LECTURE 9: BASIC HOMOLOGICAL ASPECTS OF CW COMPLEXES

We begin this lecture by a brief introduction to some basic aspects of CW complexes (subcomplexes, quotient comlexes, and the subtleties concerning products). In the remainder of the lecture we establish a few basic homological properties of CW complexes. This will make necessary a brief discussion of the compatibility of singular homology and directed colimits.

Recall from the previous lecture that CW complexes are spaces which can be inductively obtained form discrete spaces by attaching cells. We now turn to *subcomplexes* of CW complexes.

Proposition 1. *Let X be a CW complex and let $Y \subseteq X$ be a closed subspace such that the intersection $Y \cap (X^{(n)} - X^{(n-1)})$ is the union of open n-cells. The filtration*

$$Y^{(0)} \subseteq Y^{(1)} \subseteq \ldots \subseteq Y$$

given by $Y^{(n)} = Y \cap X^{(n)}$ then defines a CW decomposition on Y. Moreover, the inclusion $Y \to X$ is then a cellular map.

As a special case, this proposition suggests how to define pointed CW complexes and, more generally, pairs of CW complexes.

Definition 2. In the notation of the above proposition, we refer to Y as a **CW subcomplex** of X and to (X, Y) as a **CW pair**. A **pointed CW complex** (X, x_0) is a CW complex X together with a chosen base point $x_0 \in X^{(0)}$.

In the obvious way, this gives us the category of pointed CW complexes and CW pairs whose definitions are left to the reader.

Example 3.
(1) For an arbitrary CW complex X, we have CW pairs $(X, X^{(n)})$ for all n and similarly $(X^{(n)}, X^{(m)})$ for $n \geq m$.
(2) We have CW pairs (S^n, S^m), $(\mathbb{R}P^n, \mathbb{R}P^m)$ and similarly in the complex case for $n \geq m$. If we endow the unions

$$S^\infty = \bigcup_n S^n, \qquad \mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n, \qquad \text{and} \qquad \mathbb{C}P^\infty = \bigcup_n \mathbb{C}P^n$$

with the weak topology then each of the three spaces carries canonically a CW structure. Moreover, we have CW pairs (S^∞, S^n), $(\mathbb{R}P^\infty, \mathbb{R}P^n)$. and $(\mathbb{C}P^\infty, \mathbb{C}P^n)$ for all n.

Exercise 4. Let (X, Y) be a CW pair. Then the quotient space X/Y can be turned in a CW complex such that the quotient map $X \to X/Y$ is cellular.

We will now establish a few more closure properties of CW complexes. Let us begin with a more difficult one, namely the product. Recall that we observed that each CW complex is obtained from a disjoint union of cells by passing to a quotient space. Namely, for a CW complex X we have a quotient map:

$$\bigsqcup_n J_n \times e^n \to X$$

Given two CW complexes X and Y one might now try to take two such presentations

$$\bigsqcup_n J_n(X) \times e^n \to X \qquad \text{and} \qquad \bigsqcup_m J_m(Y) \times e^m \to Y$$

and use homeomorphisms $e^n \times e^m \cong e^{n+m}$ to obtain a map

$$\bigsqcup_k J_k(X \times Y) \times e^k \to X \times Y$$

where $J_k(X \times Y) = \sqcup_{k'+k''=k} J_{k'}(X) \times J_{k''}(Y)$. However, this map is, in general, not a quotient map. More conceptually, the problem is that the formation of products and quotients in the category of spaces are not compatible in general. Nevertheless, under certain 'finiteness conditions' one can obtain a positive result.

Proposition 5. *Let X, K be CW complexes such that K is finite. Then the product $X \times K$ is again a CW complex with the above CW decomposition.*

Using the previous proposition we can establish additional closure properties for the class of CW complexes.

Corollary 6.
(1) *The disjoint union of two CW complexes is again a CW complex such that the inclusions of the respective summands are cellular.*
(2) *Given a CW complex X then the cylinder $X \times I$ is again a CW complex. For each n-cell e^n_σ of X we obtain three cells for $X \times I$, namely two n-cells $e^n_\sigma \times \{0\}, e^n_\sigma \times \{1\}$, and an $(n+1)$-cell $e^n_\sigma \times e^1$. Moreover, the cylinder comes with cellular maps $i_0, i_1 \colon X \to X \times I$ and $p \colon X \times I \to X$.*
(3) *Given a CW complex X, then the unreduced suspension SX is again a CW complex. In fact, we know that the cylinder of X is a CW complex, and SX is obtained in two steps by passing to the quotient of a subcomplex.*

Proof. The first statement is immediate while the other ones follow from Proposition 5 and the examples of the previous lecture. □

In the definition of a CW complex X, the first condition we imposed was that $X^{(0)}$ is to be a discrete space and then that the higher skeleta are obtained from the lower ones by attaching n-cells for $n \geq 1$. We can also think of $X^{(0)}$ as being obtained from the empty space by attaching 0-cells; in fact, using the convention that $\partial e^0 = \emptyset$ we have a pushout:

$$\begin{tikzcd}
X^{(0)} \times \partial e^0 = \bigsqcup_{\sigma \in X_0} \partial e^0_\sigma \arrow[r, "\cong"] \arrow[d] & X^{(-1)} = \emptyset \arrow[d] \\
X^{(0)} \times e^0 = \bigsqcup_{\sigma \in J_0} e^0_\sigma \arrow[r, "\cong"'] & X^{(0)}
\end{tikzcd}$$

This observation is more than only a rather picky remark since it motivates the following generalization of the notion of CW complex.

Definition 7. Let (X, A) be a pair of spaces. Then X is a **CW complex relative to** A, if there is a filtration of X,

$$A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X,$$

such that the following two properties are satisfied:

(1) The space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching n-cells for $n \geq 0$.

(2) The space X is the union $\bigcup_{n\geq -1} X^{(n)}$ endowed with the weak topology.

In this situation, the pair (X, A) is called a **relative CW complex**.

Example 8. (1) Let X be a CW complex and $x_0 \in X_0$. Then we have a relative CW complex (X, x_0).
(2) More generally, every CW pair is a relative CW complex.

We now turn to basic homological aspects of CW complexes. Recall that we have a good understanding of the homology of the spheres,

$$H_k(S^n, *) \cong \begin{cases} \mathbb{Z} & , \quad k = n \\ 0 & , \quad \text{otherwise.} \end{cases}$$

The n-sphere is obtained from a point by attaching an n-cell. Since CW complexes are obtained inductively by attaching cells, a first step towards an understanding of the homology of CW complexes would consist of understanding the effect of the attachment of a cell at the level at homology.

Lemma 9. *Let X be obtained from A by attaching a n-cell along $f\colon \partial e^n \to A$, $X = A \cup_f e^n$. Then*

$$H_k(X, A) \cong \begin{cases} \mathbb{Z} & , \quad k = n \\ 0 & , \quad \textit{otherwise.} \end{cases}$$

Moreover, the attaching map applied to any orientation class of $(e^n, \partial e^n)$ gives us a generator of $H_n(X, A)$.

Proof. Let $N \subseteq e^n$ be a small collar around the boundary of e^n, e.g., $N = \partial e^n \times (1 - \epsilon, 1]$ for some small $\epsilon > 0$. Then by gradually shrinking this collar back to the boundary, we obtain a homotopy equivalences $(X, A) \simeq (X, A \cup N)$, and hence isomorphisms $H_k(X, A) \cong H_k(X, A \cup N)$ for all $k \geq 0$ by homotopy invariance. But $A \subseteq (A \cup N)^\circ$ so by excision the map induced by the inclusion in homology is an isomorphism,

$$H_k(X - A, N - A) \cong H_k(X, A \cup N), \qquad k \geq 0.$$

Now $X - A$ is an open ball of dimension n and $N - A$ is a collar, so clearly by collapsing this collar to a sphere we obtain an additional homotopy equivalence

$$(X - A, N - A) \simeq (e^n, \partial e^n).$$

Thus, again by homotopy invariance, we obtain isomorphisms

$$H_k(X - A, N - A) \cong H_k(e^n, \partial e^n)$$

for all $k \geq 0$, and the first statement follows by the long exact sequence associated to this latter pair. We leave it to the reader to go through the construction and to check the statement about the generators of $H_n(X, A)$. $\square$

Thus this lemma tells us that if a space is obtained by attaching an n-cell then there is a unique copy of the integer in the corresponding homology group, and a generator is given by the cell itself. Using this lemma, we can draw some consequences for the homology of CW complexes by 'induction on the number of the cells'. In this lecture we will only achieve a first step and continue with the program in the next lecture.

As the case of finite CW complexes requires less machinery, we treat that case independently.

Proposition 10. *For any finite CW complex X, $H_k(X^{(n)}, X^{(n-1)}) \cong 0$ for all $k \neq n$.*

Proof. The proof will be given by induction. The assertion is true if X has dimension 0. Suppose $X = A \cup_f e^i$ and the assertion is true for A. We claim it also holds for X. There are three cases to be discussed:

(1) If $n < i$ then $(X^{(n)}, X^{(n-1)}) = (A^{(n)}, A^{(n-1)})$ and there is nothing to prove.
(2) If $n > i$ then $(X^{(n)}, X^{(n-1)}) = (A^{(n)} \cup_\chi e^i, A^{(n-1)} \cup_\chi e^i)$ where χ denotes the attaching map of the i-cell. If B is a closed ball inside the interior of e^i, then the B is also contained in the interior of $X^{(n-1)}$. Thus by excision we deduce that the inclusion

$$(X^{(n)} - B, X^{(n-1)} - B) \to (X^{(n)}, X^{(n-1)})$$

induces isomorphisms in homology. But by gradually expanding this ball to fill e^i, we obtain a homotopy equivalence $(X^{(n)} - B, X^{(n-1)} - B) \simeq (A^{(n)}, A^{(n-1)})$. The homotopy invariance implies that the inclusion $(A^{(n)}, A^{(n-1)}) \to (X^{(n)}, X^{(n-1)})$ induces isomorphisms is homology so that the induction assumption on A establishes this case.
(3) So the remaining case is where $i = n$ in which case there are inclusions

$$A^{(n-1)} = X^{(n-1)} \subseteq A^{(n)} \subseteq A^{(n)} \cup_\chi e^i = X^{(n)}.$$

For this sequence of spaces, the inclusions of pairs

$$(A^{(n)}, A^{(n-1)}) \subseteq (X^{(n)}, A^{(n-1)}) = (X^{(n)}, X^{(n-1)}) \subseteq (X^{(n)}, A^{(n)})$$

induces a long exact sequence in homology (*the long exact sequence of a triple*)

$$\ldots \to H_{k+1}(X^{(n)}, A^{(n)}) \xrightarrow{\Delta} H_k(A^{(n)}, A^{(n-1)}) \to H_k(X^{(n)}, X^{(n-1)}) \to H_k(X^{(n)}, A^{(n)}) \xrightarrow{\Delta} \ldots.$$

We show that $H_k(X^{(n)}, X^{(n-1)}) \cong 0$ for all $k \neq 0$ by showing that in this range the two groups next to it in the above long exact sequence vanish. But for $k \neq n$, the group $H_k(A^{(n)}, A^{(n-1)})$ is already known to vanish by induction hypothesis. In order to obtain the vanishing of $H_k(X^{(n)}, A^{(n)})$ for $k \neq 0$ it suffices to note that $X^{(n)}$ is obtained from $A^{(n)}$ by attaching an n-cell. Thus we can conclude since $H_k(X^{(n)}, X^{(n-1)})$ sits between zeros in the sequence, hence must itself be zero.

This concludes the inductive step and hence the proof since our CW complexes were assumed to be finite. □

In order to obtain a similar result for not necessarily finite CW complexes, let us first include a short detour on *directed colimits.* Let us begin by establishing some terminology.

Definition 11. (1) A partially ordered set $(P, \leq)$ is **directed** if it is non-empty and if for every two elements $i, j \in P$ there is an element $k \in P$ such that $i \leq k$ and $j \leq k$.
(2) A **directed system** in a category $\mathcal{C}$ over a directed poset P consists of a family of objects C_i, $i \in P$, and morphisms $f_{ij}\colon C_j \to C_i$ for every pair of elements $i, j \in P$, $i \geq j$, which satisfy the relations

$$f_{ii} = id_{C_i}\colon C_i \to C_i,\ i \in P, \qquad \text{and} \qquad f_{ij} \circ f_{jk} = f_{ik},\ i \geq j \geq k.$$

(3) A **directed colimit** of such a directed system (C_i, f_{ij}) consists of an object $C \in \mathcal{C}$ together with morphisms $f_i\colon C_i \to C$ such that $f_j = f_i \circ f_{ij}$ whenever $i \geq j$. Moreover, this datum is supposed to be universal with respect to this property in the following sense: whenever there

is an object D together with morphisms $g_i\colon C_i \to D$ which also satisfy $g_j = g_i \circ f_{ij}$, $i \geq j$, then there is a unique morphism $g\colon C \to D$ such that $g_i = g \circ f_i$.

$$\begin{tikzcd}
C_j \arrow[rr, "f_{ij}"] \arrow[dr, "f_j"] \arrow[ddr, "g_j"'] & & C_i \arrow[dl, "f_i"'] \arrow[ddl, "g_i"] \\
& C \arrow[d, dashed] & \\
& D &
\end{tikzcd}$$

(4) A category $\mathcal{C}$ **has directed colimits** if there is a directed colimit for every directed system in $\mathcal{C}$.

(5) A **morphism of directed systems** $(C_i, f_{ij}) \to (C'_i, f'_{ij})$ consists of morphisms $C_i \to C'_i$ which commute with the maps f_{ij} and f'_{ij}, i.e., such the following squares commute

$$\begin{tikzcd}
C_i \arrow[r] \arrow[d] & C_j \arrow[d] \\
C'_i \arrow[r] & C'_j.
\end{tikzcd}$$

Exercise 12. Show that given two directed colimits (C, f_i) and (C', f'_i) of the same directed system (C_i, f_{ij}) then there is a unique isomorphism $g\colon C \to C'$ such that $f'_i = g \circ f_i$. This justifies that we talk about *the* directed colimit and we write $C = \mathrm{colim}_{i \in P} C_i$ for it. Conclude that a morphism of directed systems induces a morphism of directed colimits (provided both directed colimits exist).

Lemma 13. *The categories of abelian groups, of chain complexes of abelian groups, and of topological spaces have directed colimits.*

Proof. Let P be a directed partially ordered set and let $(A_i, f_{ij}\colon A_j \to A_i)$ be a directed system of abelian groups (over P). Then we can form the direct sum $\bigoplus_{i \in P} A_i$ which comes with the subgroup R generated by

$$f_{ij}(x) - x, \quad i, j \in P, x \in A_i.$$

We define C to be the quotient of $\bigoplus_i A_i/R$. The natural inclusions $A_j \to \bigoplus A_i$ induce homomorphisms $f_j\colon A_j \to C$. Note that any $c \in C$ can be represented as $f_j(a_j)$ for suitable $a_j \in A_j$. In fact, by definition of C, every element is a coset represented by a finite sum $\Sigma_{k=1}^n a_{i_k}$ for some $a_{i_k} \in A_{i_k}$. Since P is directed we can find an element $j \in P$ such that $i_k \leq j$ for all $k = 1, \ldots, n$. Thus we can consider the element $\Sigma_{k=1}^n f_{j i_k}(a_{i_k}) \in A_j$ and it is immediate that this element represents $c \in C$. It follows from the definition of R that the relations $f_{ij} \circ f_j = f_i$ are satisfied.

Now, let D be an abelian group coming with similar maps $g_i\colon A_i \to D$ such that $g_i \circ f_{ij} = g_j$. The induced map $g = (g_i)\colon \bigoplus A_i \to B$ sends the generators of R to zero and hence factors uniquely through C. This concludes the construction of the colimit of the directed system.

The case of chain complexes follows more or less directly from this (using that the differentials can be considered as defining a morphism of directed systems). The details about this case and the category of topological spaces will be discussed in the exercises. □

From now on given a directed system $(A_i, f_{ij})_{i \in P}$ of abelian groups, chain complexes, or topological spaces, we will write $\mathrm{colim}_{i \in P} A_i$ for the directed colimit.

Exercise 14. Let $(A_i, f_{ij})_{i\in P}$ be a directed system of abelian groups and let $f_j\colon A_j \to \mathrm{colim}_{i\in P} A_i$ be the canonical map. Then an element $x_j \in A_j$ satisfies $f_j(a_j) = 0 \in \mathrm{colim}_{i\in P} A_i$ if and only if there is an element $k \in P, j < k$, such that $f_{kj}(x_j) = 0 \in A_k$.

Thus, elements which are sent to zero in the directed colimit already vanish at some finite stage. Using this exercise we establish the following lemma.

Lemma 15. *Let $(A'_i, f'_{ij}) \to (A_i, f_{ij}) \to (A''_i, f''_{ij})$ be morphisms of directed systems of abelian groups. If the sequences $A'_i \xrightarrow{\iota_i} A_i \xrightarrow{\pi_i} A''_i$ are exact for all $i \in P$, then also the induced sequence*

$$\mathrm{colim}_{i\in P} A'_i \xrightarrow{\iota} \mathrm{colim}_{i\in P} A_i \xrightarrow{\pi} \mathrm{colim}_{i\in P} A''_i$$

of homomorphisms of abelian groups is exact.

Proof. We use the explicit description of elements in a directed colimit of abelian groups to prove this result. Let us consider an element $a' \in \mathrm{colim}_{i\in P} P'_i$. Then there is an index $j \in P$ such that a' can be represented by some $a'_j \in A'_j$, i.e., $a' = f'_j(a'_j)$ where $f'_j\colon A'_j \to \mathrm{colim}_{i\in P} A'_i$ is the canonical map to the colimit (we will use similar notation for the canonical maps in the other two cases). By definition of the induced maps between colimits we obtain

$$\pi\iota(a') = \pi\iota(f'_j(a'_j)) = f''_j(\pi_j\iota_j(a'_j)) = f''_j(0) = 0,$$

showing that the image of ι lies in the kernel of π.

Conversely, let $a \in \mathrm{colim}_{i\in P} A_i$ lie in the kernel of π, $\pi(a) = 0$. By definition of the induced homomorphism we can find an element $a_j \in A_j$ representing the element a, $f_j(a_j) = a$, and such that $\pi_j(a_j)$ becomes zero in the colimit $\mathrm{colim}_{i\in P} A''_i$. But $\pi_j(a_j)$ has then already to vanish at a finite stage. More precisely, there is an index $k \in P$, $j < k$, such that $f''_{kj}(\pi_j(a_j)) = 0$. But we also have $0 = f''_{kj}(\pi_j(a_j)) = \pi_k(f_{kj}(a_j))$. Using the exactness of the morphisms of directed systems in the k-th level, we conclude that there is an element $a'_k \in A'_k$ such that $\iota_k(a'_k) = f_{kj}(a_j)$. But for the element $a' = f'_k(a'_k) \in \mathrm{colim}_{i\in P} A'_i$ we then calculate

$$\iota(a') = \iota(f'_k(a'_k)) = f_k(\iota_k(a'_k)) = f_k(f_{kj}(a_j)) = f_j(a_j) = a,$$

as intended. □

Corollary 16. *Homology of chain complexes commutes with directed colimits. More precisely, for every n and every directed system of chain complexes (C^i, f_{ij}) there is a natural isomorphism*

$$\mathrm{colim}_{i\in P} H_n(C^i) \cong H_n(\mathrm{colim}_{i\in P} C^i).$$

Proof. This follows from the previous lemma, and the reader is asked to fill in the details in the exercises. □