Algebras over coloured operads and localization functors

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(joint work with C. Casacuberta, I. Moerdijk and R. M. Vogt)
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Outline of the talk

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Precedents

Abelian groups

- If R is a (commutative) ring, then LR is a (commutative) ring and the localization map is a ring morphism.
- If M is an R-module, then LM is an R-module and the localization map is a morphism of R-modules.

Topological spaces

- If X is an H-space, then LX is homotopy equivalent to an H-space and the localization map is equivalent to an H-map.
- If X is a loop space, then LX is homotopy equivalent to a loop space and the localization map is equivalent to a loop map. (In fact, $L_f\Omega X \simeq \Omega L_{\Sigma f}X$ [Farjoun, 1996].)

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Stable homotopy category

- If R is a connective ring spectrum and LR is connective, then LR is a ring spectrum and the localization map is a map of ring spectra.
- If M is an R-module spectrum and R is connective, then LM is an R-module spectrum and the localization map is a map of R-modules [Casacuberta-G, 2005].

S-modules

- If R is an S-algebra and E_* is a homology theory, then the Bousfield localization $L_E R$ is an S-algebra and the localization map is a map of S-algebras [EKMM, 1997].

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Objectives

Symmetric spectra

- Study the preservation of strict ring spectra and module spectra under localizations (by viewing them as algebras over A_∞ or E_∞) .

Monoidal model categories

- Study the preservation under localizations of structures defined as algebras over coloured operads.

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- Study the preservation under localizations of structures defined as algebras over coloured operads.

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Coloured operads

- Let $\mathcal{E}$ be a cocomplete closed symmetric monoidal category. Let C be a set, whose elements will be called **colours**. A C**-coloured collection** is a set P of objects $P(c_1, \ldots, c_n; c)$ in $\mathcal{E}$ for every $n \geq 0$ and each tuple $(c_1, \ldots, c_n; c)$ of colours, together with maps

$$\sigma^* : P(c_1, \ldots, c_n; c) \longrightarrow P(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c)$$

for all permutations $\sigma \in \Sigma_n$, yielding together a right action.

- A C**-coloured operad** is a C-coloured collection P equipped with unit maps $I \to P(c; c)$ and *composition product* maps

$$P(c_1, \ldots, c_n; c) \otimes P(a_{1,1}, \ldots, a_{1,k_1}; c_1) \otimes \cdots \otimes P(a_{n,1}, \ldots, a_{n,k_n}; c_n) \longrightarrow P(a_{1,1}, \ldots, a_{1,k_1}, a_{2,1}, \ldots, a_{2,k_2}, \ldots, a_{n,1}, \ldots, a_{n,k_n}; c)$$

compatible with the action of the symmetric groups and subject to associativity and unitary compatibility relations.

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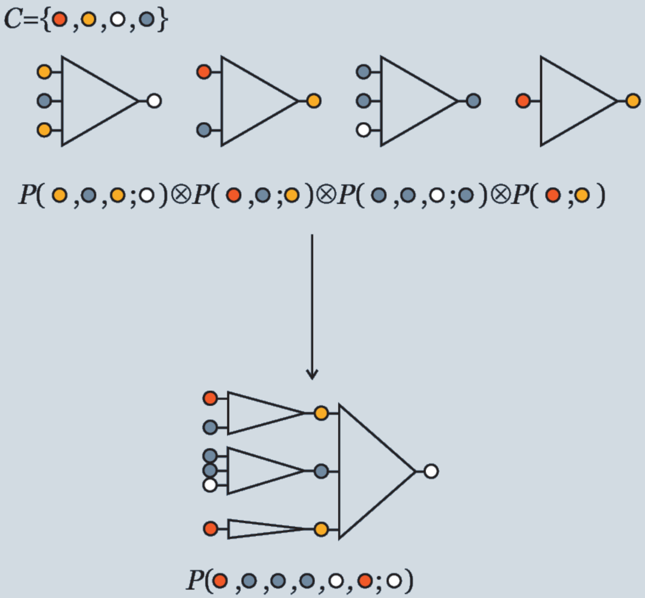

Algebras over coloured operads

- If P is a C-coloured operad, a P-**algebra** is an object $\mathbf{X} = (X(c))_{c \in C}$ in $\mathcal{E}^C$ together with a morphism of C-coloured operads

$$P \longrightarrow \mathrm{End}(\mathbf{X})$$

where the C-coloured operad $\mathrm{End}(\mathbf{X})$ is defined as

$$\mathrm{End}(\mathbf{X})(c_1, \ldots, c_n; c) = \mathrm{Hom}_{\mathcal{E}}(X(c_1) \otimes \cdots \otimes X(c_n), X(c)).$$

Examples

- An operad is a coloured operad with only one colour.
- The **associative operad** A is defined as $A(n) = I[\Sigma_n]$ for all n, where $I[\Sigma_n]$ is a coproduct of copies of the unit I of $\mathcal{E}$ indexed by Σ_n. The **commutative operad** Com is defined as $Com(n) = I$. A-algebras are monoids and Com-algebras are commutative monoids in $\mathcal{E}$.

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Example (Modules)

- Let $C = \{r, m\}$ and P an operad. Define a C-coloured operad Mod_P whose only nonzero terms are

$$\mathrm{Mod}_P(r, \overset{(n)}{\ldots}, r; r) = P(n)$$

and

$$\mathrm{Mod}_P(c_1, \ldots, c_n; m) = P(n)$$

when exactly one c_i is m and the rest (if any) are r. Then an algebra over Mod_P is a pair (R, M) where R is a P-algebra and M is an R-module. By using non-symmetric operads, one obtains left R-modules and right R-modules similarly.

Hence, **modules over P-algebras can be viewed as algebras over coloured operads.**

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Example (Morphisms of algebras)

- Let P be a C-coloured operad and choose $D = \{0,1\} \times C$. Define a D-coloured operad Mor_P whose value on $((i_1,c_1),\ldots,(i_n,c_n);(i,c))$ is

$$\begin{cases} 0 & \text{if } i=0 \text{ and } i_k=1 \text{ for some } k; \\ P(c_1,\ldots,c_n;c) & \text{otherwise.} \end{cases}$$

Then an algebra over Mor_P consists of two P-algebras $\mathbf{X}_0 = (X(0,c))_{c\in C}$ and $\mathbf{X}_1 = (X(1,c))_{c\in C}$ and a map of P-algebras $\mathbf{f}\colon \mathbf{X}_0 \longrightarrow \mathbf{X}_1$ defined for every $c \in C$ as the composite

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Model structures

Model structure for coloured operads

- Let $\mathcal{E}$ be a monoidal model category ▸ Definition. A model structure on the category of C-coloured operads in $\mathcal{E}$ for a fixed C was described by [Berger-Moerdijk, 2007].
- A map of operads $P \longrightarrow Q$ is a weak equivalence (resp. fibration) if for every $(c_1, \ldots, c_n; c)$ the map

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Outline of the talk

Localization functors

A **homotopical localization** on a model category $\mathcal{M}$ with homotopy function complexes $map(-,-)$ is a functor $L\colon \mathcal{M} \longrightarrow \mathcal{M}$ that preserves weak equivalences and takes fibrant values, together with a natural transformation $\eta\colon \mathrm{Id}_{\mathcal{M}} \longrightarrow L$ such that, for every object X, the following hold:

- $L\eta_X : LX \longrightarrow LLX$ is a weak equivalence;
- η_{LX} and $L\eta_X$ are equal in the homotopy category $\mathrm{Ho}(\mathcal{M})$;
- $\eta_X : X \longrightarrow LX$ is a cofibration such that the map

$$map(\eta_X, LY)\colon\ map(LX, LY) \longrightarrow map(X, LY)$$

is a weak equivalence of simplicial sets for all Y.

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Localization functors

- Every homotopical localization functor is an idempotent functor on the homotopy category $\mathrm{Ho}(\mathcal{M})$.
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- Morphisms f such that Lf is a weak equivalence are called **L-equivalences**.

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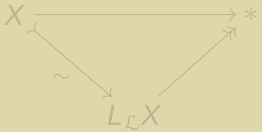
Example (Left Bousfield localizations)

Let $\mathcal{M}$ be a left proper cellular model category and $\mathcal{L}$ a set of morphisms in $\mathcal{M}$. Then there exists a model structure $\mathcal{M}_{\mathcal{L}}$ such that

- Cofibrations in $\mathcal{M}$ $=$ Cofibrations in $\mathcal{M}_{\mathcal{L}}$.
- Fibrant objects in $\mathcal{M}_{\mathcal{L}}$ $=$ $\mathcal{L}$-local objects.
- Weak equivalences in $\mathcal{M}_{\mathcal{L}}$ $=$ $\mathcal{L}$-equivalences.

A functorial factorization in $\mathcal{M}_{\mathcal{L}}$ of a map as a trivial cofibration followed by a fibration gives a homotopical localization functor on $\mathcal{M}$:



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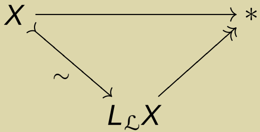

Extending localization functors

If (L, η) is a homotopical localization functor in $\mathcal{M}$, how can we apply L to an object $\mathbf{X} = (X(c))_{c\in C}$ of $\mathcal{M}^C$?

Definition

The **extension of (L, η) over $\mathcal{M}^C$ away from $J \subseteq C$** is the coaugmented functor given by:

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Extending localization functors

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Ideals

Example

Let L be a localization functor on abelian groups. If M is an R-module, then the pair (R, M) is an algebra over Mod_A.

- (LR, LM) and (R, LM) are algebras over Mod_A
- (LR, M) is not an algebra over Mod_A in general

Definition

If P is a C-coloured operad, a subset $J \subseteq C$ is called an **ideal relative to** P if $P(c_1, \ldots, c_n; c) = 0$ whenever $n \geq 1$, $c \in J$, and $c_i \notin J$ for some $i \in \{1, \ldots, n\}$.

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Consider the coloured operads of the previous examples:

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$$\emptyset,\ \{r\} \text{ and } \{r,m\}.$$

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Outline of the talk

Localization of algebras

Theorem (CGMV)

Let (L, η) be a homotopical localization on a simplicial monoidal model category $\mathcal{M}$ ▸ Definition.

Let P be a cofibrant C-coloured operad in simplicial sets, and consider the extension of (L, η) over $\mathcal{M}^C$ away from an ideal $J \subseteq C$ relative to P.

Let $\mathbf{X}$ be a P-algebra such that $X(c)$ is cofibrant in $\mathcal{M}$ for every $c \in C$.

Suppose that

$$(\eta_{\mathbf{X}})_{c_1} \otimes \cdots \otimes (\eta_{\mathbf{X}})_{c_n}$$

is an L-equivalence.

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Proof.

For all $(c_1, \ldots, c_n; c)$, the map

$$X(c_1) \otimes \cdots \otimes X(c_n) \longrightarrow L_{c_1}X(c_1) \otimes \cdots \otimes L_{c_n}X(c_n)$$

is an L-equivalence by assumption, and it is also a cofibration since $X(c)$ is cofibrant for all c. Hence, the map

$$\mathrm{Map}(L_{c_1}X(c_1) \otimes \cdots \otimes L_{c_n}X(c_n), L_cX(c)) \longrightarrow \mathrm{Map}(X(c_1) \otimes \cdots \otimes X(c_n), L_cX(c))$$

is a fibration and a weak equivalence. By definition,

$$\mathrm{Map}(L_{c_1}X(c_1) \otimes \cdots \otimes L_{c_n}X(c_n), L_cX(c)) = \mathrm{End}(L\mathbf{X})(c_1, \ldots, c_n; c)$$

$$\mathrm{Map}(X(c_1) \otimes \cdots \otimes X(c_n), L_cX(c)) = \mathrm{Hom}(\mathbf{X}, L\mathbf{X})(c_1, \ldots, c_n; c).$$

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Define a C-coloured operad $\mathrm{End}(\eta_{\mathbf{X}})$ as the following pull-back:

$$\begin{array}{ccc} \mathrm{End}(\eta_{\mathbf{X}}) & \dashrightarrow & \mathrm{End}(L\mathbf{X}) \\ \vdots & & \downarrow \\ \mathrm{End}(\mathbf{X}) & \longrightarrow & \mathrm{Hom}(\mathbf{X}, L\mathbf{X}). \end{array}$$

The right-hand vertical arrow is a trivial fibration. Hence, the left-hand vertical arrow is also a trivial fibration.

Then, since P is cofibrant, there is a lifting

$$\begin{array}{cccccc} & & \mathrm{End}(\eta_{\mathbf{X}}) & \longrightarrow & \mathrm{End}(L\mathbf{X}) \\ & \nearrow & \downarrow & & \downarrow \\ P & \longrightarrow & \mathrm{End}(\mathbf{X}) & \longrightarrow & \mathrm{Hom}(\mathbf{X}, L\mathbf{X}). \end{array}$$

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Applications

Topological spaces

- If X is a cofibrant A_∞-space, then LX has a homotopy unique A_∞-space structure such that the localization map $\eta_X : X \longrightarrow LX$ is a map of A_∞-spaces.
- Same is true for A_∞-maps, E_∞-spaces and E_∞-maps.

Symmetric spectra

- Let M be a left A_∞-module over an A_∞-ring R, and assume that both R and M are cofibrant as spectra. Let (L, η) be a localization functor that commutes with suspension. Then LM has a homotopy unique left A_∞-module structure over R such that $\eta_M : M \longrightarrow LM$ is a morphism of A_∞-modules.
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Rectification of homotopy algebras

Any map of C-coloured operads $\varphi\colon P \longrightarrow Q$ induces a pair of adjoint functors

$$\varphi_!\colon \mathrm{Alg}_P(\mathcal{M}) \rightleftarrows \mathrm{Alg}_Q(\mathcal{M})\colon \varphi^*$$

This is a Quillen equivalence if φ is a weak equivalence and P and Q are well-pointed Σ-cofibrant [Berger-Moerdijk, 2007].

Let $\mathcal{M}$ be the category of symmetric spectra with the *positive* model structure. Then

- For every C-coloured operad P in simplicial sets, the category $\mathrm{Alg}_P(\mathcal{M})$ admits a model structure [Elmendorf-Mandell, 2006].
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Let (L,η) be a homotopical localization on the model category $\mathcal{M}$ of symmetric spectra with the positive model structure.

Let P be a C-coloured operad in simplicial sets and consider the extension of (L,η) over $\mathcal{M}^C$ away from an ideal $J \subseteq C$ relative to P.

Let $\mathbf{X}$ be a P-algebra such that $X(c)$ is cofibrant for each $c \in C$, and let $\eta_{\mathbf{X}} \colon \mathbf{X} \longrightarrow L\mathbf{X}$ be the localization map.

Suppose that $(\eta_{\mathbf{X}})_{c_1} \wedge \cdots \wedge (\eta_{\mathbf{X}})_{c_n}$ is an L-equivalence.

Then there is a map $\xi_{\mathbf{X}} \colon D\mathbf{X} \longrightarrow T\mathbf{X}$ of P-algebras, depending functorially on $\mathbf{X}$, such that:

- *$\mathbf{X}$ and $D\mathbf{X}$ are naturally weakly equivalent as P-algebras;*
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Let $\mathbf{X}$ be a P-algebra such that $X(c)$ is cofibrant for each $c \in C$, and let $\eta_{\mathbf{X}} \colon \mathbf{X} \longrightarrow L\mathbf{X}$ be the localization map.

Suppose that $(\eta_{\mathbf{X}})_{c_1} \wedge \cdots \wedge (\eta_{\mathbf{X}})_{c_n}$ is an L-equivalence.

Then there is a map $\xi_{\mathbf{X}} \colon D\mathbf{X} \longrightarrow T\mathbf{X}$ of P-algebras, depending functorially on $\mathbf{X}$, such that:

- *$\mathbf{X}$ and $D\mathbf{X}$ are naturally weakly equivalent as P-algebras;*
- *$L\mathbf{X}$ and $T\mathbf{X}$ are naturally weakly equivalent as P_∞-algebras;*
- *$\eta_{\mathbf{X}}$ and $\xi_{\mathbf{X}}$ are naturally weakly equivalent as $(\mathrm{Mor}_P)_\infty$-algebras.*

Theorem (CGMV)

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Rectification results for spectra

Symmetric spectra

- Let R be a ring spectrum and M a left R-module. Suppose either that L commutes with suspension or that R is connective. Then $\eta_M : M \longrightarrow LM$ is naturally weakly equivalent to a morphism $\xi_M : DM \longrightarrow TM$ of left R-modules where $DM \simeq M$ as R-modules.
- Same is true for ring spectra, ring maps, algebras over commutative ring spectra . . .
- For every commutative connective ring spectrum R, each connective R-algebra has a Postnikov tower consisting of R-algebras (previously proved by [Lazarev, 2001]).

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Outline of the talk

Generalizations and further results

- Consider operads with values in symmetric spectra.
- Homotopical colocalizations (right Bousfield localizations).
- More general situation. Consider an $\mathcal{E}$-enriched Quillen pair

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : U$$

where $\mathcal{E}$ is a monoidal model category, $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{E}$-enriched monoidal model categories and F is strict monoidal.
If P is a cofibrant C-coloured operad in $\mathcal{E}$ and $\mathbf{X}$ is a P-algebra that is cofibrant in $\mathcal{M}^C$, then $\underline{R}U\underline{L}F\mathbf{X}$ admits a P-algebra structure such that

$$\mathbf{X} \longrightarrow \underline{R}U\underline{L}F\mathbf{X}$$

is a map of P-algebras.
(The case of a left Bousfield localization is $\mathrm{Id} : \mathcal{M} \leftrightarrows \mathcal{M}_{\mathcal{L}} : \mathrm{Id}$.)

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Monoidal model categories

Definition

A model category $\mathcal{E}$ is **monoidal** if it has an associative internal product $\otimes$ with a unit I and an internal hom $\mathrm{Hom}_{\mathcal{E}}(-,-)$, satisfying the *pushout-product axiom*, that is, if $f\colon X \to Y$ and $g\colon U \to V$ are cofibrations in $\mathcal{E}$, then the induced map

$$(X \otimes V) \coprod_{X \otimes U} (Y \otimes U) \longrightarrow Y \otimes V$$

is a cofibration which is a weak equivalence if f or g are.

Back

Simplicial model categories

Definition

A model category $\mathcal{E}$ is **simplicial** if it is enriched, tensored and cotensored over simplicial sets in such a way that Quillen's *SM7 axiom* holds, namely, if $f\colon X \to Y$ is a cofibration and $g\colon U \to V$ is a fibration in $\mathcal{E}$, then the induced map

$$\mathrm{Map}(Y,U) \longrightarrow \mathrm{Map}(Y,V) \times_{\mathrm{Map}(X,V)} \mathrm{Map}(X,U)$$

is a fibration which is a weak equivalence if f or g are.

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