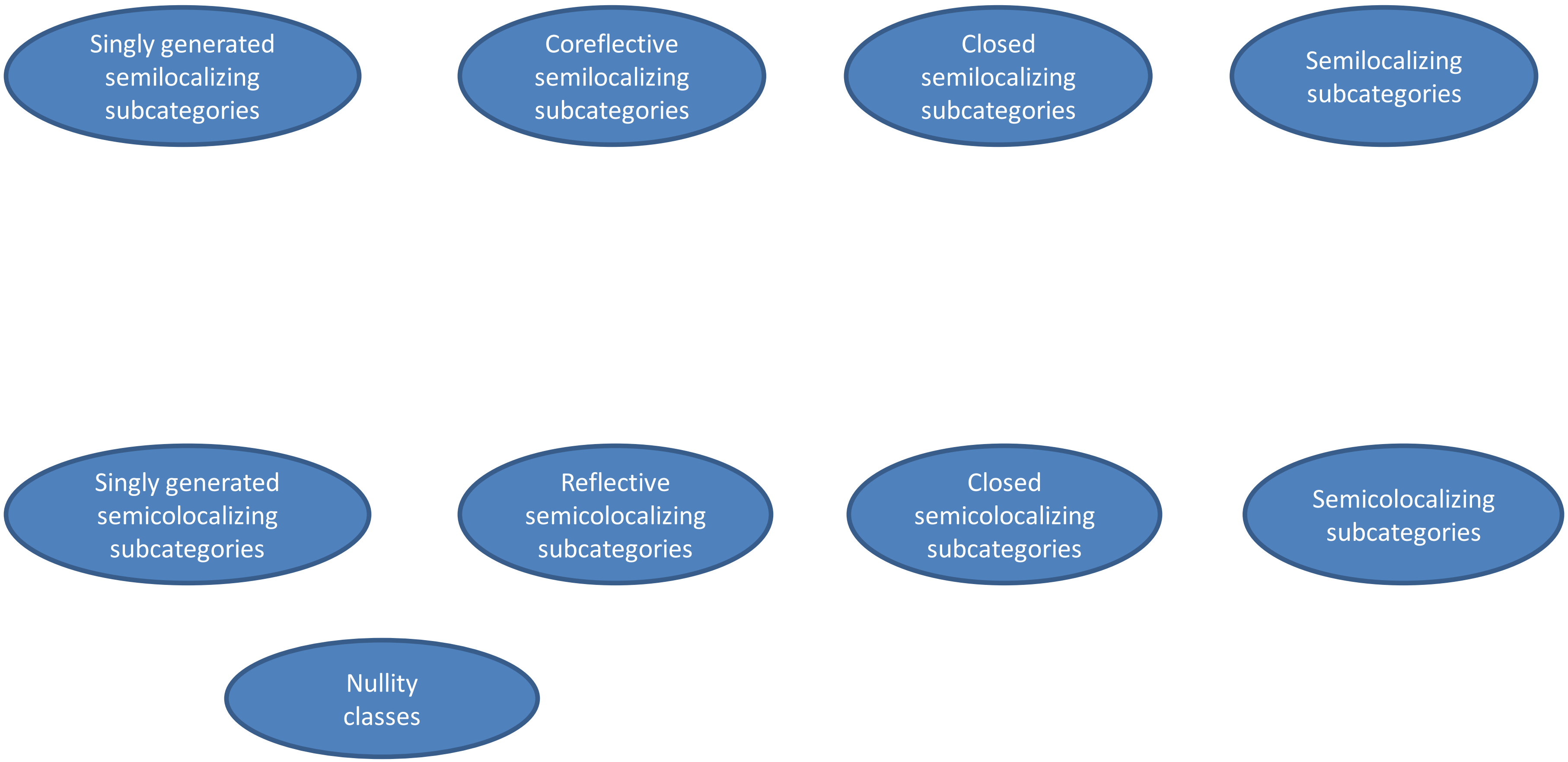
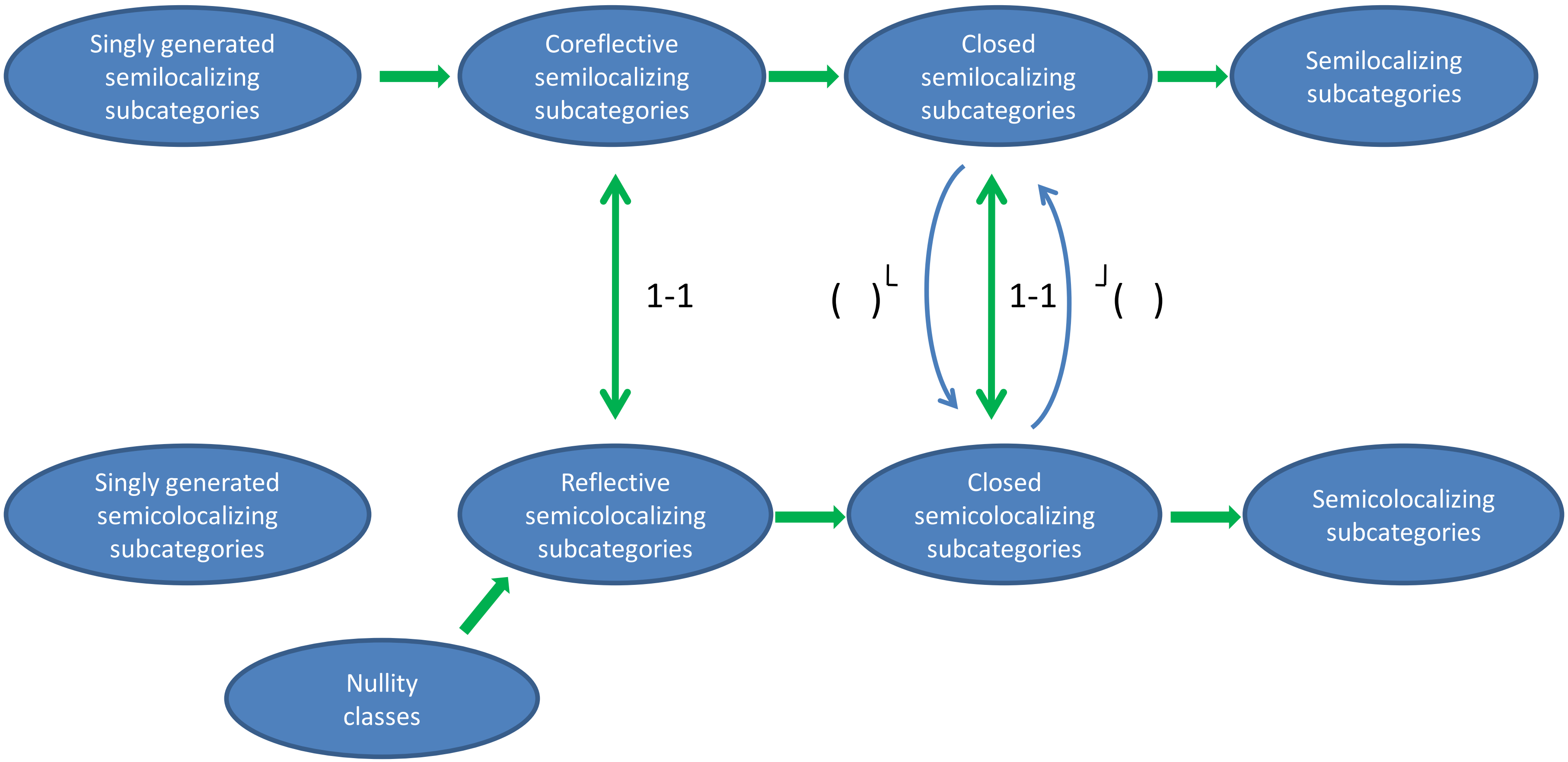
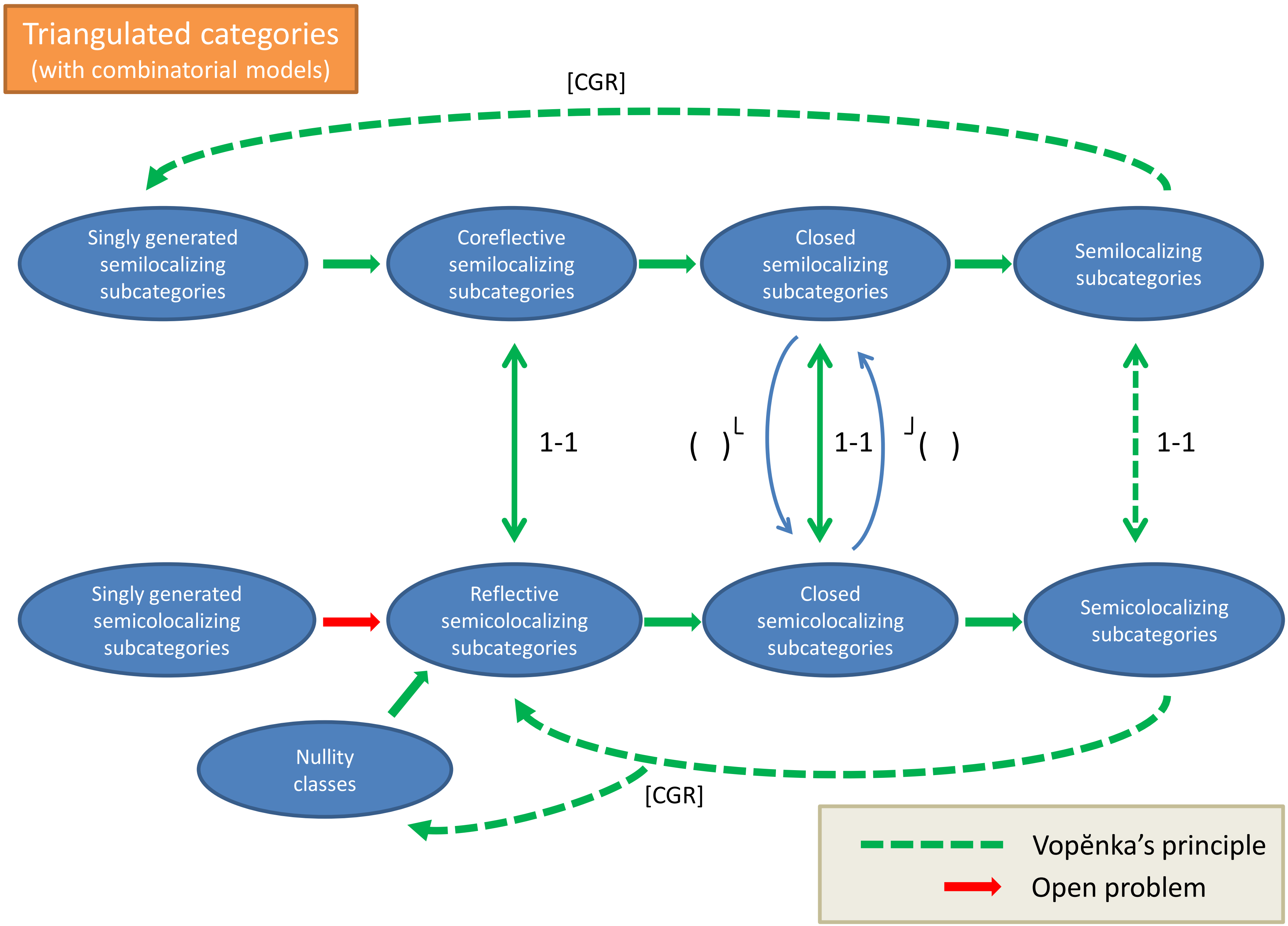
Semicolocalizing subcategories

Triangulated categories (with combinatorial models)



Triangulated categories
(with combinatorial models)




Triangulated categories
(with combinatorial models)
[CGR]
Singly generated semilocalizing subcategories
Coreflective semilocalizing subcategories
Closed semilocalizing subcategories
Semilocalizing subcategories
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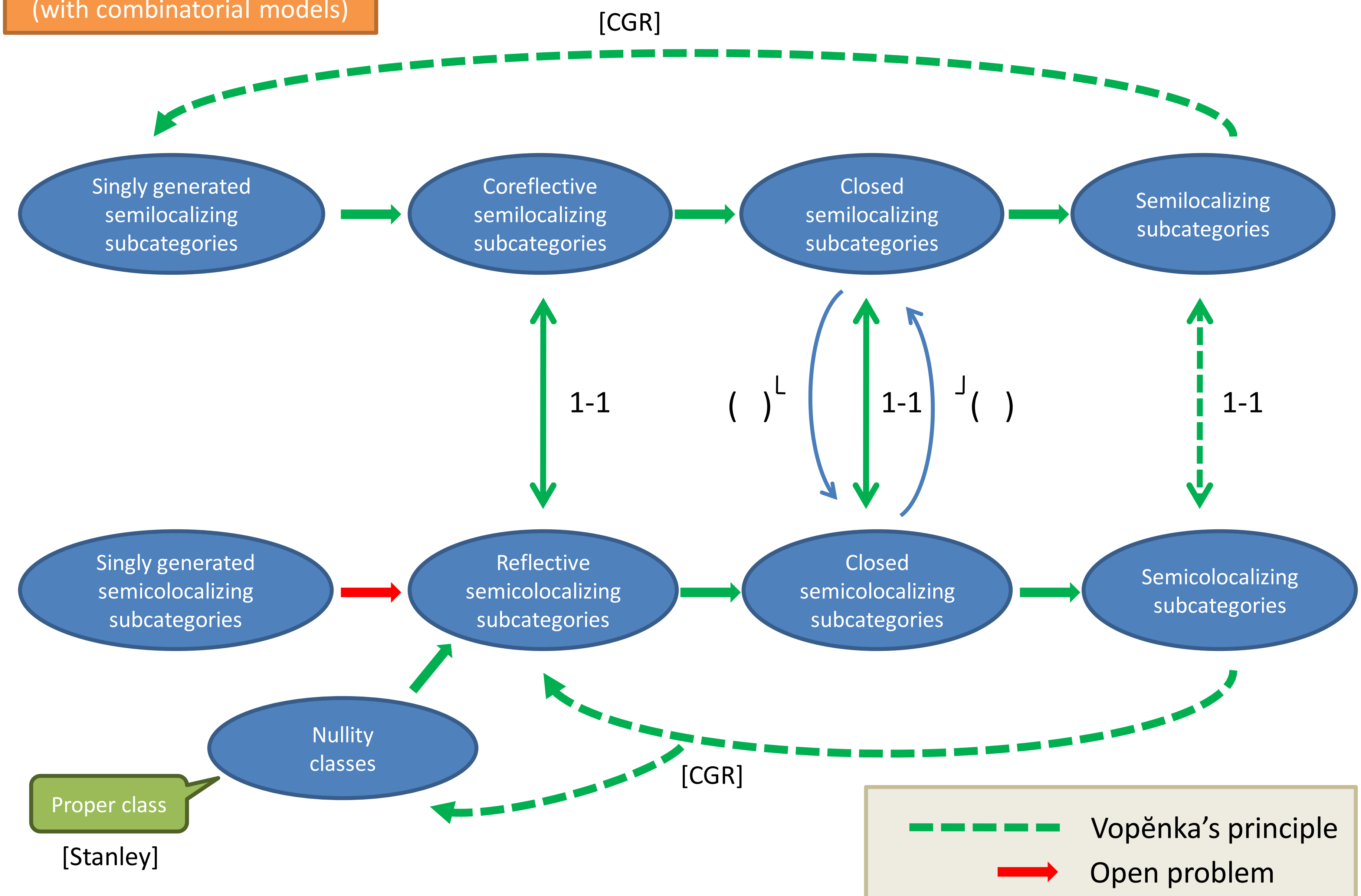
Singly generated semicolocalizing subcategories
Reflective semicolocalizing subcategories
Closed semicolocalizing subcategories
Semicolocalizing subcategories
Nullity classes
[CGR]
Vopěnka's principle
Open problem

Triangulated categories
(with combinatorial models)



Tensor triangulated categories
(with combinatorial models)

Localizing ideals

Colocalizing coideals

Tensor triangulated categories
(with combinatorial models)

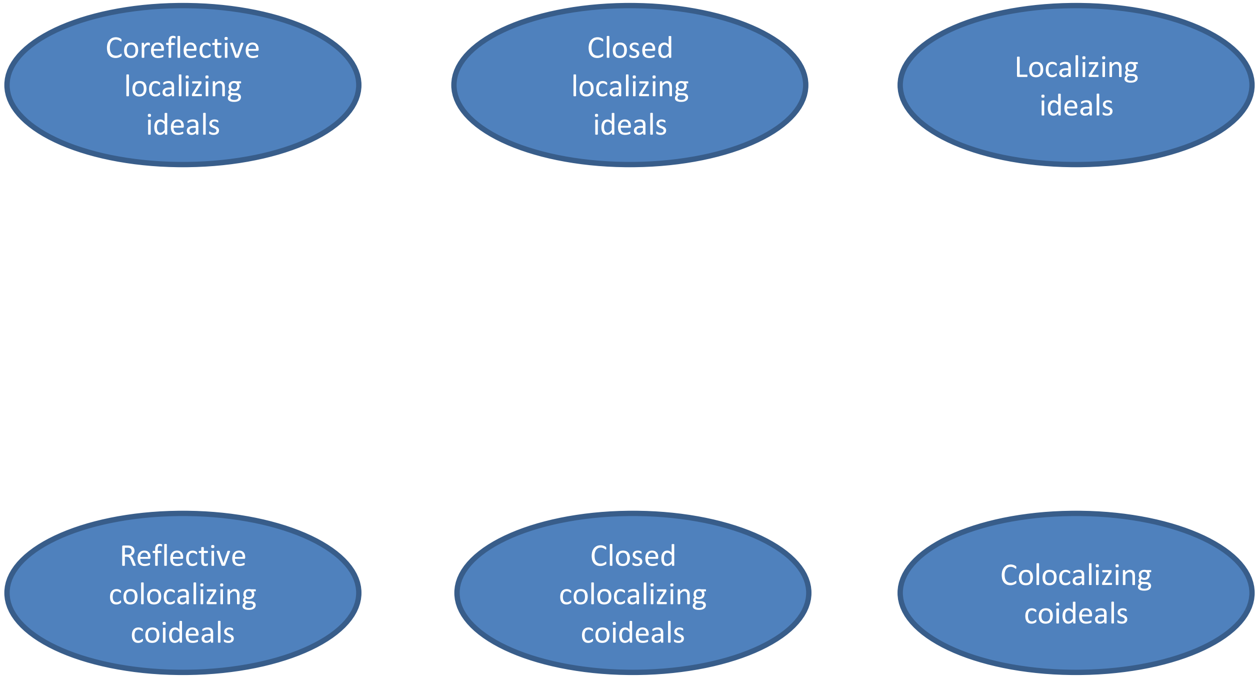
Closed localizing ideals

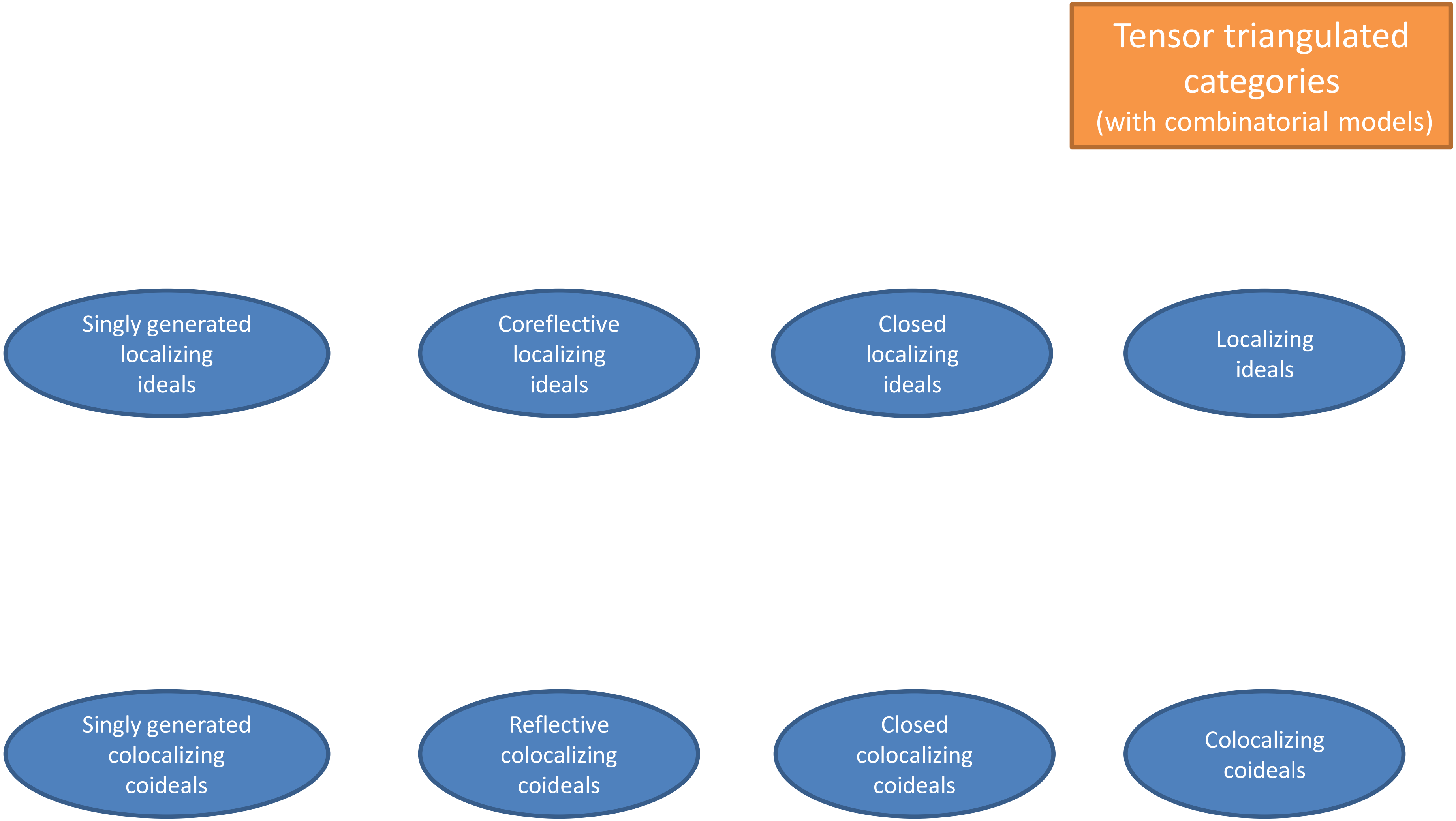
Localizing ideals

Closed colocalizing coideals

Colocalizing coideals

Tensor triangulated categories
(with combinatorial models)




Tensor triangulated categories
(with combinatorial models)
Singly generated localizing ideals
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Singly generated colocalizing coideals
Reflective colocalizing coideals
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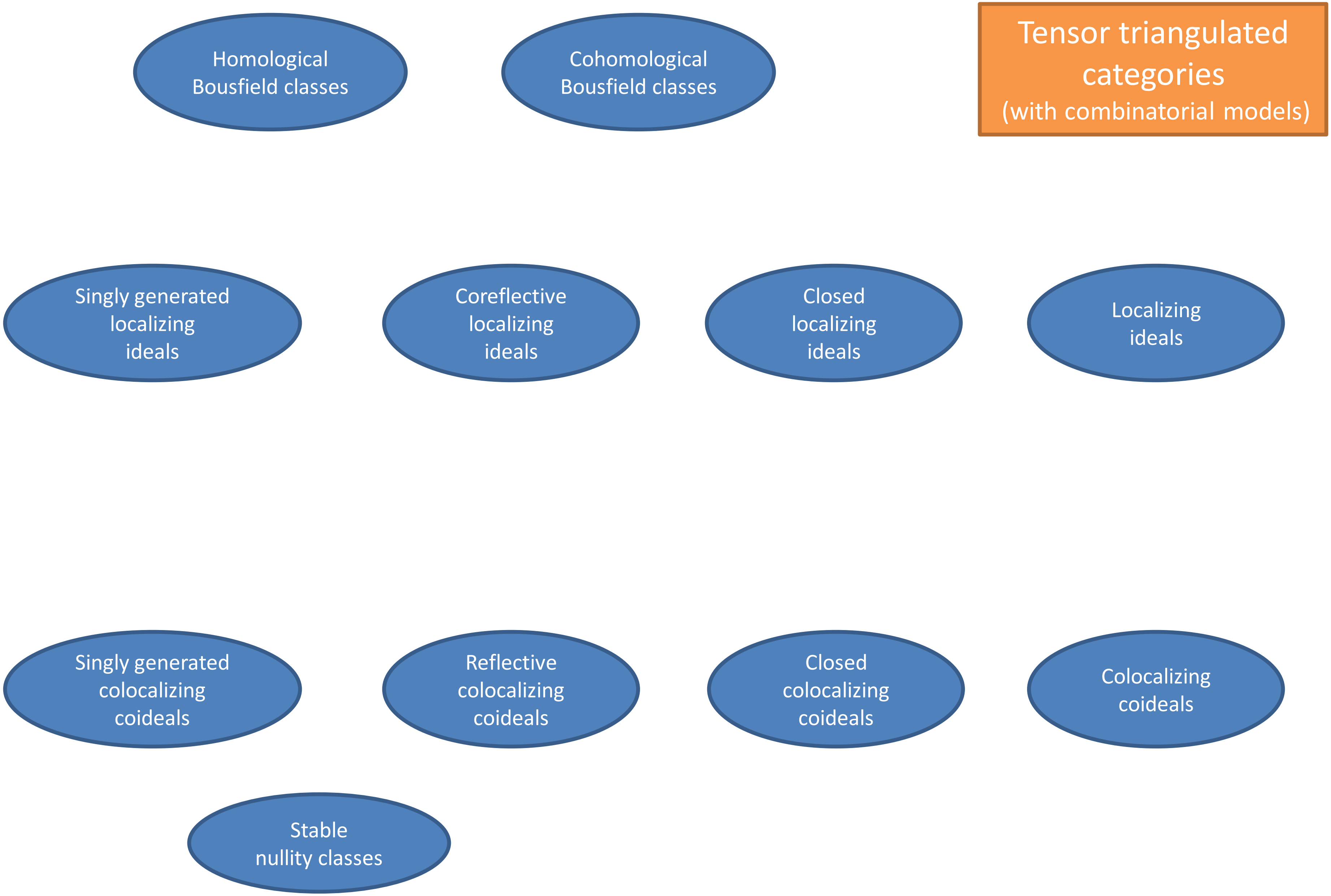
Tensor triangulated categories
(with combinatorial models)
Homological Bousfield classes
Cohomological Bousfield classes
Singly generated localizing ideals
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Localizing ideals
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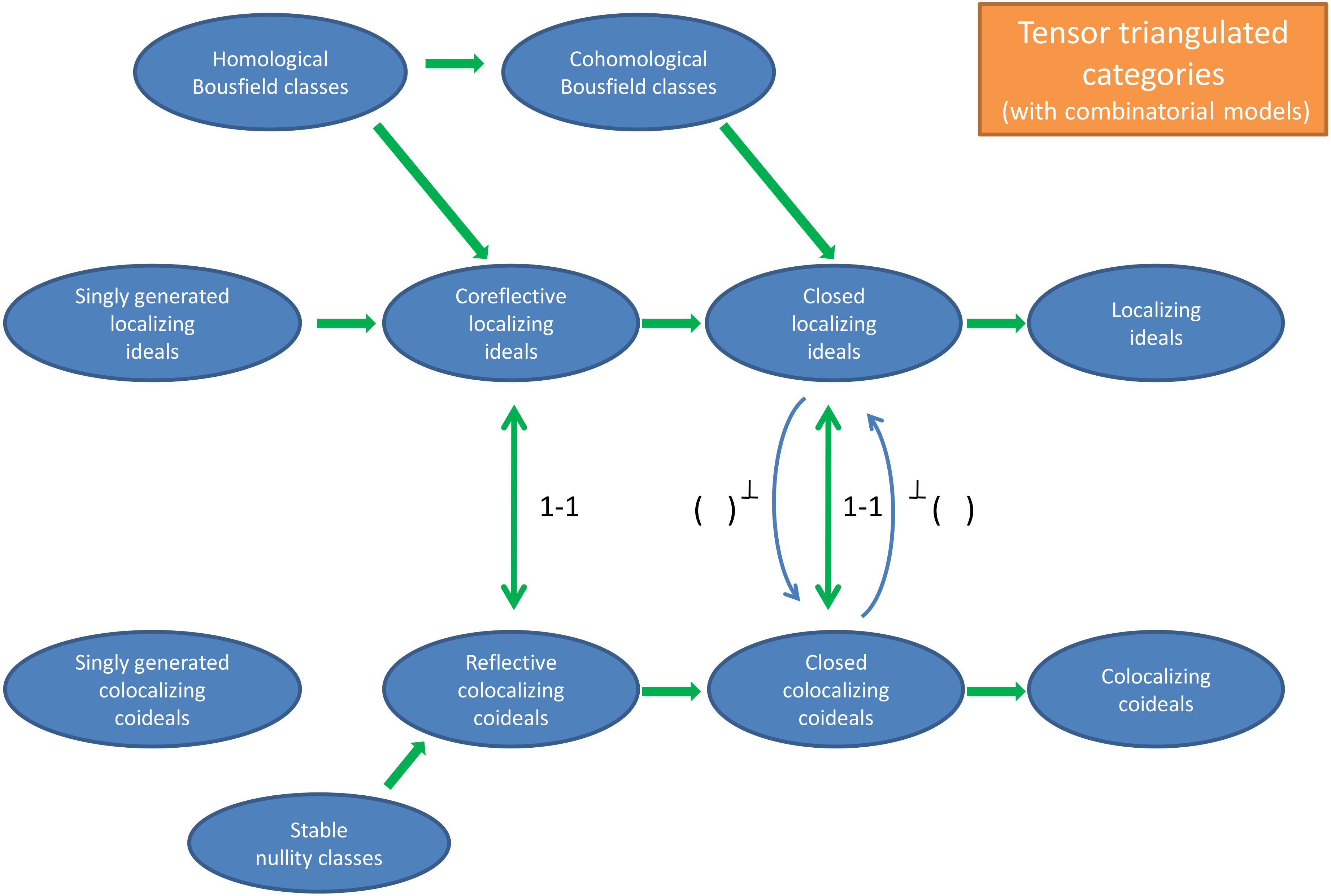
Colocalizing coideals
Stable nullity classes



Tensor triangulated categories (with combinatorial models)
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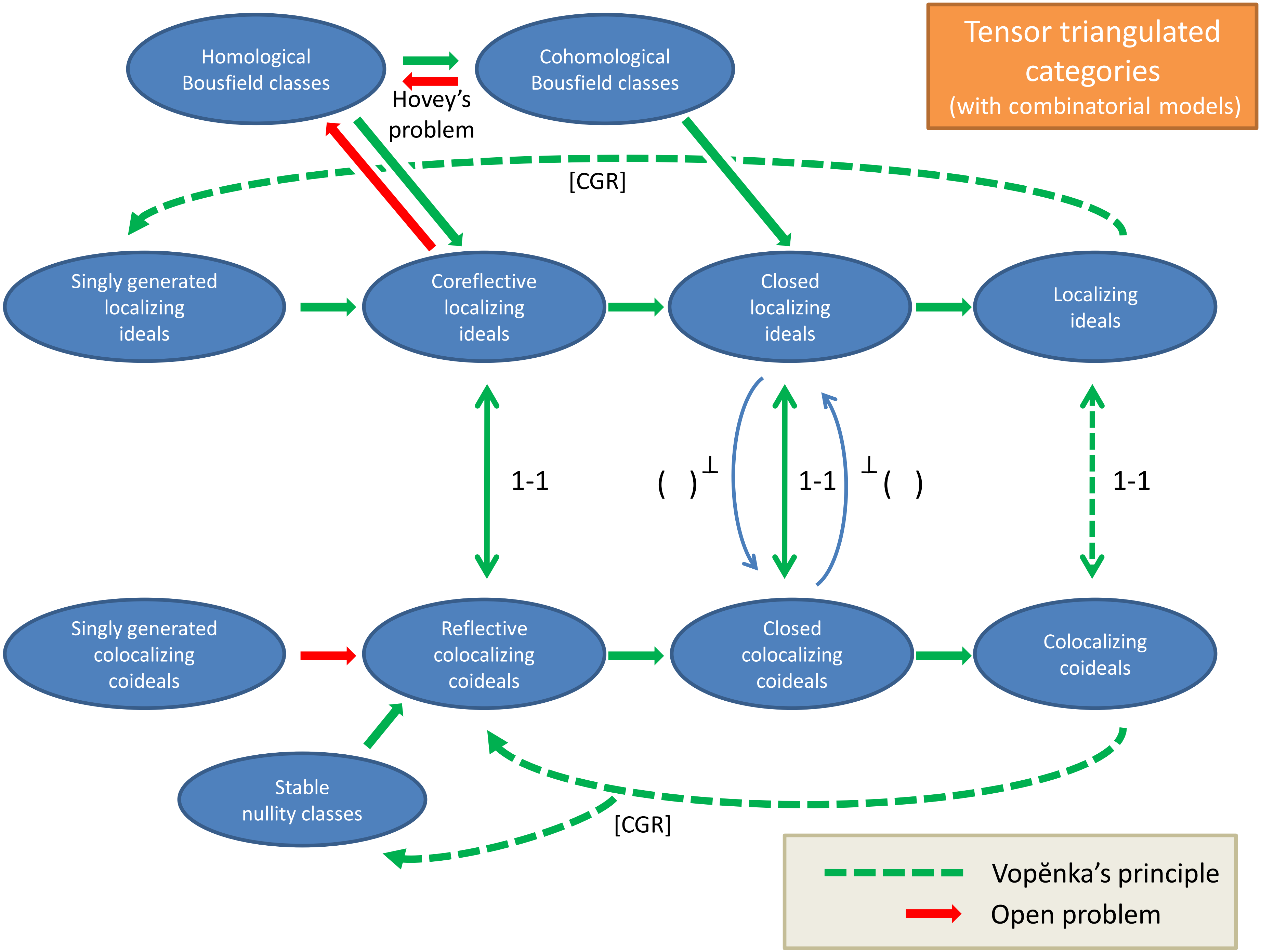
Tensor triangulated categories
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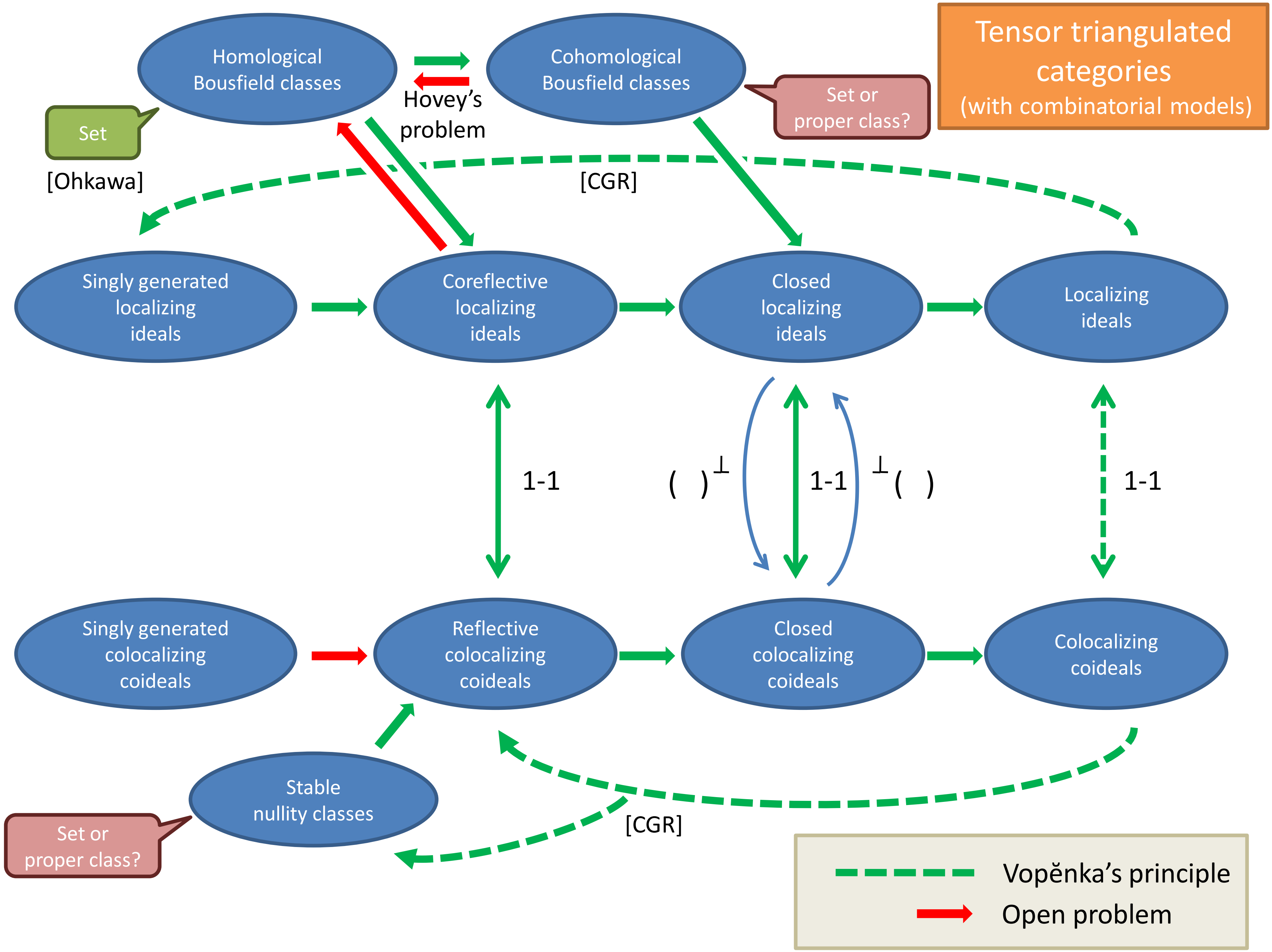
Tensor triangulated categories
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Set
[Ohkawa]
Set or proper class?
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Vopěnka's principle
Open problem

Triangulated categories with combinatorial models

A model category $\mathcal{K}$ is called

(i) *Combinatorial* if it is locally presentable and cofibrantly generated.

(ii) *Stable* if it is pointed and the suspension and loop operator are inverse equivalences on the homotopy category $\mathrm{Ho}(\mathcal{K})$. In this case $\mathrm{Ho}(\mathcal{K})$ is triangulated.

We are interested in triangulated categories that appear as homotopy categories of combinatorial stable (monoidal) model categories. Such triangulated categories are well-generated [Rosicky].

Examples

- The homotopy category of spectra
- The derived category of a commutative ring

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Vopěnka's principle

Vopěnka's principle (Categorical formulation)

Given any family of objects X_s of an accessible category indexed by the class of all ordinals, there is a morphism $X_s \to X_t$ for some ordinal $s < t$.

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Main results

Theorem 1 (CGR)

Let $\mathcal{K}$ be a locally presentable category with a stable model category structure. If Vopěnka's principle holds, then every full subcategory $\mathcal{L}$ of $\mathrm{Ho}(\mathcal{K})$ closed under fibres and products is reflective. If $\mathcal{L}$ is semicolocalizing, then the reflection is semiexact. If $\mathcal{L}$ is colocalizing, then the reflection is exact.

Corollary

Let $\mathcal{K}$ be a locally presentable stable model category. If Vopěnka's principle holds, then every closed semilocalizing subcategory of $\mathrm{Ho}(\mathcal{K})$ is coreflective.

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Main results

Theorem 2 (CGR)

Let $\mathcal{K}$ be a stable combinatorial model category. If Vopěnka's principle holds, then every semilocalizing subcategory of $\mathrm{Ho}(\mathcal{K})$ is singly generated and coreflective. The same result holds for localizing subcategories and for localizing ideals.

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Main results

Corollary

Let $\mathcal{T}$ be a triangulated category with combinatorial models. Assuming Vopěnka's principle, every semicolocalizing subcategory of $\mathcal{T}$ is equal to $E^{\llcorner}$ for some object E (i.e., a nullity class) and every colocalizing subcategory is equal to $E^{\perp}$ for some E (i.e., a stable nullity class).

Summary

Assuming **Vopěnka's principle**:

- Every semilocalizing subcategory of a triangulated category with combinatorial models is part of a t-structure and the same happens with every semicolocalizing subcategory.
- In every triangulated category with combinatorial models there is a bijective correspondence between the class of (semi)localizing subcategories and (semi)colocalizing subcategories.
- In every tensor triangulated category with combinatorial models any localizing ideal is the kernel of a localization functor and every colocalizing coideal is the kernel of a colocalization functor.

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