

## LECTURE 7: PROOF OF EXCISION PROPERTY OF SINGULAR HOMOLOGY

In this lecture we will give a proof of the excision property of singular homology. For convenience, let us quickly recall the statement.

**Theorem 1.** *Let  $U \subset A \subset X$  be subspaces such that the closure  $\bar{U}$  of  $U$  lies in the interior  $A^\circ$  of  $A$ . Then the inclusion  $(X \setminus U, A \setminus U) \rightarrow (X, A)$  induces isomorphisms on relative homology groups:*

$$H_n(X \setminus U, A \setminus U) \xrightarrow{\cong} H_n(X, A), \quad n \geq 0$$

The proof of this theorem uses the notion of *small chains* in  $X$ . In the situation of the theorem, let us call a generator  $\alpha: \Delta^n \rightarrow X$  **small** if:

$$\alpha(\Delta^n) \subseteq A \quad \text{or} \quad \alpha(\Delta^n) \cap U = \emptyset$$

The second condition is obviously equivalent to  $\alpha(\Delta^n) \subseteq X \setminus U$ . This notion is extended to singular  $n$ -chains in the obvious way: such a chain is **small** if it is a linear combination of small generators. Let us denote the subgroup of small singular  $n$ -chains by:

$$C'_n(X) \subseteq C_n(X)$$

This clearly defines a subcomplex of  $C(X)$  since the boundary of a small chain is again small. Let us define  $C'_n(X, A)$  by the following short exact sequence of abelian groups:

$$0 \rightarrow C_n(A) \rightarrow C'_n(X) \rightarrow C'_n(X, A) \rightarrow 0$$

The inclusion  $C_n(A) \subseteq C'_n(X)$  is part of a morphism of chain complexes. It is thus immediate that the  $C'_n(X, A)$  can be uniquely assembled into a chain complex such that we have a short exact sequence of chain complexes:

$$0 \rightarrow C(A) \rightarrow C'(X) \rightarrow C'(X, A) \rightarrow 0$$

There are now two quotient complexes in sight: the relative singular chain complex  $C(X \setminus U, A \setminus U)$  and  $C'(X, A)$ . The two defining short exact sequences are related in the following way:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_n(A \setminus U) & \longrightarrow & C_n(X \setminus U) & \longrightarrow & C_n(X \setminus U, A \setminus U) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_n(A) & \longrightarrow & C'_n(X) & \longrightarrow & C'_n(X, A) \longrightarrow 0 \end{array}$$

The two solid vertical arrows are the obvious ones and since the square commutes we get an induced morphism of chain complexes  $C(X \setminus U, A \setminus U) \rightarrow C'(X, A)$ .

**Lemma 2.** *The induced chain map  $C(X \setminus U, A \setminus U) \rightarrow C'(X, A)$  is an isomorphism of chain complexes.*

*Proof.* We have to check that we have an isomorphism in each degree. The injectivity is left as an easy exercise. For the surjectivity each element  $[\alpha] \in C'_n(X, A)$  can be represented by a small chain  $\alpha \in C'_n(X)$ . But such an  $\alpha$  can be decomposed as a sum  $\alpha = \beta + \gamma$  where  $\beta \in C_n(A)$  and  $\gamma \in C_n(X \setminus U)$ . Thus, the element of  $C_n(X \setminus U, A \setminus U)$  represented by  $\gamma$  is sent to  $[\alpha]$ .  $\square$

The proof of excision is now based on the following proposition.

**Proposition 3.** *The inclusion of chain complexes  $C'(X) \rightarrow C(X)$  induces isomorphisms in homology.*

Let us assume this proposition for the moment and let us see how we can deduce Theorem 1 from this.

*Proof.* (of Theorem 1 assuming Proposition 3)

By definition of the chain complexes  $C'(X, A)$  and  $C(X, A)$  there are the following two short exact sequences of chain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(A) & \longrightarrow & C'(X) & \longrightarrow & C'(X, A) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow & & \downarrow \text{---} & & \\ 0 & \longrightarrow & C(A) & \longrightarrow & C(X) & \longrightarrow & C(X, A) & \longrightarrow & 0 \end{array}$$

Since the square on the left commutes we get an induced map on the quotients as indicated by the dashed arrow. By the last proposition, the vertical map in the middle induces isomorphisms on all homology groups. Moreover, this is obviously also the case for the vertical map on the left. The naturality of the long exact sequence induced in homology can now be used to conclude that also the chain map  $C'(X, A) \rightarrow C(X, A)$  induces isomorphisms in homology. Finally, the chain map  $C(X \setminus U, A \setminus U) \rightarrow C(X, A)$  which we consider in the statement of the excision theorem factors as a composition:

$$C(X \setminus U, A \setminus U) \rightarrow C'(X, A) \rightarrow C(X, A)$$

By the above and by Lemma 2 we know that both maps induce isomorphisms in homology which concludes the proof.  $\square$

The main work is thus to establish Proposition 3. The proof of the proposition is based on the construction of two *natural* maps: a morphism of chain complexes

$$\text{bs}^X : C(X) \rightarrow C(X)$$

for any space  $X$  (bs stands for *barycentric subdivision*) and a chain homotopy

$$R^X : C_\bullet(X) \rightarrow C_{\bullet+1}(X)$$

between  $\text{bs}^X$  and the identity. Thus, for any  $\alpha \in C_n(X)$  we want to have:

$$\partial R_n^X(\alpha) + R_{n-1}^X(\partial\alpha) = \text{bs}_n^X(\alpha) - \alpha$$

Note that the existence of the chain homotopy implies that  $\text{bs}^X$  induces the identity in homology, i.e.,

$$[\text{bs}_n^X(\alpha)] = [\alpha] \in H_n(X)$$

for any  $\alpha \in C_n(X)$  with  $\partial\alpha = 0$ . It will follow from the construction that

- (1) For any  $\alpha \in C_n(X)$ , if we apply  $\text{bs}^X$  to  $\alpha$  sufficiently many –say  $k$ – times, we get a *small* chain, i.e.,

$$(\text{bs}_n^X)^k(\alpha) \in C'_n(X), \quad k \text{ large enough.}$$

- (2) For any small  $\alpha$  the chain  $R_n^X(\alpha)$  is also small.

**Exercise 4.** Show that the existence of such maps would indeed imply Proposition 3.

Let us begin with a preliminary construction, the **cone construction**. Let  $K$  be a convex set (in some  $\mathbb{R}^d$ , say) and let  $p \in K$ . For  $\alpha: \Delta^n \rightarrow K$ , let

$$\text{Cone}_p(\alpha): \Delta^{n+1} \rightarrow K$$

be the map:

$$\text{Cone}_p(\alpha)(t_0, \dots, t_{n+1}) = t_0 p + (1 - t_0)\alpha(t'_1, \dots, t'_{n+1})$$

Here,  $t'_i = t_i/(1 - t_0)$ ,  $t_0 < 1$ . More geometrically,  $\Delta^{n+1}$  is the convex hull of its zeroth vertex and the  $n$ -simplex opposite to that vertex (which is spanned by the remaining  $n + 1$  vertices). We want  $\text{Cone}_p(\alpha)$  to map the zeroth vertex to  $p$  and we want it to be  $\alpha$  on the opposite  $n$ -simplex. This is forced by defining  $\text{Cone}_p(\alpha)$  to be the convex linear extension in the  $t_0$ -direction of these two maps. The linear extension of  $\text{Cone}_p$  to chains will be denoted by the same notation:

$$\text{Cone}_p: C_n(K) \rightarrow C_{n+1}(K)$$

Notice the important *cone formula*:

$$(1) \quad \partial \text{Cone}_p(\alpha) = \alpha - \text{Cone}_p(\partial \alpha), \quad \alpha \in C_n(K)$$

This formula and these constructions should remind you of the way we prepared the proof of the homotopy invariance of singular homology (see Lecture 5, Lemma 5 and Proposition 6).

We now turn to the actual construction of the maps  $\text{bs}^X$  and  $R^X$ . Note first that *naturality* means that the following squares commute for any  $f: X \rightarrow Y$ :

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\text{bs}_n^X} & C_n(X) \\ f_* \downarrow & & \downarrow f_* \\ C_n(Y) & \xrightarrow{\text{bs}_n^Y} & C_n(Y) \end{array} \quad \begin{array}{ccc} C_n(X) & \xrightarrow{R_n^X} & C_{n+1}(X) \\ f_* \downarrow & & \downarrow f_* \\ C_n(Y) & \xrightarrow{R_n^Y} & C_{n+1}(Y) \end{array}$$

This naturality together with the linearity of these maps has the consequence that  $\text{bs}$  and  $R$  are completely determined by their effect on the identity maps  $\eta_n$ :

$$\eta_n = (\text{id}: \Delta^n \rightarrow \Delta^n) \in C_n(\Delta^n)$$

Indeed, given a generator  $\alpha \in C_n(X)$ ,  $\alpha: \Delta^n \rightarrow X$ , we have:

$$(2) \quad \text{bs}_n^X(\alpha) = \alpha_*(\text{bs}_n^{\Delta^n}(\eta_n)) \quad \text{and} \quad R_n^X(\alpha) = \alpha_*(R_n^{\Delta^n}(\eta_n))$$

We begin by defining  $\text{bs}_n^X$  for all  $X$  by induction on  $n$ . For  $n = 0$  let us put  $\text{bs}_0^{\Delta^0}(\eta_0) = \eta_0$ . This defines  $\text{bs}_0^X(\alpha)$  for all spaces  $X$  and all  $\alpha \in C_0(X)$ . For the induction step, let us suppose that  $\text{bs}_n^X(\alpha)$  has already been defined for all  $X$  and all  $\alpha$ . Define

$$\text{bs}_{n+1}^{\Delta^{n+1}}(\eta_{n+1}) = \text{Cone}_z(\text{bs}_n^{\Delta^{n+1}}(\partial \eta_{n+1}))$$

where  $z = z_{n+1}$  is the **barycenter** of  $\Delta^{n+1}$ . Naturality forces the definition of  $\text{bs}_{n+1}^X(\alpha)$  for any generator  $\alpha \in C_{n+1}(X)$  by the above formulas, which is linearly extended to arbitrary  $(n+1)$ -chains. This concludes the definition of the **barycentric subdivision operators**  $\text{bs}_n^X: C_n(X) \rightarrow C_n(X)$ .

**Exercise 5.** Draw some low-dimensional pictures to convince yourself that this is a good definition for a barycentric subdivision. Do the exercise!

**Lemma 6.** The maps  $\text{bs}_n^X: C_n(X) \rightarrow C_n(X)$ ,  $n \geq 0$ , define a chain map  $C(X) \rightarrow C(X)$ .

*Proof.* We prove  $\partial \circ \text{bs}_n^X(\alpha) = \text{bs}_{n-1}^X \circ \partial(\alpha)$  by induction on  $n$ . For  $n = 0$  it is clear. Suppose the formula holds for all  $X$  and  $\alpha$ , for a fixed  $n$ . Then for  $n + 1$  we have the following chain of identities where the first three are given by definition and by the fact that  $\alpha_*$  is a chain map:

$$\begin{aligned}
\partial \circ \text{bs}_{n+1}^X(\alpha) &= \partial \circ \alpha_* \circ \text{bs}_{n+1}^{\Delta^{n+1}}(\eta_{n+1}) \\
&= \partial \circ \alpha_* \circ \text{Cone}_z(\text{bs}_n^{\Delta^{n+1}}(\partial\eta_{n+1})) \\
&= \alpha_* \circ \partial \circ \text{Cone}_z(\text{bs}_n^{\Delta^{n+1}}(\partial\eta_{n+1})) \\
&= \alpha_* \circ \text{bs}_n^{\Delta^{n+1}}(\partial\eta_{n+1}) - \alpha_* \circ \text{Cone}_z(\partial \circ \text{bs}_n^{\Delta^{n+1}}(\partial\eta_{n+1})) \\
&= \alpha_* \circ \text{bs}_n^{\Delta^{n+1}}(\partial\eta_{n+1}) \\
&= \text{bs}_n^X \circ \alpha_*(\partial\eta_{n+1}) \\
&= \text{bs}_n^X \circ \partial(\alpha)
\end{aligned}$$

The fourth and the fifth step are given by the cone formula (1) and the induction assumption respectively while the remaining steps again follow from the definition.  $\square$

The next step is to define the chain homotopies  $R_n^X: C_n(X) \rightarrow C_{n+1}(X)$ , again by induction on  $n$  and in such a way that the homotopy formula will hold. In this construction we use the so-called *method of acyclic models*. Recall from Lecture 4, Proposition 6 that contractible spaces have trivial homology groups in positive dimensions which applies, in particular, to simplices. In dimension 0 we set:

$$R_0^{\Delta^0}(\eta_0) = (\Delta^1 \rightarrow \Delta^0) \in C_1(\Delta^0)$$

Since the boundary of  $R_0^{\Delta^0}(\eta_0)$  is zero the homotopy formula is satisfied in this dimension. This defines  $R_0^X$  for all spaces  $X$  (by means of (2)). For the inductive step, let us now suppose that  $R_n^X$  has already been defined for all  $X$ , in such a way that the homotopy formula

$$\partial \circ R_n^X + R_{n-1}^X \circ \partial = \text{bs}_n^X - \text{id}$$

holds for all  $X$ . As we already know, to define  $R_{n+1}^X$  for all  $X$ , it is enough to find an element

$$\beta = R_{n+1}^{\Delta^{n+1}}(\eta_{n+1}) \in C_{n+2}(\Delta^{n+1}).$$

This  $\beta$  should satisfy the formula  $\partial\beta + R_n^{\Delta^{n+1}}(\partial\eta_{n+1}) = \text{bs}_{n+1}^{\Delta^{n+1}}(\eta_{n+1}) - \eta_{n+1}$ , i.e.,

$$\partial\beta = -R_n^{\Delta^{n+1}}(\partial\eta_{n+1}) + \text{bs}_{n+1}^{\Delta^{n+1}}(\eta_{n+1}) - \eta_{n+1} \quad \text{in } C_{n+1}(\Delta^{n+1}).$$

To prove that such a  $\beta$  exists, it is enough to show that the right-hand-side is a cycle. We can then use that  $H_{n+1}(\Delta^{n+1}) \cong 0$  (since  $n$  is at least 0!) in order to conclude that this cycle has to be a boundary, i.e., that such a  $\beta$  exists. The fact that the right-hand-side is a cycle follows from the following calculation using the homotopy formula in dimension  $n$  and the fact that  $\text{bs}_{n+1}^{\Delta^{n+1}}$  is a chain map:

$$\begin{aligned}
&\partial(-R_n^{\Delta^{n+1}}(\partial\eta_{n+1}) + \text{bs}_{n+1}^{\Delta^{n+1}}(\eta_{n+1}) - \eta_{n+1}) \\
&= (R_{n-1}^{\Delta^{n+1}}(\partial\partial\eta_{n+1}) - \text{bs}_n^{\Delta^{n+1}}(\partial\eta_{n+1}) + \partial\eta_{n+1}) + \partial\text{bs}_{n+1}^{\Delta^{n+1}}(\eta_{n+1}) - \partial\eta_{n+1} \\
&= 0
\end{aligned}$$

Thus, we can find such a  $\beta$  and this concludes the inductive construction of the natural chain homotopy  $R^X$  for all  $X$ .

As a final step, given an arbitrary generator  $\alpha: \Delta^n \rightarrow X$  it only remains to show that  $(\text{bs}_n^X)^k(\alpha)$  is small for  $k$  large enough and that  $R_n^X$  sends small simplices to small simplices. Let us observe first that  $(\text{bs}_n^{\Delta^n})^k(\eta_n)$  is a linear combination of affine maps  $\Delta^n \rightarrow \Delta^n$ . Moreover, the diameter of the images of these affine maps becomes arbitrarily small as  $k$  increases. Thus, the smallness of  $(\text{bs}_n^X)^k(\alpha)$  will follow from the existence of a Lebesgue number (see the next lemma) applied to the open cover of  $\Delta^n$  given by

$$\alpha^{-1}(A^\circ) \quad \text{and} \quad \alpha^{-1}(X - \bar{U}).$$

(This is an open cover by our assumption in the excision theorem:  $\bar{U} \subseteq A^\circ$ .) The fact that  $R_n^X(\alpha)$  is small if  $\alpha$  is small follows immediately from the fact that  $R_n^X(\alpha)$  lies in the image of  $\alpha_*$ . Thus, this concludes the proof of Proposition 3 and hence of the excision property.

**Lemma 7.** *Let  $(Y, d)$  be a compact metric space and let  $(U_i)_{i \in I}$  be an open cover of  $Y$ . Then there is a positive real number  $\lambda$ , called a **Lebesgue number of the cover**, such that every subset of  $Y$  of diameter less than  $\lambda$  is entirely contained in  $U_i$  for some  $i$ .*

*Proof.* This is one of the exercises of the exercise sheet. □

It might be enlightening for the reader to again have a look at the proof of the homotopy invariance in Lecture 5. That proof was based on the construction of the chain-level cross product which in turn was also given by the method of acyclic models. Thus, two of the key features of singular homology (homotopy invariance and excision) have been established by this method.

Moreover, the method of acyclic models itself uses in an essential way that the homology of simplices (or contractible spaces) is trivial in positive dimensions. To get this vanishing result we already used the cone construction in Lecture 4. Thus, judged from this perspective the cone construction is one of the essential ingredients at least in our treatment of singular homology theory.