ALGEBRAIC TOPOLOGY, EXERCISE SHEET 10, 04.12.2015

Exercise 1. A pair (X, A) of spaces is a *good pair* if A is a deformation retract of some open neighborhood of itself in X.

(1) If A is a subcomplex of a CW-complex X show that (X, A) is a good pair.
Hint: construct the neighborhood inductively.

(2) For a good pair (X, A) show that the quotient map $q\colon (X, A) \to (X/A, A/A)$ induces an isomorphism

$$q_*\colon H_n(X, A) \to H_n(X/A, A/A) \cong \tilde{H}_n(X/A), \quad n \geq 0.$$

Hint: If A is a retract of its neighborhood V, use the excision on $A \subset V \subset X$ and also on $A/A \subset V/A \subset X/A$.

Exercise 2. Compute the cellular homology of $\mathbb{R}P^n$ for all $n \geq 0$.

Exercise 3. Let $f\colon S^n \to S^n$ be a map of degree p and let $C(f)$ be the mapping cone of f. Show that $C(f)$ has a CW-structure and compute the cellular homology of $C(f)$.

Exercise 4. We say that a partially ordered set $(P, \leq)$ is *directed* if it is non-empty and if for every two elements $i, j \in P$ there is an element $k \in P$ such that $i \leq k$ and $j \leq k$. A *directed system* in a category $\mathcal{C}$ over a directed poset P consists of a family of objects C_i, $i \in P$, and morphisms $f_{ij}\colon C_j \to C_i$ for every pair of elements $i, j \in P$, $i \geq j$, which satisfy the relations:

$$f_{ii} = id_{C_i}\colon C_i \to C_i,\ i \in P, \qquad \text{and} \qquad f_{ij} \circ f_{jk} = f_{ik},\ i \geq j \geq k$$

A *directed colimit* of such a directed system (C_i, f_{ij}) consists of an object $C \in \mathcal{C}$ together with morphisms $g_i\colon C_i \to C$ such that $g_j = g_i \circ f_{ij}$ whenever $i \geq j$. Moreover, it is supposed to be universal with respect to this property in the following sense: whenever there is an object D together with morphisms $h_i\colon C_i \to D$ which also satisfy $h_j = h_i \circ f_{ij}$, $i \geq j$, then there is a unique morphism $h\colon C \to D$ such that $h_i = h \circ g_i$ (draw a diagram!).

(1) Show that given two directed colimits (C, g_i) and (C', g'_i) of the same directed system (C_i, f_{ij}) then there is a unique isomorphism $h\colon C \to C'$ such that $g'_i = h \circ g_i$. This justifies that we talk about *the* directed colimit and we write $C = \mathrm{colim}_{i \in P} C_i$ for it.

(2) We say that a category $\mathcal{C}$ *has directed colimits* if there is a directed colimit of every directed system in $\mathcal{C}$. Show that the categories of abelian groups, of chain complexes, and of topological spaces have directed colimits.
Hint: proceed in exactly the same way as in the case of a *sequence* of spaces/abelian groups/chain complexes.

(3) Show that for every every directed system of chain complexes $(C^i_\bullet, f_{ij})$ and every natural number n there is a natural isomorphism

$$\mathrm{colim}_{i \in P} H_n(C^i) \cong H_n(\mathrm{colim}_{i \in P} C^i).$$

(4) Show that if a space X is a directed colimit of subspaces $Y_i \subseteq X, i \in P$, such that any compact subset $K \subseteq X$ is contained in some Y_i then

$$\mathrm{colim}_{i \in P} H_n(Y_i) \cong H_n(X).$$

(5) Compute the homology groups of the infinity sphere S^∞ and of the infinite projective spaces $\mathbb{C}P^\infty$ and $\mathbb{R}P^\infty$.

Exercise 5.

(1) If X is a CW-complex with countably many cells, prove that there are finite subcomplexes $Y_k, k \in \mathbb{N}$, of X
$$Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y_k \subseteq \ldots$$
such that X is homeomorphic to the union of Y_k. Prove that we can arrange this in such a way that Y_k contains l-cells only for $l \leq k$.

(2) Let X be a CW complex and $\mathcal{F}$ be the family of finite subcomplexes of X partially ordered by inclusion. Note that $\mathcal{F}$ is a directed system (over itself) and show that X is the directed colimit of $\mathcal{F}$. Conclude that there is an isomorphism
$$\mathrm{colim}_{Y \in \mathcal{F}} H_n(Y) \cong H_n(X).$$