LECTURE 5: HOMOTOPY INVARIANCE OF SINGULAR HOMOLOGY

The aim of this lecture is to show that singular homology is homotopy-invariant. In the last lecture we showed that the singular homology of a contractible space is trivial in positive dimensions. This result (applied to products of simplices) is essential to our approach to the homotopy-invariance of singular homology.

Given X, $Y \in \mathsf{Top}$ we want to relate the singular chain complexes $C(X)$, $C(Y)$, and $C(X \times Y)$ to each other and this relation is to be natural in the spaces. Recall that given maps $f\colon X \to X'$ and $g\colon Y \to Y'$ then there is the product map $(f,g)\colon X \times Y \to X' \times Y'$ which sends (x,y) to $(f(x), g(y))$.

Theorem 1. *Associated to topological spaces X and Y there are bilinear maps*

$$\times\colon C_p(X) \times C_q(Y) \to C_{p+q}(X \times Y)\colon \quad (c,d) \mapsto c \times d$$

for all p, $q \geq 0$, called cross product maps*, with the following properties:*
i) *For $x \in X$, $y \in Y$, $\sigma\colon \Delta^p \to X$, and $\tau\colon \Delta^q \to Y$ we have:*

$$x \times \tau\colon \Delta^q \cong \Delta^0 \times \Delta^q \overset{(x,\tau)}{\to} X \times Y \qquad \textit{and} \qquad \sigma \times y\colon \Delta^p \cong \Delta^p \times \Delta^0 \overset{(\sigma,y)}{\to} X \times Y$$

ii) *The cross product is natural in X and Y, i.e., for maps $f\colon X \to X'$ and $g\colon Y \to Y'$ we have:*

$$(f,g)_*(c \times d) \quad = \quad f_*(c) \times g_*(d) \qquad \textit{in} \qquad C_{p+q}(X' \times Y')$$

iii) *The boundary ∂ is a* derivation *with respect to $\times$ in the sense that for $c \in C_p(X)$ and $d \in C_q(Y)$ we have:*

$$\partial(c \times d) \quad = \quad \partial(c) \times d \ + \ (-1)^p c \times \partial(d) \qquad \textit{in} \qquad C_{p+q-1}(X \times Y)$$

Proof. The proof will be given by induction on $n = p + q$. By condition i) there is no choice if $p = 0$ or $q = 0$. So let us assume we already constructed a cross product map for all $p' + q' \leq n-1$ and let us consider $p + q = n$ with $p > 0$, $q > 0$. Let us begin with the special case where $X = \Delta^p$ and $Y = \Delta^q$. In this case there are the special singular simplices $\sigma = id_p\colon \Delta^p \to \Delta^p$ and $\tau = id_q\colon \Delta^q \to \Delta^q$. Whatever $id_p \times id_q \in C_{p+q}(\Delta^p \times \Delta^q)$ will be, condition iii) forces its boundary to be:

$$\partial(id_p \times id_q) \quad = \quad \partial(id_p) \times id_q \ + \ (-1)^p id_p \times \partial(id_q)$$

Using this relation again and our induction assumption we can calculate the boundary of the expression on the right-hand-side to be zero, i.e.,:

$$\partial(id_p) \times id_q \ + \ (-1)^p id_p \times \partial(id_q) \in Z_{p+q-1}(\Delta^p \times \Delta^q)$$

Since $p+q \geq 2$ we can use that $H_{p+q-1}(\Delta^p \times \Delta^q) \cong 0$ to deduce that this cycle must be a boundary of some chain in $C_{p+q}(\Delta^p \times \Delta^q)$. Let us choose an arbitrary such chain and use this as the definition of $id_p \times id_q \in C_{p+q}(\Delta^p \times \Delta^q)$. We have now defined the cross product in the *universal example* which forces the definition on all other simplices: Given a pair of simplices $\sigma\colon \Delta^p \to X$ and $\tau\colon \Delta^q \to Y$ the definition of their cross product is forced by property ii):

$$\sigma \times \tau \quad = \quad \sigma_*(id_p) \times \tau_*(id_q) \quad \overset{!}{=} \quad (\sigma, \tau)_*(id_p \times id_q)$$

A bilinear extension concludes the definition of $\times\colon C_p(X)\times C_q(Y)\to C_{p+q}(X\times Y)$. We still have to verify that property iii) is satisfied for arbitrary basis elements σ and τ. But this is done by the following calculation:

$$\begin{aligned}
\partial(\sigma\times\tau) &= \big(\partial\circ(\sigma,\tau)_*\big)(id_p\times id_q)\\
&= \big((\sigma,\tau)_*\circ\partial\big)(id_p\times id_q)\\
&= (\sigma,\tau)_*\big(\partial(id_p)\times id_q \;+\; (-1)^p id_p\times\partial(id_q)\big)\\
&= \sigma_*(\partial(id_p))\times\tau_*(id_q) \;+\; (-1)^p\sigma_*(id_p)\times\tau_*(\partial(id_q))\\
&= \partial(\sigma_*(id_p))\times\tau_*(id_q) \;+\; (-1)^p\sigma_*(id_p)\times\partial(\tau_*(id_q))\\
&= \partial(\sigma)\times\tau \;+\; (-1)^p\sigma\times\partial(\tau)
\end{aligned}$$

Thus we have checked that the map $\times\colon C_p(X)\times C_q(Y)\to C_{p+q}(X\times Y)$ satisfies the relations ii) and iii) which finishes the induction step. □

There were many choices made during the construction of the cross product. Nevertheless, it can be shown that the collection of all such maps is essentially unique in a certain precise sense. Since we do not have the algebraic toolkit necessary to attack this statement we will not be pursuing this any further.

Recall that a pair of spaces (X,A) is given by a space X together with a subspace A. Given two such pairs (X,A) and (Y,B) their product is defined to be $(X,A)\times(Y,B)=(X\times Y, A\times Y\cup X\times B)$. It is straightforward to generalize the construction of the cross product to the context of pairs of spaces. Namely, given $(X,A),\ (Y,B)\in\mathsf{Top}^2$ we obtain an induced cross product

$$\times\colon C_p(X,A)\times C_q(Y,B)\to C_{p+q}((X,A)\times(Y,B)) = C_{p+q}(X\times Y, A\times Y\cup X\times B)$$

with similar formal properties as in the absolute case. Let us only give the definition of this map. For this purpose let $c''\in C_p(X,A)$ be represented by $c\in C_p(X)$ so that each other representative of c'' is of the form $c+c'$ for some $c'\in C_p(A)$ (we allow ourselves to be a bit sloppy in that we drop the inclusions from notation!). Similarly, all representatives of $d''\in C_q(Y,B)$ can be written as $d+d'$. The idea is to define $c''\times d''$ to be the relative chain represented by $c\times d$. To see that this is well-defined it suffices to make the following calculation:

$$(c+c')\times(d+d') \quad=\quad c\times d \;+\; c\times d' \;+\; c'\times d \;+\; c'\times d'$$

But the sum of the last three chains lives in $C_{p+q}(A\times Y\cup X\times B)$ so that both $(c+c')\times(d+d')$ and $c\times d$ define the same relative chain in $C_{p+q}((X,A)\times(Y,B))$. For later reference we collect the following consequence of our last result.

Corollary 2. *Given two pairs of spaces (X,A) and (Y,B) then the cross product at the level of singular chain groups induces a bilinear map:*

$$\times\colon H_p(X,A)\times H_q(Y,B)\to H_{p+q}((X,A)\times(Y,B))$$

This homomorphism is natural in (X,A) and (Y,B) and is called the (homology) cross product.

Proof. Let us consider relative homology classes $\omega\in H_p(X,A)$ and $\omega'\in H_q(Y,B)$. We can represent ω by a chain z in X which is a cycle relative to A, i.e., $\partial(z)\in C_{p-1}(A)$. Similarly, ω' can be represented by a $z'\in C_q(Y)$ with $\partial(z')\in C_{q-1}(B)$. Now, the boundary of $z\times z'$ lies in:

$$\partial(z\times z') \quad=\quad \partial(z)\times z' \;+\; (-1)^p z\times\partial(z') \quad\in\quad C_{p+q-1}(A\times Y\cup X\times B)$$

In other words, $z \times z'$ defines a relative cycle in $(X, A) \times (Y, B)$. Let us show that the relative homology class represented by $z \times z'$ is well-defined. If we take a different representing relative cycle $z + \partial(c)$ for ω then we calculate:

$$(z + \partial(c)) \times z' \quad = \quad z \times z' \; + \; \partial(c) \times z' \quad = \quad z \times z' \; + \; \partial(c \times z') \; - \; (-1)^{p+1} c \times \partial(z')$$

But since $c \times \partial(z') \in C_{p+q}(X \times B)$ this expression represents the same relative homology class. A similar calculation shows that the class is independent of the chosen representing relative cycle for ω'. Thus the assignment

$$\times\colon H_p(X, A) \times H_q(Y, B) \to H_{p+q}((X, A) \times (Y, B))\colon \quad ([z], [z']) \mapsto [z \times z']$$

is well-defined. The bilinearity and naturality follow immediately from the corresponding properties of the cross product at the level of singular chains. $\square$

With this preparation we can now establish the homotopy invariance of singular homology.

Theorem 3. *Let X be a space and let $I = [0, 1]$ be the interval. The inclusions $i_j\colon X \to I \times X$ which send x to (j, x), $j = 0, 1$, induce chain homotopic maps $(i_0)_* \simeq (i_1)_*\colon C(X) \to C(I \times X)$. Similarly, if $(X, A) \in \mathsf{Top}^2$ then the induced chain maps $(i_0)_*$, $(i_1)_*\colon C(X, A) \to C(I \times X, I \times A)$ are chain homotopic.*

Proof. We give the proof in the absolute case. Let $\iota\colon \Delta^1 \to I$ be the affine isomorphism with $e_0 \mapsto 0$ and $e_1 \mapsto 1$. This singular 1-simplex has boundary $\partial(\iota) = \epsilon_1 - \epsilon_0$ where we wrote $\epsilon_j\colon \Delta^0 \to I$ for the 0-simplex corresponding to $j \in I$. We can use the cross product to obtain group homomorphisms:

$$s_n = \iota \times -\colon C_n(X) \to C_{n+1}(I \times X)\colon \quad c \mapsto \iota \times c$$

A calculation of the boundary of $s_n(c)$ gives:

$$\partial(s_n(c)) \quad = \quad \partial(\iota \times c) \quad = \quad \partial(\iota) \times c \; - \; \iota \times \partial(c) \quad = \quad \epsilon_1 \times c \; - \; \epsilon_0 \times c \; - \; s_{n-1}(\partial(c))$$

In the special case of $n = 0$ the right-hand-side simply reads as $\epsilon_1 \times c \; - \; \epsilon_0 \times c$. The 'normalization' of the cross products shows that $(i_j)_* = \epsilon_j \times -\colon C_n(X) \to C_n(I \times X)$. Thus, these equations tell us that the homomorphisms $s_n = \iota \times -$ assemble to a chain homotopy $s\colon (i_0)_* \simeq (i_1)_*$. Thus the induced maps in homology are the same. $\square$

Recall that a *homotopy of maps* $f, g\colon (X, A) \to (Y, B)$ *mod* A is a map $H\colon I \times X \to Y$ such that $H_0 = f$, $H_1 = g$, and $H(I \times A) \subset B$. If we agree that $I \times (X, A)$ is the pair $(I \times X, I \times A)$ then such a homotopy is, in particular, a map $H\colon I \times (X, A) \to (Y, B)$.

Corollary 4. *Let f, $g\colon X \to Y$ be homotopic maps then the maps f_*, $g_*\colon C(X) \to C(Y)$ are chain homotopic. In particular, homotopic maps induce the same map in singular homology. Similarly, if $f, g\colon (X, A) \to (Y, B)$ are maps which are homotopic relative to A then f_*, $g_*\colon C(X, A) \to C(Y, B)$ are chain homotopic. Thus, relative singular homology is homotopy-invariant.*

Proof. We again content ourselves to give a proof in the absolute case. Let $H\colon I \times X \to Y$ be a homotopy from f to g, i.e., we have $f = H \circ i_0\colon X \to I \times X \to Y$ and $g = H \circ i_1\colon X \to I \times X \to Y$. It suffices to do the following short calculation to conclude the proof:

$$f_* \quad = \quad H_* \circ i_{0*} \quad \simeq \quad H_* \circ i_{1*} \quad = \quad g_*$$

Here we used that the chain homotopy relation behaves nicely with compositions (see the exercises sheet). $\square$

Corollary 5. *A homotopy equivalence $f\colon X \to Y$ induces isomorphisms $f_*\colon H_n(X) \to H_n(Y)$.*

Proof. By definition of a homotopy equivalence we can find a map $g\colon Y \to X$ and homotopies $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. But the last corollary then implies that we have:

$$g_* \circ f_* = id \qquad \text{and} \qquad f_* \circ g_* = id$$

□

We want to finish this lecture by summarizing our main results so far. But before that let us quickly mention the naturality of the connecting homomorphisms associated to a short exact sequence. Let us sketch the construction of this homomorphism from Lecture 3. In the context of a short exact sequence $0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$ of chain complexes we defined a homomorphism $\delta_n\colon H_n(C'') \to H_{n-1}(C')$ by considering the following diagram:

$$\begin{tikzcd}
 & C_n \arrow[r, "p_n"] \arrow[d, "\partial_n"] & C''_n \\
C'_{n-1} \arrow[r, "i_{n-1}"'] & C_n &
\end{tikzcd}$$

If $\omega \in H_n(C'')$ is represented by $z''_n \in Z_n(C'')$ then we showed that the expression '$i_{n-1}^{-1} \circ \partial_n \circ p_n^{-1}(z''_n)$' makes sense and defines a well-defined homology class $\delta_n(\omega) \in H_{n-1}(C')$. We now want to show that this homomorphism is natural with respect to morphisms of short exact sequences.

Proposition 6. *Let us consider the following diagram of chain complexes in which the rows are short exact sequences:*

$$\begin{tikzcd}
0 \arrow[r] & C' \arrow[r, "i"] \arrow[d, "f'"'] & C \arrow[r, "p"] \arrow[d, "f"] & C'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & D' \arrow[r, "j"'] & D \arrow[r, "q"'] & D'' \arrow[r] & 0
\end{tikzcd}$$

The connecting homomorphism is natural in the short exact sequence in the sense that for all $n \geq 1$ the following diagram commutes:

$$\begin{tikzcd}
H_n(C'') \arrow[r, "\delta_n"] \arrow[d, "f''_*"'] & H_{n-1}(C') \arrow[d, "f'_*"] \\
H_n(D'') \arrow[r, "\delta_n"'] & H_{n-1}(D')
\end{tikzcd}$$

Proof. The proof is left as an exercise and follows more or less directly from the fact that the connecting homomorphism is well-defined. □

In the situation of the proposition it is immediate that the long exact sequences in homology associated to the respective short exact sequences of chain complexes assemble to the following commutative 'ladder':

$$\begin{tikzcd}
\ldots \arrow[r] & H_{n+1}(C'') \arrow[r, "\delta_{n+1}"] \arrow[d, "f''_*"'] & H_n(C') \arrow[r, "i_*"] \arrow[d, "f'_*"'] & H_n(C) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(C'') \arrow[r] \arrow[d, "f''_*"] & \ldots \\
\ldots \arrow[r] & H_{n+1}(D'') \arrow[r, "\delta_{n+1}"'] & H_n(D') \arrow[r, "j_*"'] & H_n(D) \arrow[r, "q_*"'] & H_n(D'') \arrow[r] & \ldots
\end{tikzcd}$$

Corollary 7. *Let $f\colon (X,A) \to (Y,B)$ be a map of pairs of topological spaces. Then the long exact sequences in homology of the respective pairs fit together to give the following commutative diagram:*

$$\begin{tikzcd}
\ldots \arrow[r] & H_{n+1}(X,A) \arrow[r, "\delta_{n+1}"] \arrow[d, "f_*"'] & H_n(A) \arrow[r] \arrow[d, "f_*"'] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r] \arrow[d, "f_*"] & \ldots \\
\ldots \arrow[r] & H_{n+1}(Y,B) \arrow[r, "\delta_{n+1}"'] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & \ldots
\end{tikzcd}$$

Proof. This is immediate from the above algebraic fact. In fact, it suffices to observe that f induces three chain maps $C(A) \to C(B)$, $C(X) \to C(Y)$, and $C(X,A) \to C(Y,B)$. These chain maps taken together define a morphism between the short exact sequences of chain complexes which are used to define the respective relative singular chain complexes. $\square$

Let us summarize the most important results which we obtained during the last five lectures. We defined the singular homology groups of a space and also of a pair of spaces. We showed that singular homology is functorial in the pair and that the connecting homomorphisms assemble to a certain natural transformation. Moreover, this datum consisting of the singular homology functors and the connecting homomorphisms has certain key properties:

- homotopy invariance: Corollary 4
- long exact sequence in homology associated to a pair: Lecture 3, Corollary 11
- homology of a point is concentrated in degree zero: Lecture 1, Example 12(2); exercises sheet
- additivity: Lecture 1, Example 13ii); exercises sheet

There is one additional important property, namely the so-called *excision property* of singular homology. The reason why we emphasize these properties of singular homology is the following: It can be shown that the singular homology theory is essentially *characterized* by these five properties! In the next lecture we will state the remaining property. This is the last important ingredient necessary for many interesting applications. Some of them will be discussed in the next lecture so that we are sufficiently motivated to attack the proof of the excision property one week later.