LECTURE 11: CELLULAR HOMOLOGY

In this lecture we continue the study of homological properties of CW complexes, culminating in the definition of *cellular homology* for such complexes, and the proof that this alternative homology theory is naturally isomorphic to singular homology and that it is useful in explicit calculations.

We begin by recalling some basics about (homological) orientations. Recall that $H_n(S^n) \cong \mathbb{Z}$. An *orientation* of S^n is a choice of generator in $H_n(S^n)$; so there are two orientations. The boundary $\partial\Delta^n$ of the n-simplex is a model of S^{n-1}, and has a canonical orientation given by the order of its vertices

$$v_0, \ldots, v_n$$

where $v_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ with 1 in the i-th place, $i = 0, \ldots, n$. More precisely, the $(n-1)$-cycle $\sum(-1)^i\partial_i$ is a generator, where $\partial_i \colon \Delta^{n-1} \to \Delta^n$ is the face opposite to the i-th vertex,

$$\partial_i(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}).$$

An *orientation* of the n-cell e^n is a generator of $H_n(e^n, \partial e^n)$ (note that $H_n(e^n, \partial e^n) \cong \mathbb{Z}$ by the long exact sequence of the pair and the contractibility of e^n). Each homeomorphism $\alpha\colon \Delta^n \to e^n$ determines an orientation, since the map α itself is a cycle and represents an element of $H_n(e^n, \partial e^n)$.

An **oriented n-cell** in a CW complex X is a pair (e, θ) consisting of an n-cell e in X and an orientation θ of e. We write $C_n^{\mathrm{or}}(X)$ for the free abelian group generated by the oriented n-cells of X. Let $C_n^{\mathrm{cell}}(X)$ be the quotient of $C_n^{\mathrm{or}}(X)$ obtained by identifying (e, θ) and $-(e, \theta')$ if θ and θ' are the two possible orientations of e. So $C_n^{\mathrm{cell}}(X)$ is isomorphic to the free abelian group on the set of n-cells (but the isomorphism would require a choice of orientations).

In the final lecture we will prove the following theorem.

Theorem 1. *Let X be a CW complex. The abelian groups $C_\bullet^{\mathrm{cell}}(X)$ can be turned into a chain complex, the homology of which is isomorphic to the singular homology $H_n(X)$ of X.*

Of course a complete statement of the theorem, and its proof, requires an explicit description of the (cellular) boundary operator

$$\partial\colon C_n^{\mathrm{cell}}(X) \to C_{n-1}^{\mathrm{cell}}(X).$$

This description will be given in the next lecture and is based on the homological degree of maps $S^n \to S^n$. But even as it stands, it is already clear that the theorem is useful in calculations. For example, for the complex projective space $\mathbb{CP}^n$ we had a CW decomposition with one $2i$-cell for each $0 \leq i \leq n$. Thus, for the homology we obtain

$$H_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & , \quad k = 2l, 0 \leq l \leq n, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Let us say that a CW complex has **dimension bounded by** n if it has no cells in dimension larger than n. Using this terminology, the following is immediate for a CW complex X:

(1) If $\dim(X) \leq n$, then $H_k(X) \cong 0$ for all $k > n$.
(2) If X is dimension-wise finite, then all $H_k(X)$ are finitely generated.

For our further study of the homology of CW complexes let us recall the following two results.

Lemma 2. *Let X be obtained from A by attaching an n-cell along $f\colon \partial e^n \to A$, $X = A \cup_f e^n$. Then*
$$H_k(X,A) \cong \begin{cases} \mathbb{Z} &, \quad k=n \\ 0 &, \quad \text{otherwise.} \end{cases}$$
Moreover, the attaching map applied to any orientation class of $(e^n, \partial e^n)$ gives us a generator of $H_n(X,A)$.

Using this lemma, we can draw some consequences for the homology of CW complexes by 'induction on the number of cells'. In the case of finitely many n-cells the following result was already established in the previous lecture. We leave it to the reader to deduce the general case from this using filtered colimits.

Proposition 3. *For any CW complex X, $H_k(X^{(n)}, X^{(n-1)}) \cong 0$ for all $k \neq n$.*

We now continue establishing some interesting facts about the singular homology of CW complexes.

Proposition 4. *For any CW-complex X and any $n \geq 0$ we have $H_i(X, X^{(n)}) \cong 0$, for $i \leq n$.*

Proof. It suffices to prove this result for finite CW complexes X. The general case will follow by an argument using filtered colimits. The proof will be by induction over the number of cells in X. If X has dimension 0, the assertion is clear. Let us suppose that the proposition holds for A, and let us consider $X = A \cup e^k$. Then by excision as in the previous lecture, if $n \geq k$ then
$$H_i(X, X^{(n)}) \cong H_i(A, A^{(n)}).$$
If $k > n$ let us again consider the long exact sequence in singular homology associated to the triple
$$A^{(n)} = X^{(n)} \overset{\subseteq}{\to} A \overset{\subseteq}{\to} A \cup e^k = X,$$
a part of which looks like:
$$\ldots \to H_i(A, A^{(n)}) \to H_i(X, A^{(n)}) = H_i(X, X^{(n)}) \to H_i(X,A) \to \ldots$$
By Lemma 2 the group $H_i(X,A)$ is nonzero only for $i = k > n$. Moreover, by induction the group $H_i(A, A^{(n)})$ is zero for $i \leq n$. So surely the group in the middle is zero for $i \leq n$ as intended. □

Let us now show that the range in which the singular homology of a CW complex is possibly nontrivial is bounded by its dimension.

Proposition 5. *Let X be a CW complex of dimension $\leq n$. Then $H_i(X) \cong 0$ for $i > n$.*

Proof. Again, we prove the result for finite CW complexes by induction on the number of cells. The case of an infinite CW complex will follow by a colimit-argument. If $\dim(X) = 0$ then the proposition is clear. Suppose the proposition holds for A, and let $X = A \cup e^k$, so in particular $k \leq n$. A typical part of the long exact homology sequence of the pair (X, A) looks like:
$$\ldots \to H_i(A) \to H_i(X) \to H_i(X,A) \to \ldots$$
Now the group $H_i(X,A)$ is trivial for $i \neq k$ (so surely for $i > n$) as is the group $H_i(A)$ for $i > n$ by induction assumption. Thus also $H_i(X) \cong 0$ for $i > n$ concluding the proof. □

We can now prove a more explicit addendum to Proposition 3. If X is a CW complex, let us choose for each n-cell a characteristic map f and a homeomorphism α as in
$$\Delta^n \overset{\alpha}{\to} e^n \overset{f}{\to} X.$$

Then $f\circ\alpha$ is a cycle in $C_n(X^{(n)},X^{(n-1)})$, so we obtain a homology class $[f\circ\alpha]\in H_n(X^{(n)},X^{(n-1)})$. Doing this for each n-cell gives a well-defined homomorphism

$$\phi_n\colon C_n^{\mathrm{cell}}(X)\to H_n(X^{(n)},X^{(n-1)}).$$

It might be helpful to refamiliarize yourself with the proof of Proposition 3 (as given in the previous lecture) before reading the proof of the following proposition.

Proposition 6. *For all CW complexes X and all n, the map $\phi_n\colon C_n^{\mathrm{cell}}(X)\to H_n(X^{(n)},X^{(n-1)})$ is an isomorphism.*

Proof. Again, it suffices to prove this for finite CW complexes X, the case where X has dimension zero is clear, and we consider only the induction step $X=A\cup e^k$. If $k\neq n$ then

$$H_n(X^{(n)},X^{(n-1)})\cong H_n(A^{(n)},A^{(n-1)})$$

(as we saw in the proof of Proposition 3 in the previous lecture) and also $C_n^{\mathrm{cell}}(A)=C_n^{\mathrm{cell}}(X)$. So we only need to look at the case $k=n$. But here we have a commutative diagram of the following form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_n(A^{(n)},A^{(n-1)}) & \longrightarrow & H_n(X^{(n)},X^{(n-1)}) & \longrightarrow & H_n(X^{(n)},A^{(n)}) & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle\phi_n} & & \uparrow{\scriptstyle\phi_n} & & \uparrow{\scriptstyle\cong} & & \\
0 & \longrightarrow & C_n^{\mathrm{cell}}(A) & \longrightarrow & C_n^{\mathrm{cell}}(X) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0.
\end{array}$$

It is obvious that the last row is exact but also the first row is exact: for this consider the long exact sequence associated to the triple

$$A^{(n-1)}=X^{(n-1)}\overset{\subseteq}{\to}A^{(n)}\overset{\subseteq}{\to}X^{(n)}$$

and use that both groups $H_{n+1}(X^{(n)},X^{(n-1)})$ and $H_{n-1}(A^{(n)},A^{(n-1)})$ vanish. The fact that this diagram commutes follows from the explicit description of the isomorphism $\mathbb{Z}\cong H_n(X^{(n)},A^{(n)})$. But by our induction assumption the vertical map on the left is an isomorphism. Thus we can deduce by the 5-lemma that also the vertical map in the middle is an isomorphism, completing the induction step. □

Thus, these relative homology groups are just free abelian groups generated by the various indexing sets of the cell structure. We now want to show that these relative homology groups themselves assemble into a chain complex, and in the next lecture we show that the homology of this new complex again calculates the homology of the space. A priori this does not seem to be an efficient idea: we build a complex consisting of relative homology groups of a space in order to calculate the homology groups of that same space. However, as we saw these relative homology groups have an easy explicit description and we will see that this alternative way of calculating the homology is very convenient. This is also due to the fact that the differential can be given in quite explicit geometric terms. If one has a good understanding of the attaching maps of a given CW complex, then this allows for the calculation of its homology.

Here, we give an abstract description of the differentials. The translation into more geometric terms will be given in the following lecture. Let us recall that associated to a *triple of spaces* (X,A,B) there is a *connecting homomorphism*

$$\Delta_n\colon H_n(X,A)\overset{\delta}{\to}H_{n-1}(A)\to H_{n-1}(A,B),$$

were δ is the connecting homomorphism of the pair (X, A) and the undecorated morphism belongs to the long exact sequence of the pair (A, B).

Let X be a CW complex. For each $n \geq 1$ there is the triple of spaces $(X^{(n)}, X^{(n-1)}, X^{(n-2)})$ (we use the standard convention $X^{(-1)} = \emptyset$). Let us denote the connecting homomorphism of this triple by

$$\partial_n^{\text{cell}} \colon H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)}).$$

A key property of these maps is given by the following lemma.

Lemma 7. *For a CW complex X and $n \geq 2$ we have*

$$0 = \partial_{n-1}^{\text{cell}} \circ \partial_n^{\text{cell}} \colon H_n(X^{(n)}, X^{(n-1)}) \to H_{n-2}(X^{(n-2)}, X^{(n-3)}).$$

Proof. This follows since the composition of these cellular boundary homomorphisms is given by

$$\begin{array}{ccccc}
H_n(X^{(n)}, X^{(n-1)}) & & & & \\
\downarrow{\scriptstyle\delta} & & & & \\
H_{n-1}(X^{(n-1)}) & \longrightarrow & H_{n-1}(X^{(n-1)}, X^{(n-2)}) & & \\
& \overset{0}{\searrow} & \downarrow{\scriptstyle\delta} & & \\
& & H_{n-2}(X^{(n-2)}) & \longrightarrow & H_{n-2}(X^{(n-2)}, X^{(n-3)}).
\end{array}$$

But the composition of the second and the third morphism is trivial since these are two subsequent morphisms belonging to the long exact homology sequence of the pair $(X^{(n-1)}, X^{(n-2)})$. □

From now on we will use these relative homology groups as *definitions* of $C_\bullet^{\text{cell}}(X)$, but keep in mind that these are isomorphic to the groups described at the beginning of this lecture. Thus we make the following definition.

Definition 8. The **cellular chain complex** $C_\bullet^{\text{cell}}(X)$ of a CW complex X is given by the **cellular chain groups**

$$C_n^{\text{cell}}(X) = H_n(X^{(n)}, X^{(n-1)}), \quad n \geq 0,$$

together with the **cellular boundary homomorphisms**

$$\partial_n^{\text{cell}} \colon C_n^{\text{cell}}(X) = H_n(X^{(n)}, X^{(n-1)}) \to C_{n-1}^{\text{cell}}(X) = H_{n-1}(X^{(n-1)}, X^{(n-2)}).$$

The **cellular homology** $H_n^{\text{cell}}(X)$ of X is given by

$$H_n^{\text{cell}}(X) = H_n(C^{\text{cell}}(X)), \quad n \geq 0.$$

Note that cellular homology is functorial with respect to *cellular maps* of CW complexes. This follows from the naturality of the connecting homomorphism of a triple since any cellular map $f \colon X \to Y$ induces maps of triples

$$f \colon (X^{(n)}, X^{(n-1)}, X^{(n-2)}) \to (Y^{(n)}, Y^{(n-1)}, Y^{(n-2)}).$$

Thus, cellular homology defines a functor on the category of CW complexes and cellular maps.

Note that this definition of the cellular chain complex of a CW complex does not only depend on the underlying space but also on the chosen CW structure. In fact, by definition the cellular chain groups are relative homology groups of subsequent filtration steps in the skeleton filtration. Thus, one might wonder whether the resulting cellular homology is an invariant of the underlying space only (in that it would be independent of the actual choice of a CW structure). Theorem 10 tells us, in particular, that this is indeed the case.

We split off a preliminary lemma.

Lemma 9. *Let X be a CW complex. The canonical map $H_{n+1}(X^{(n+1)}, X^{(n)}) \to H_{n+1}(X, X^{(n)})$ is surjective for every $n \geq 0$.*

Proof. For this it suffices to consider the long exact homology sequences associated to the triple $(X, X^{(n+1)}, X^{(n)})$. The relevant part of it is given by

$$H_{n+1}(X^{(n+1)}, X^{(n)}) \to H_{n+1}(X, X^{(n)}) \to H_{n+1}(X, X^{(n+1)}).$$

But by Proposition 4, the group $H_{n+1}(X, X^{(n+1)})$ is trivial, concluding the proof. □

Theorem 10. ***(Singular and cellular homology are isomorphic.)***
Let X be a CW complex. Then there is an isomorphism $H_n(X) \cong H_n^{\mathrm{cell}}(X), n \geq 0$, which is natural with respect to cellular maps.

Proof. Let us begin by identifying the *cellular cycles*, i.e., the kernel of the cellular boundary operator,

$$Z_n^{\mathrm{cell}}(X) = \ker(H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)})).$$

By definition, this boundary operator factors as

$$H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)}).$$

But the second map in this factorization is injective as one easily checks using the long exact homology sequence of the pair $(X^{(n-1)}, X^{(n-2)})$ together with the fact that $H_{n-1}(X^{(n-2)})$ vanishes. This implies that $Z_n^{\mathrm{cell}}(X)$ is simply the kernel of $H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)})$. If we consider the long exact homology sequence of $(X^{(n)}, X^{(n-1)})$, then the interesting part reads as

$$H_n(X^{(n-1)}) \to H_n(X^{(n)}) \to H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)}).$$

Using that $H_n(X^{(n-1)})$ is trivial, we conclude that there is a canonical isomorphism

$$H_n(X^{(n)}) \xrightarrow{\cong} Z_n^{\mathrm{cell}}(X),$$

and that this isomorphism is induced by the map $H_n(X^{(n)}) \to H_n(X^{(n)}, X^{(n-1)})$.

Let us now describe the *cellular boundaries*, i.e., the image of the cellular boundary operator,

$$B_n^{\mathrm{cell}}(X) = \mathrm{im}(H_{n+1}(X^{(n+1)}, X^{(n)}) \to H_n(X^{(n)}, X^{(n-1)})).$$

Again, by definition this map is $H_{n+1}(X^{(n+1)}, X^{(n)}) \to H_n(X^{(n)}) \to H_n(X^{(n)}, X^{(n-1)})$. By the first part of this proof, we know that $H_n^{\mathrm{cell}}(X)$ is canonically isomorphic to the cokernel of the first map $H_{n+1}(X^{(n+1)}, X^{(n)}) \to H_n(X^{(n)})$, the connecting homomorphism of the pair $(X^{(n+1)}, X^{(n)})$. Recall that these connecting homomorphisms are natural with respect to maps of pairs, hence applied to the map $(X^{(n+1)}, X^{(n)}) \to (X, X^{(n)})$ this yields the following commutative diagram

$$\begin{array}{ccccccc}
H_{n+1}(X^{(n+1)}, X^{(n)}) & \longrightarrow & H_n(X^{(n)}) & & & & \\
\downarrow & & \downarrow{=} & & & & \\
H_{n+1}(X, X^{(n)}) & \longrightarrow & H_n(X^{(n)}) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, X^{(n)}),
\end{array}$$

in which the lower row is part of the long exact sequence of the pair $(X, X^{(n)})$. By Lemma 1, the vertical map on the left is surjective, and $H_n^{\mathrm{cell}}(X)$ is thus canonically isomorphic to the cokernel of

$H_{n+1}(X, X^{(n)}) \to H_n(X^{(n)})$. But since $H_n(X, X^{(n)})$ vanishes, the above exact sequence allows us to conclude that $H_n^{\mathrm{cell}}(X)$ is isomorphic to $H_n(X)$. It follows from this proof that the isomorphism is compatible with cellular maps. □