LECTURE 2: LOW-DIMENSIONAL IDENTIFICATIONS

The aim of this lecture is to show that the zeroth singular homology group $H_0(X)$ can be constructed from $\pi_0(X)$ in a purely algebraic way. Similarly, for a *connected* pointed space (X, x_0) the first singular homology group $H_1(X)$ can be obtained from $\pi_1(X, x_0)$ by algebraic means only.

We begin with the case of $H_0(X)$. Let us recall from the last lecture that associated to a connected space X we have the following (natural) *augmentation map* ϵ:

$$\epsilon\colon C_0(X) \to \mathbb{Z}\colon \qquad \sum_{i=1}^{k} n_i x_i \mapsto \sum_{i=1}^{k} n_i$$

By our convention, we have $C_0(X) = Z_0(X)$. Since the augmentation map vanishes on all 0-boundaries there is a unique induced group homomorphism ϵ_* as indicated in:

$$\begin{tikzcd}
B_0(X) \arrow[r, "i"] \arrow[dr, bend right, "0"'] \arrow[dr, phantom, "="] & Z_0(X) \arrow[r, "q"] \arrow[d, "\epsilon"] & H_0(X) \arrow[dl, bend left, "\epsilon_*"] \arrow[dl, phantom, "="] \\
& \mathbb{Z} &
\end{tikzcd}$$

Proposition 1. (1) *Let X be a path-connected topological space. Then the augmentation induces a (natural) isomorphism $\epsilon_*\colon H_0(X) \to \mathbb{Z}$.*

(2) *Let X be a topological space. Then we have a (natural) isomorphism $H_0(X) \cong \mathbb{Z}\pi_0(X)$.*

Proof. (1): Let us show that ϵ_* is surjective. Given an integer $n \in \mathbb{Z}$, we can choose an arbitrary point $x \in X$ and consider the 0-cycle $z = nx$. We then have $\epsilon_*([z]) = \epsilon(z) = n$.
We now show that ϵ_* is injective. So let us assume that the homology class $[z]$ represented by $z = \sum_{j=1}^{k} n_j x_j$ lies in the kernel of ϵ_*, i.e., that we have $\sum_{j=1}^{k} n_j = 0$. Since X is connected we can find a point $x_0 \in X$ and paths $\sigma_j\colon \Delta^1 \to X$ such that $\sigma_j(0) = x_0$ and $\sigma_j(1) = x_j$. Let us form the singular 1-chain $\sigma = \sum_{j=1}^{k} n_j \sigma_j$. The following calculation shows that the homology class $[z]$ is trivial:

$$\partial(\sigma) \quad = \quad \sum_{j=1}^{k} n_j (x_j - x_0) \quad = \quad \sum_{j=1}^{k} n_j x_j - \sum_{j=1}^{k} n_j x_0 \quad = \quad \sum_{j=1}^{k} n_j x_j \quad = \quad z$$

(2): The proof of this part is very similar and uses the additivity of singular homology. It is left as an exercise to the reader. $\square$

We now continue with the relation between π_1 and H_1. Let us begin by recalling some basic terminology concerning the manipulation of paths in a space. Two paths $\gamma_0, \gamma_1\colon \Delta^1 \to X$ are called *composable* if they satisfy $\gamma_0(1) = \gamma_1(0)$. If we have two such composable paths γ_0 and γ_1, then we denote their **concatenation** by

$$\gamma_0 * \gamma_1\colon \Delta^1 \to X.$$

This is the path obtained by first running through γ_0 and then through γ_1, and both at a double speed. The **inverse path** γ^{-1} of a path γ is defined by

$$\gamma^{-1}\colon \Delta^1 \to X\colon \qquad t \mapsto \gamma(1-t).$$

Let us next construct the so-called *Hurewicz homomorphism* $\pi_1(X,x_0)\to H_1(X)$ associated to a pointed space (X,x_0). So, let us consider a homotopy class $\alpha=[\gamma]\in\pi_1(X,x_0)$ represented by a pointed loop $\gamma\colon S^1\to X,\ 1\mapsto x_0$. The quotient map

$$e\colon \Delta^1\to S^1=\Delta^1/\partial\Delta^1$$

allows us to associate a singular 1-chain $\gamma\circ e\in C_1(X)$ to such a loop γ:

$$\gamma\circ e\colon \Delta^1\xrightarrow{e} S^1\xrightarrow{\gamma} X$$

Since γ is a loop it is immediate that $\gamma\circ e$ is a 1-cycle and hence represents a homology class. Thus, we could try to associate the homology class $[\gamma\circ e]\in H_1(X)$ to the homotopy class $\alpha=[\gamma]\in\pi_1(X,x_0)$. In order to see that this assignment is well-defined we have to check the following: if two loops γ_0 and γ_1 are homotopic relative to the base point, then the singular 1-cycles $\gamma_0\circ e$ and $\gamma_1\circ e$ are *homologous*, i.e., their difference is a boundary. But a 2-chain realizing this can be constructed from such a pointed homotopy $H\colon\Delta^1\times\Delta^1\to X$. In fact, the homotopy satisfies the relations

$$H(0,-)=\gamma_0\circ e,\quad H(1,-)=\gamma_1\circ e,\quad\text{and}\quad H(t,0)=H(t,1)=x_0,\ t\in\Delta^1.$$

If we set $w(t)=H(t,t)$ then the above relations of the homotopy can be graphically depicted by

$$\begin{tikzcd} {} \arrow[r, "\kappa_{x_0}"] & {} \\ {} \arrow[u, "\gamma_0\circ e"] \arrow[r, "\kappa_{x_0}"'] \arrow[ur, "w"] & {} \arrow[u, "\gamma_1\circ e"'] \end{tikzcd}$$

where κ_{x_0} denotes the constant map with value x_0. The restrictions of the homotopy H to the upper left and the lower right 2-simplex give us maps $\sigma_1\colon\Delta^2\to X$ and $\sigma_2\colon\Delta^2\to X$ respectively. We can conclude by calculating the boundary of $\sigma_H=\sigma_1-\sigma_2\in C_2(X)$:

$$\begin{aligned}\partial(\sigma_H) &= \partial\sigma_1-\partial\sigma_2\\ &= (\kappa_{x_0}-w+\gamma_0\circ e)-(\gamma_1\circ e-w+\kappa_{x_0})\\ &= \gamma_0\circ e-\gamma_1\circ e\end{aligned}$$

Thus, we obtain $\gamma_0\circ e\sim\gamma_1\circ e$ as intended.

Proposition 2. *If (X,x_0) is a pointed topological space then the assignment*

$$h\colon\pi_1(X,x_0)\to H_1(X)\colon\quad [\gamma]\mapsto[\gamma\circ e]$$

defines a (natural) homomorphism of groups, the ***Hurewicz homomorphism*** *of (X,x_0).*

Proof. By the above discussion the map of sets $h\colon\pi_1(X,x_0)\to H_1(X)$ is well-defined. Let us now check that it is a group homomorphism. The homotopy class of the constant loop κ_{S^1,x_0} at x_0 is the neutral element of $\pi_1(X,x_0)$. It's image under h is the homology class of the 1-cycle $\kappa_{S^1,x_0}\circ e=\kappa_{\Delta^1,x_0}$ where the latter denotes the constant path at x_0. But the boundary of the constant 2-simplex $\kappa_{\Delta^2,x_0}\colon\Delta^2\to X$ is given by

$$\partial(\kappa_{\Delta^2,x_0})\quad=\quad\kappa_{\Delta^1,x_0}-\kappa_{\Delta^1,x_0}+\kappa_{\Delta^1,x_0}\quad=\quad\kappa_{\Delta^1,x_0}.$$

We thus deduce that $h(1)=0$ as intended.

For the compatibility with the group structures let us consider two homotopy classes $\alpha_1=[\gamma_1]$ and $\alpha_2=[\gamma_2]$ in $\pi_1(X,x_0)$. Note that the paths $\gamma_1\circ e$, $\gamma_2\circ e$, and $(\gamma_1*\gamma_2)\circ e$ can be used to define a map from the (geometric) boundary of Δ^2 to X. This is indicated in the next diagram (strictly speaking we have to use some linear reparametrizations of the paths but we will ignore this issue):

$$\begin{tikzcd} & \cdot \arrow[dr, "\gamma_2\circ e"] & \\ \cdot \arrow[ur, "\gamma_1\circ e"] \arrow[rr, "(\gamma_1*\gamma_2)\circ e"'] & & \cdot \end{tikzcd}$$

We can extend this to a continuous map $\sigma\colon \Delta^2 \to X$ which is constant along the 'vertical lines'. But this singular 2-simplex σ implies the intended relation $h(\alpha_1\alpha_2) = h(\alpha_1) + h(\alpha_2)$. □

As a preparation for the main theorem of this lecture, let us collect a few convenient facts.

Lemma 3. *Let X be a topological space.*

1. *A constant path $\kappa\colon \Delta^1 \to X$ is a boundary.*
2. *If two paths γ_0, $\gamma_1\colon \Delta^1 \to X$ are homotopic relative to the boundary then $\gamma_0 \sim \gamma_1$.*
3. *If $\gamma_0, \gamma_1\colon \Delta^1 \to X$ are composable then $\gamma_0 + \gamma_1 - \gamma_0 * \gamma_1$ is a boundary.*
4. *If $\gamma_0\colon \Delta^1 \to X$ is a path then $\gamma_0 + \gamma_0^{-1}$ is a boundary.*

Proof. (1): We proved this already when we showed that the Hurewicz homomorphism h preserves neutral elements.
(2): This was already proved when we checked that the Hurewicz homomorphism is well-defined.
(3): We established the corresponding result for loops when we showed that h is multiplicative. But that proof never used that we considered loops as opposed to more general paths.
(4): This is a combination of the previous results:

$$\gamma_0 + \gamma_0^{-1} \quad \overset{iii)}{\sim} \quad \gamma_0 * \gamma_0^{-1} \quad \overset{ii)}{\sim} \quad \kappa_{\gamma_0(0)} \quad \overset{i)}{\sim} \quad 0$$ □

The Hurewicz homomorphism is a group homomorphism whose target is an abelian group while the source itself is not necessarily abelian. Let us shortly abstract from this specific situation and consider a group G, an abelian group A, and a group homomorphism $f\colon G \to A$. The group G will be written multiplicatively while A will be written additively. Then, for two elements g_1 and g_2 of G we obtain:

$$f(g_1g_2) \quad = \quad f(g_1) + f(g_2) \quad = \quad f(g_2) + f(g_1) \quad = \quad f(g_2g_1)$$

Hence, all elements of the form $g_1g_2g_1^{-1}g_2^{-1} \in G$ are sent to the neutral element $0 \in A$. Such an element is called a **commutator** and we will denote by $[G, G]$ the subgroup of G *generated* by the commutators:

$$[G, G] = \langle\{g_1g_2g_1^{-1}g_2^{-1} \mid g_1, g_2 \in G\}\rangle \subseteq G$$

Lemma 4. *In the above notation we have the following facts:*

1. *The subset $[G, G]$ is a normal subgroup of G and the quotient group $G^{\mathrm{ab}} = G/[G, G]$ is abelian. The subgroup $[G, G]$ is the* ***commutator subgroup*** *of G and the quotient group $G^{\mathrm{ab}} = G/[G, G]$ is called the* ***abelianization*** *of G.*
2. *The pair (G^{ab}, q) consisting of the abelianization G^{ab} and the canonical group homomorphism $q\colon G \to G^{\mathrm{ab}}$ has the following universal property: Given a further pair (A, r) consisting of an abelian group A and a group homomorphism $r\colon G \to A$ then there is unique group homomorphism $g\colon G^{\mathrm{ab}} \to A$ such that $g \circ q = r$.*

Proof. Exercise. □

More diagrammatically, this universal property can be visualized as follows:

$$\begin{tikzcd} G \arrow[r, "q"] \arrow[dr, bend right, "\forall r"'] & G^{ab} \arrow[d, dashed, "g"] \\ \arrow[ur, phantom, "="] & A \end{tikzcd} \qquad \begin{tikzcd} G^{ab} \arrow[d, "\exists! g"] \\ A \end{tikzcd}$$

Note again that the two parts of this diagram take place in different categories: the diagram on the left lives in the category of groups while the one on the right is a diagram in the category of abelian groups. Thus, we can think of the abelianization as 'the' best approximation of an arbitrary group by an abelian group.

Let us now return to the context of the Hurewicz homomorphism

$$h\colon \pi_1(X, x_0) \to H_1(X)$$

associated to a pointed space (X, x_0). The above lemma implies that h factors uniquely through a homomorphism $\tilde{h}\colon \pi_1(X, x_0)^{\mathrm{ab}} \to H_1(X)$, i.e., we have

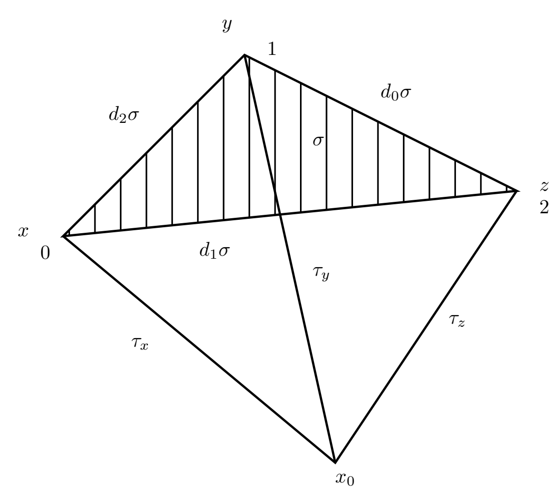
$$h\colon \pi_1(X, x_0) \to \pi_1(X, x_0)^{\mathrm{ab}} \xrightarrow{\tilde{h}} H_1(X).$$

Theorem 5. *Let (X, x_0) be a* path-connected *pointed space. Then the (natural) group homomorphism $\tilde{h}\colon \pi_1(X, x_0)^{\mathrm{ab}} \to H_1(X)$ induced by the Hurewicz homomorphism is an isomorphism.*

Proof. We have seen that h induces a well-defined group homomorphim $\tilde{h}$, since $H_1(X)$ is an abelian group. To construct a map in the opposite direction, *choose* first for each $x \in X$ a path τ_x from the basepoint x_0 to x. Then, to every path α in X from x to y we can associate a loop $\varphi(\alpha)$ at x_0 by defining $\varphi(\alpha) = \tau_x * \alpha * \tau_y^{-1}$. This induces a homomorphism $\varphi\colon C_1(X) \to \pi_1(X, x_0)^{\mathrm{ab}}$. Moreover, φ induces a group homomorphism

$$\tilde{\varphi}\colon H_1(X) \longrightarrow \pi_1(X, x_0)^{\mathrm{ab}}$$

because it vanishes on the image of $C_2(X) \xrightarrow{\partial} C_1(X)$. Indeed, if $\sigma \in C_2(X)$, then we can define a homotopy from $d_2\sigma * d_0\sigma$ to $d_1\sigma$:



so that
$$\tau_x * d_2\sigma * \tau_y^{-1} * \tau_y * d_0\sigma * \tau_z^{-1} \simeq \tau_x * d_2\sigma * d_0\sigma * \tau_z^{-1} \simeq \tau_x * d_1\sigma * \tau_z^{-1}.$$
That is, $\tilde{\varphi}(d_2\sigma) \cdot \tilde{\varphi}(d_0\sigma) = \tilde{\varphi}(d_1\sigma)$ in $\pi_1(X, x_0)$, so $\tilde{\varphi}(\partial\sigma) = 0$ in $\pi_1(X, x_0)^{\mathrm{ab}}$.

Now we will prove that $\tilde{h}$ and $\tilde{\varphi}$ are mutually inverses. Remember that we choose τ_x to be a path from x_0 to x. Let's agree to choose τ_x to be the constant path κ_{x_0} if $x = x_0$. Then clearly, for a loop α based at x_0 we have that
$$(\tilde{\varphi} \circ \tilde{h})(\alpha) = \kappa_{x_0} * \alpha * \kappa_{x_0} \simeq \alpha.$$
On the other hand, let $\alpha \in C_1(X)$. Then we have that $(\tilde{h} \circ \varphi)(\alpha) = \tilde{h}(\tau_x * \alpha * \tau_y^{-1}) = \tau_x + \alpha - \tau_y = \alpha + \tau_x - \tau_y$. Let's take a class $[\beta]$ in $H_1(X)$ represented by $\beta = \sum n_i \alpha_i \in C_1(X)$ with $\partial\beta = 0$. Then
$$(\tilde{h} \circ \varphi)(\beta) = \sum n_i \alpha_i + \sum n_i(\tau_{x_i} - \tau_{y_i}) = \beta + \sum n_i(\tau_{x_i} - \tau_{y_i}) = \beta,$$
the latter because $\partial\beta = \sum n_i(x_i - y_i) = 0$. Thus $(\tilde{h} \circ \tilde{\varphi})([\beta]) = [\beta]$ also, which concludes the proof. $\square$

The theorem allows us to do some first calculations for several spaces, if you already know their fundamental groups. Let $\mathbb{R}P^n$ denote the real projective space of dimension n and let $\mathbb{T}^n$ denote the n-dimensional torus, i.e., $\mathbb{T}^n$ is an n-fold product of 1-spheres S^1.

Corollary 6. *The Hurewicz homomorphism induces the following identifications:*

(1) $H_1(S^1) \cong \mathbb{Z}$,
(2) $H_1(S^n) \cong 0, \quad n \geq 2$,
(3) $H_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}, \quad n \geq 2$,
(4) $H_1(\mathbb{T}^n) \cong \mathbb{Z}^n$.