LECTURE 3: RELATIVE SINGULAR HOMOLOGY

In this lecture we want to cover some basic concepts from homological algebra. These prove to be very helpful in our discussion of singular homology. The following definition abstracts the key ingredients which were necessary to introduce the singular homology groups of a topological space.

Definition 1. A **chain complex** (of abelian groups) C consists of abelian groups $C_n,\ n \geq 0$, together with group homomorphisms $\partial\colon C_n \to C_{n-1},\ n \geq 1$, such that

$$\partial \circ \partial = 0\colon C_n \to C_{n-2}, \quad n \geq 2.$$

The homomorphisms ∂ are called **boundary homomorphisms** or **differentials**.

Thus, a chain complex of abelian groups can be depicted as

$$\ldots \stackrel{\partial}{\to} C_{n+1} \stackrel{\partial}{\to} C_n \stackrel{\partial}{\to} \ldots \stackrel{\partial}{\to} C_1 \stackrel{\partial}{\to} C_0.$$

The elements of C_n are said to be of **degree** n and are called n-**chains** of C. Given such a chain complex C, we call

$$Z_n = Z_n(C) = \ker(\partial\colon C_n \to C_{n-1}) \subseteq C_n$$

the subgroup of n-**cycles** and

$$B_n = B_n(C) = \mathrm{im}(\partial\colon C_{n+1} \to C_n)$$

the subgroup of n-**boundaries**. By convention, we set $Z_0 = C_0$, i.e., we define all 0-chains to be 0-cycles. The fundamental relation $\partial \circ \partial = 0$ implies that we have an inclusion $B_n \subseteq Z_n$ for all $n \geq 0$. The n-th **homology group** $H_n = H_n(C)$ of a chain complex C is the quotient group

$$H_n(C) = Z_n(C)/B_n(C).$$

Elements of $H_n(C)$ are cosets $z_n + B_n(C)$ which satisfy $z_n \in Z_n(C)$. Such an element is also denoted by $[z_n]$ and is called the **homology class** of degree n represented by z_n.

Example 2. Associated to a topological space $X \in \mathsf{Top}$ we earlier constructed the singular chain complex $C(X) = C_*(X)$. In degree n it is given by the free abelian group generated by the singular n-simplices $\sigma\colon \Delta^n \to X$ in X. The singular boundary operator $C_n(X) \to C_{n-1}(X)$ is induced by the face maps $\Delta^{n-1} \to \Delta^n$. By definition, the homology groups of this chain complex are the singular homology groups of our given space.

Using the universal property of free abelian groups generated by a set we saw already that the assignment $X \mapsto C_n(X)$ is functorial. But also the singular boundary operators behave nicely with respect to maps of spaces. Let us consider topological spaces X and Y and a continuous map $f\colon X \to Y$ between them. Given a singular n-simplex $\sigma\colon \Delta^n \to X$ then the associativity of the composition law for maps of spaces implies the relation

$$(f \circ \sigma) \circ d^i = f \circ (\sigma \circ d^i)\colon \Delta^{n-1} \to Y, \qquad n \geq 1, \quad 0 \leq i \leq n.$$

By linearity, this implies that the square

$$\begin{array}{ccc} C_n(X) & \xrightarrow{d_i} & C_{n-1}(X) \\ {\scriptstyle C_n(f)}\downarrow & & \downarrow{\scriptstyle C_{n-1}(f)} \\ C_n(Y) & \xrightarrow[d_i]{} & C_{n-1}(Y) \end{array}$$

commutes. Since the singular boundary operators $C_n(X) \to C_{n-1}(X)$ and $C_n(Y) \to C_{n-1}(Y)$ are obtained from these face maps by forming alternating sums this implies that the various $(C_n(f))_{n \geq 0}$ assemble into a *chain map* $C(X) \to C(Y)$ in the following sense.

Definition 3. Let C and D be chain complexes of abelian groups. A **chain map** $f\colon C \to D$ consists of group homomorphisms $f_n\colon C_n \to D_n,\ n \geq 0$, which commute with the boundary operators in the sense that

$$\partial \circ f_n = f_{n-1} \circ \partial\colon C_n \to D_{n-1}, \qquad n \geq 1.$$

Thus, if we depict chain complexes by diagrams as above, then a morphism of chain complexes gives us a 'commutative ladder' as described by the next diagram:

$$\begin{array}{ccccccccccccc} \cdots & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & C_1 & \xrightarrow{\partial} & C_0 & \qquad & C \\ & & {\scriptstyle f_{n+1}}\downarrow & & {\scriptstyle f_n}\downarrow & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ \cdots & \xrightarrow[\partial]{} & D_{n+1} & \xrightarrow[\partial]{} & D_n & \xrightarrow[\partial]{} & \cdots & \xrightarrow[\partial]{} & D_1 & \xrightarrow[\partial]{} & D_0 & & D \end{array}$$

Given chain maps $f\colon C \to D$ and $g\colon D \to E$ then it is immediate to see that the degreewise compositions $g_n \circ f_n\colon C_n \to E_n$ assemble into a chain map $g \circ f\colon C \to E$. It is similarly obvious that the identity maps $C_n \to C_n$ assemble to a chain map $\mathrm{id}\colon C \to C$. Moreover, it follows from the definition that a chain map sends cycles to cycles and boundaries to boundaries.

Lemma 4. (1) *Chain complexes of abelian groups together with chain maps assemble into a category which is denoted by* $\mathsf{Ch} = \mathsf{Ch}(\mathbb{Z})$.

(2) *The formation of cycles, boundaries, and homology in a fixed dimension is functorial, i.e., for all* $n \geq 0$ *the assignments*

$$C \mapsto Z_n(C), \qquad C \mapsto B_n(C), \qquad \textit{and} \qquad C \mapsto H_n(C)$$

extend to functors $Z_n, B_n, H_n\colon \mathsf{Ch} \to \mathsf{Ab}$.

Proof. This is straightforward. Let us only mention that, given a chain map $f\colon C \to D$, then the induced map in homology is defined by

$$f_* = H_n(f)\colon H_n(C) \to H_n(D)\colon \quad [c_n] \mapsto [f_n(c_n)].$$

We leave it to the reader to check the details. □

Corollary 5. *The singular chain group functors* $C_n\colon \mathsf{Top} \to \mathsf{Ab}$ *and the singular boundary operators together define a singular chain complex functor* $C\colon \mathsf{Top} \to \mathsf{Ch}$. *In particular, there is a singular homology functor* $H_n\colon \mathsf{Top} \to \mathsf{Ab}$ *defined as the composition*

$$H_n\colon \mathsf{Top} \xrightarrow{C} \mathsf{Ch} \xrightarrow{H_n} \mathsf{Ab}.$$

A formal consequence of having a functor $H_n\colon \mathsf{Top} \to \mathsf{Ab}$ is that singular homology groups form *topological invariants*, i.e., *homeomorphic* spaces have isomorphic singular homology groups (more precisely, any such homeomorphism induces an isomorphism in singular homology). There is a much stronger statement though: even spaces which are only *homotopy equivalent* have canonically isomorphic homology groups. We refer to this statement by saying that 'singular homology is *homotopy-invariant*'. A proof of this important result will be given later in the course.

Let us be given a (not necessarily bounded) sequence of abelian groups A_n together with group homomorphisms $\partial\colon A_n \to A_{n-1}$ for all n. Thus, we have a diagram of abelian groups as follows:

$$\ldots \xrightarrow{\partial} A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1} \xrightarrow{\partial} \ldots$$

Such a sequence is a *chain complex* if we always have $\partial\circ\partial = 0$. In light of this more general definition, we sometimes refer to objects considered in Definition 1 as *non-negative* or *non-negatively graded* chain complexes. We say that such a sequence is **exact at** A_n if we have the equality of subgroups

$$\mathrm{im}(\partial\colon A_{n+1} \to A_n) = \ker(\partial\colon A_n \to A_{n-1}) \subseteq A_n.$$

Such a sequence is called an **exact sequence** if it is exact at all A_n. Note that such an exact sequence is, in particular, a chain complex since the relation $\partial \circ \partial = 0$ is equivalent to the fact that we have inclusions of subgroups

$$\mathrm{im}(\partial\colon A_{n+1} \to A_n) \subseteq \ker(\partial\colon A_n \to A_{n-1})$$

for all n. A chain complex is exact if we also have inclusions in the opposite directions, namely

$$\mathrm{im}(\partial\colon A_{n+1} \to A_n) \supseteq \ker(\partial\colon A_n \to A_{n-1})$$

for all n. Note also that this is equivalent to the vanishing of all homology groups.

A particularly important special case is given by so-called **short exact sequences** of abelian groups. By definition, this is an exact sequence of abelian groups of the form

$$0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0.$$

The following lemma makes precise what kind of structure is encoded by such a short exact sequence.

Lemma 6. (1) *The sequence of abelian groups* $0 \to A' \xrightarrow{i} A$ *is exact at* A' *if and only if* i *is injective.*

(2) *The sequence of abelian groups* $A \xrightarrow{p} A'' \to 0$ *is exact at* A'' *if and only if* p *is surjective.*

(3) *The sequence of abelian groups* $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ *is exact if and only if* i *is injective,* p *is surjective, and we have* $\mathrm{im}(i) = \ker(p)$.

(4) *The sequence of abelian groups* $0 \to A \xrightarrow{f} B \to 0$ *is exact if and only if* f *is an isomorphism.*

Thus, a short exact sequence basically only encodes an inclusion of a subgroup together with its quotient map. Nevertheless, this notion proves to be very useful. It is straightforward to extend it to chain complexes. A **short exact sequence of chain complexes** is a diagram of chain complexes and chain maps

$$0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$$

which induces a short exact sequence of abelian groups

$$0 \to C'_n \xrightarrow{i_n} C_n \xrightarrow{p_n} C''_n \to 0$$

in each degree. As in the case of abelian groups, also short exact sequences of chain complexes basically only encode the inclusion of a subcomplex together with the corresponding projection to the quotient complex. Let us make these notions precise. Given a chain complex C, a **subcomplex** of C is given by a family of subgroups $C'_n \subseteq C_n$ such that the boundary operator $\partial\colon C_n \to C_{n-1}$ restricts to a homomorphism $C'_n \to C'_{n-1}$ for all n. In this situation the family of subgroups C'_n can be uniquely turned into a chain complex C' such that the inclusions of the subgroups $C'_n \subseteq C_n$ assemble into a morphism of chain complexes $i\colon C' \to C$ (do this as an exercise!). In particular, given a chain map $f\colon C \to D$, the kernels

$$\ker(f_n\colon C_n \to D_n) \subseteq C_n$$

assemble into a chain complex $\ker(f)$, the **kernel** of f. As in the case of abelian groups, the kernel of a chain map comes with a canonical map $\ker(f) \to C$ which satisfies an appropriate universal property.

Exercise 7. Define the **image** $\mathrm{im}(f) \in \mathsf{Ch}$ of a chain map $f\colon C \to D$. Show that any chain map $f\colon C \to D$ factors as a composition

$$f\colon C \to \mathrm{im}(f) \to D.$$

We already discussed subobjects of chain complexes, namely subcomplexes. We now turn to the dual concept which is given by quotient complexes.

Lemma 8. *Let $i\colon C' \to C$ be the inclusion of a subcomplex and let $p_n\colon C_n \to C''_n = C_n/C'_n$ be the (levelwise) quotient map. Then there is a unique way to turn the $(C''_n)_{n\geq 0}$ into a chain complex C'' such that the $(p_n)_{n\geq 0}$ assemble into a chain map $p\colon C \to C''$. Moreover, it follows that the sequence $0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$ is exact.*

Proof. This is left to the reader as an exercise. □

The chain complex C'' constructed in the lemma is the **quotient complex** associated to the inclusion $i\colon C' \to C$. Similarly, using pointwise definitions, once can construct the **cokernel** $\mathrm{cok}(f)$ of a chain map $f\colon C \to D$. This is a chain complex which comes with a chain map $D \to \mathrm{cok}(f)$ and this pair satisfies the usual universal property.

As a punchline of this lengthy discussion you should take away that basic constructions like kernels, cokernels, subobjects and quotient objects can be extended from abelian groups to chain complexes by pointwise definitions. And these extended definitions still 'behave as expected'. However, it is tremendously important to note that the formation of homology is *not* compatible with these constructions. Let us be more specific about this and consider a short exact sequence

$$0 \to C' \to C \to C'' \to 0$$

of chain complexes. Since homology is functorial we obtain induced maps

$$H_n(C') \to H_n(C) \to H_n(C'')$$

and one might wonder if there are short exact sequences

$$0 \to H_n(C') \to H_n(C) \to H_n(C'') \to 0$$

in each dimension n. In general, this turns out to be an unreasonable demand but there is the following proposition.

Proposition 9. *Let $0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$ be a short exact sequence of chain complexes of abelian groups. Then there is a (natural)* ***connecting homomorphism***

$$\delta_n\colon H_n(C'') \to H_{n-1}(C')$$

such that the following sequence is exact:

$$\ldots \to H_{n+1}(C'') \stackrel{\delta_{n+1}}{\to} H_n(C') \stackrel{i_*}{\to} H_n(C) \stackrel{p_*}{\to} H_n(C'') \stackrel{\delta_n}{\to} H_{n-1}(C') \to \ldots$$

Proof. Let us begin with the construction of the connecting homomorphism δ_n for a given n. For this purpose let us draw the part of the short exact sequence which is relevant in that situation:

$$\begin{array}{ccccc} & & C_n & \xrightarrow{p_n} & C''_n \\ & & \big\downarrow{\scriptstyle \partial_n} & & \\ C'_{n-1} & \xrightarrow[i_{n-1}]{} & C_{n-1} & & \end{array}$$

Let us consider a homology class $\omega \in H_n(C'')$ and let us represent it by an n-cycle $z''_n \in Z_n(C'')$. By the surjectivity of p_n we can find an n-chain $c_n \in C_n$ such that $p_n(c_n) = z''_n$. For the image $\partial_n(c_n)$ of c_n under the boundary operator we calculate $p_{n-1}(\partial_n(c_n)) = \partial_n(p_n(c_n)) = \partial_n(z''_n) = 0$ since z''_n is a cycle. Thus, the fact that we have a short exact sequence in level $n-1$ implies that there is a unique $z'_{n-1} \in C'_{n-1}$ such that $i_{n-1}(z'_{n-1}) = \partial_n(c_n)$. This $(n-1)$-chain z'_{n-1} is, in fact, a cycle as the following calculation combined with the injectivity of i_{n-2} implies:

$$i_{n-2}(\partial_{n-1}(z'_{n-1})) \quad = \quad \partial_{n-1}(i_{n-1}(z'_{n-1})) \quad = \quad \partial_{n-1}(\partial_n(c_n)) \quad = \quad 0$$

Thus, z'_{n-1} represents a homology class $[z'_{n-1}] \in H_{n-1}(C')$. We define the connecting homomorphism δ_n as follows:

$$\delta_n \colon H_n(C'') \to H_{n-1}(C') \colon \qquad [z''_n] \mapsto [z'_{n-1}]$$

In the construction of the connecting homomorphism some choices were made. We leave it to the reader to check that our definition is well-defined and that δ_n is a group homomorphism.

Let us now turn to the exactness issues. However, we will only give the proof of the exactness at $H_{n-1}(C')$. In order to establish the inclusion $\mathrm{im}(\delta_n) \subseteq \ker(i_* \colon H_{n-1}(C'_*) \to H_{n-1}(C_*)$ we will still use the notation from the construction of δ_n. But his inclusion is immediate since

$$i_*(\delta_n[z''_n]) \quad = \quad i_*[z'_{n-1}] \quad = \quad [i_{n-1}(z'_{n-1})] \quad = \quad [\partial_n(c_n)] \quad = \quad 0.$$

Let us assume conversely that we have a homology class $\omega' \in H_{n-1}(C')$ such that $i_*(\omega') = 0$. Thus, if we represent ω' by z'_{n-1} we have $i_{n-1}(z'_{n-1}) = \partial_n(c_n)$ for some $c_n \in C_n$. The image $z''_n = p_n(c_n)$ of c_n under p_n is a cycle since:

$$\partial_n(z''_n) \quad = \quad \partial_n(p_n(c_n)) \quad = \quad p_{n-1}(\partial_n(c_n)) \quad = \quad p_{n-1}(i_{n-1}(z'_{n-1})) \quad = \quad 0$$

Hence, we can form the homology class $\omega'' = [z''_n] \in H_n(C'')$ and it follows from the construction of the connecting homomorphism and the fact that it is well-defined that we have $\omega' = \delta_n(\omega'')$.

The proofs of the exactness at $H_n(C)$ and $H_n(C'')$ are similar and are left to the reader as an exercise. □

This long exact sequence is referred to as the **long exact homology sequence** induced by a short exact sequence of chain complexes. It is a very powerful tool – both for theoretical and computational purposes. In the case of nonnegative chain complexes, this long exact sequence ends on

$$\ldots \to H_2(C'') \stackrel{\delta_2}{\to} H_1(C') \to H_1(C) \to H_1(C'') \stackrel{\delta_1}{\to} H_0(C') \to H_0(C) \to H_0(C'') \to 0.$$

The final aim in this lecture is to apply Proposition 9 in a topological context. More precisely, let us consider a pair of spaces (X, A), i.e., a topological space X together with a subspace $A \subseteq X$. Let us denote the inclusion of the subspace by $i \colon A \to X$. It is immediate that the induced map of singular chain complexes $i_*\colon C(A) \to C(X)$ is levelwise injective and we can hence consider the quotient complex $C(X,A) = C(X)/C(A)$. (To be completely precise we should form the quotient by the subcomplex $i_*(C(A)) \subseteq C(X)$, but we allow ourselves to blur the distinction between the *isomorphic* complexes $C(A)$ and $i_*(C(A))$.)

Definition 10. The **relative singular chain complex** $C(X,A)$ of a pair of spaces (X,A) is the quotient complex

$$C(X,A) = C(X)/C(A).$$

The homology of $C(X,A)$ is the **relative singular homology** of the pair (X,A), and it will be denoted by

$$H_n(X,A) = H_n(C(X,A)), \quad n \geq 0.$$

Thus, if we have a pair of spaces (X,A), then there is by definition a short exact sequence of chain complexes

$$0 \to C(A) \to C(X) \to C(X,A) \to 0.$$

An application of Proposition 9 implies immediately the following result.

Corollary 11. *Let (X,A) be a pair of spaces. Then there are (natural) connecting homomorphisms*

$$\delta_n\colon H_n(X,A) \to H_{n-1}(A), \quad n \geq 1,$$

such that the following sequence is exact

$$\ldots \to H_2(X,A) \xrightarrow{\delta_2} H_1(A) \to H_1(X) \to H_1(X,A) \xrightarrow{\delta_1} H_0(A) \to H_0(X) \to H_0(X,A) \to 0.$$

This is the **long exact homology sequence** associated to the pair of spaces (X,A).

Example 12. Let X be a space, $i\colon A \to X$ the inclusion of a subspace, and let $x_0 \in X$ be a point.

(1) The homology of the empty space is trivial in all dimensions. Thus, the inclusion $j\colon \emptyset \to X$ induces isomorphisms

$$H_n(X) \xrightarrow{\cong} H_n(X,\emptyset), \quad n \geq 0.$$

(2) The inclusion i induces isomorphisms in homology $i_*\colon H_n(A) \xrightarrow{\cong} H_n(X)$, $n \geq 0$, if and only if all relative homology groups $H_n(X,A)$, $n \geq 0$, vanish, i.e.,

$$H_n(X,A) \cong 0.$$

In particular, the homology groups $H_n(X,X)$ vanish for all $n \geq 0$.

(3) If the maps $H_n(A) \to H_n(X)$ are injective for $n \geq 0$ or if the maps $H_n(X) \to H_n(X,A)$ are surjective for all $n \geq 1$, then we have short exact sequences

$$0 \to H_n(A) \to H_n(X) \to H_n(X,A) \to 0, \quad n \geq 0.$$

In fact, in both cases the connecting homomorphisms are trivial and the result follows. This applies, in particular, to the case of the inclusion of a retract.

(4) Let us consider the inclusion $k\colon \{x_0\} \to X$. Our earlier calculation $H_n(x_0) \cong 0$ for all $n \geq 1$ together with the long exact homology sequence for the pair (X,x_0) implies that we have (natural) isomorphisms $H_n(X) \to H_n(X,x_0), n \geq 2$. Observe that $H_0(x_0) \to H_0(X)$ is injective since it is just the $\mathbb{Z}$-linear extension of the inclusion of the path-component of x_0 in $\pi_0(X)$. Hence the connecting homomorphism $H_1(X,x_0) \to H_0(x_0)$ is trivial and we obtain a short exact sequence $0 \to H_1(x_0) \to H_1(X) \to H_1(X,x_0) \to 0$. We conclude from this discussion that there are natural isomorphisms

$$H_n(X) \xrightarrow{\cong} H_n(X,x_0), \quad n \geq 1.$$

More generally, if A has trivial homology in positive degrees and if the map $\pi_0(A) \to \pi_0(X)$ is injective, then we have (natural) isomorphisms $H_n(X) \to H_n(X, A)$ for all $n \geq 1$.

Let us close this lecture with a short comment on the notions introduced here. With our main example of the singular chain complex associated to a space in mind, we decided to restrict attention to chain complexes of abelian groups. However, the concepts of chain complexes, chain maps, homology, and exactness make perfectly well sense in other contexts. For example, given a field k, we could consider the category $\mathsf{Ch}(k)$ of chain complexes of vector spaces over k. The only difference is that in this case all maps in sight are supposed to be k-linear. More generally, given a commutative ring R, we can consider the category $\mathsf{Ch}(R)$ of chain complexes of R-modules. From this perspective, we considered the case of the ring $R = \mathbb{Z}$ since the categories of abelian groups and $\mathbb{Z}$-modules are isomorphic.