LECTURE 12: THE DEGREE OF A MAP AND THE CELLULAR BOUNDARIES

In this lecture we will study the (homological) degree of self-maps of spheres, a notion which generalizes the usual degree of a polynomial. We will study many examples, establish basic properties of the degree, and discuss some of the typical applications. We will also see how the boundary operator of the cellular chain complex of a space can be defined in terms of the degrees of self-maps of the spheres.

1. Degrees of maps between spheres

Let us recall from Lecture 6 that for each $n \geq 1$ we have isomorphisms

$$H_k(D^n, S^{n-1}) \cong \tilde{H}_k(S^n) \cong H_k(S^n, *) \cong \begin{cases} \mathbb{Z} & , \quad k = n \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Generators of the respective free abelian groups of rank one are *fundamental classes* or *orientation classes*. Note that these are well-defined up to a sign and we will now make coherent choices which will then be denoted by

$$[D^n, S^{n-1}] \in H_n(D^n, S^{n-1}), \quad [S^n] \in \tilde{H}_n(S^n) \quad \text{or} \quad [S^n] \in H_n(S^n, *)$$

respectively. Now, the n-sphere is obtained by gluing the 'north' hemisphere D^n_N and the 'south' hemisphere D^n_S along their common boundary (the 'equator'). Both hemispheres are just copies of the unit ball D^n and as such homeomorphic to Δ^n. To be more specific, we take the homeomorphism $\sigma\colon \Delta^n \to D^n$ which is essentially given by a rescaling: D^n is homeomorphic by a translation and a rescaling to a disc of dimension n centered at the barycenter of Δ^n and with the radius chosen such that all vertices of Δ^n lie on the boundary of that disc; given this disc then we choose σ to be just the obvious homeomorphism given by rescaling. Using these homeomorphisms we obtain singular n-simplices

$$\sigma_N\colon \Delta^n \xrightarrow{\sigma} D^n \cong D^n_N \hookrightarrow S^n \qquad \text{and} \qquad \sigma_S\colon \Delta^n \xrightarrow{\sigma} D^n \cong D^n_S \hookrightarrow S^n,$$

where the undecorated homeomorphisms are obtained by projection into $\mathbb{R}^n \times \{0\}$. One can check that the formal difference $z_n = \sigma_S - \sigma_N \in Z_n(S^n)$ is a cycle which actually represents a generator $[S^n] \in \tilde{H}_n(S^n)$. Under the above isomorphisms this also defines the fundamental classes $[D^n, S^{n-1}] \in H_n(D^n, S^{n-1})$ and $[S^n] \in H_n(S^n, *)$.

Recall from Lecture 2 the definition of the *Hurewicz homomorphism*: for every pointed space (X, x_0) there is a natural group homomorphism $h\colon \pi_1(X, x_0) \to H_1(X)$. Given a homotopy class $\alpha \in \pi_1(X, x_0)$ represented by a loop $\gamma\colon S^1 \to X$, $h(\alpha) \in H_1(X)$ is the well-defined homology class represented by the cycle

$$\gamma \circ e\colon \Delta^1 \to \Delta^1/\partial\Delta^1 \cong S^1 \to X$$

where $e\colon \Delta^1 \to \Delta^1/\partial\Delta^1$ denotes the quotient map. The fundamental class $[S^1]$ which we constructed above boils down to endowing S^1 with the 'usual' counter-clockwise orientation. The quotient map $e\colon \Delta^1 \to S^1$ can be chosen to be the concatenation $e = \sigma_S * \sigma_N^{-1}$. Considering e as a 1-cycle in S^1 and using Lemma 3 of Lecture 2 we see that e is homologous to z_1:

$$e \quad = \quad \sigma_S * \sigma_N^{-1} \quad \sim \quad \sigma_S + \sigma_N^{-1} \quad \sim \quad \sigma_S - \sigma_N \quad = \quad z_1$$

Thus, the Hurewicz homomorphism associated to a pointed space (X, x_0) is given by

$$h\colon \pi_1(X, x_0) \to H_1(X)\colon \quad \alpha \mapsto \alpha_*([S^1])$$

where $\alpha_*\colon H_1(S^1) \to H_1(X)$ denotes the induced map in homology. Of course, the homotopy invariance of singular homology motivated us to write α_* (no matter which representing loop we choose we get the same map in homology!).

This description of the Hurewicz homomorphism suggests an extension to higher dimensions. Given a pointed homotopy class α of maps $(S^n, *) \to (X, x_0)$, we obtain the homology class

$$\alpha_*([S^n]) \in H_n(X).$$

In order to give a precise definition of this higher dimensional Hurewicz homomorphism, one has first to introduce higher dimensional analogues of the fundamental group. This is done in any standard course on Homotopy Theory, and the thusly defined Hurewicz homomorphisms do play an important role.

Definition 1. Let $f\colon S^n \to S^n$ be a map. The unique integer $\deg(f) \in \mathbb{Z}$ such that

$$f_*([S^n]) = \deg(f) \cdot [S^n] \in \tilde{H}_n(S^n)$$

is called the **degree** of $f\colon S^n \to S^n$.

In this definition we used of course that $\tilde{H}_n(S^n) \cong \mathbb{Z}$ so that every self-map of $\tilde{H}_n(S^n)$ is given by multiplication by an integer. Note that the definition of the degree is independent of the actual choice of fundamental classes: a different choice would amount to replacing $[S^n]$ by $-[S^n]$ and hence would give rise to the same value for $\deg(f)$.

Lemma 2. (1) *If $f, g\colon S^n \to S^n$ are homotopic, then $\deg(f) = \deg(g)$.*
(2) *For maps $g, f\colon S^n \to S^n$ we have $\deg(g \circ f) = \deg(g)\deg(f)$ and $\deg(\mathrm{id}_{S^n}) = 1$.*

Proof. This is immediate from the definition. □

The second statement of this lemma is referred to by saying that the degree is multiplicative.

Example 3. (1) Let $f\colon S^1 \to S^1\colon (x_0, x_1) = (-x_0, x_1)$ be the reflection in the axis $x_0 = 0$. Then $\deg(f) = -1$. One way to see this is as follows. Let $\sigma_W, \sigma_E\colon \Delta^1 \to S^1$ be paths from the 'south pole' to the 'north pole' in the clockwise and the counterclockwise sense respectively. Obviously $\sigma_W - \sigma_E$ is a cycle and it can be checked (using a minor variant of Lemma 4 of Lecture 2: see the exercises) that

$$[\sigma_W - \sigma_E] = -[S^1] \in H_1(S^1).$$

In particular, we can use $[\sigma_W - \sigma_E]$ to calculate degrees. Since $f_*([\sigma_W - \sigma_E]) = [\sigma_E - \sigma_W]$ we deduce $\deg(f) = -1$. It follows that a map $S^1 \to S^1$ given by a reflection in an arbitrary line through the origin has degree -1.
(2) Let $a\colon S^1 \to S^1\colon (x_0, x_1) \mapsto (-x_0, -x_1)$ be the antipodal map. Then $\deg(a) = 1$. This follows from the previous example and the multiplicativity of the degree since the antipodal map is the composition of two reflections.
(3) Let $\tau\colon S^1 \to S^1$ be given by $\tau(x_0, x_1) = (x_1, x_0)$. Then $\deg(\tau) = -1$. Indeed, τ is a reflection in a line.

In order to extend these examples to higher dimensions, let us recall that we can construct S^{n+1} as the *suspension* of S^n,

$$S^{n+1} \cong S(S^n).$$

If we write $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$ then $S^{n+1} = S(S^n)$ is homeomorphic to the quotient of $S^n \times [-1,1]$ obtained by identifying all $(x_0, \ldots, x_n, 1)$ to a point N, the north pole, and all $(x_0, \ldots, x_n, -1)$ to a point S, the south pole. The homeomorphism is induced by the map

$$S^n \times [-1,1] \to S^{n+1}\colon \quad ((x_0, \ldots, x_n), t) \mapsto (x_0', \ldots, x_n', t)$$

where $x_i' = \sqrt{1-t^2} \cdot x_i$. Under this isomorphism, it is clear that the suspension of $f\colon S^n \to S^n$ in the standard coordinates for S^{n+1} is

$$Sf\colon S^{n+1} \to S^{n+1}\colon \quad (x_0, \ldots, x_{n+1}) \mapsto (rf(r^{-1}x_0, \ldots, r^{-1}x_n), x_{n+1})$$

where $r = \sqrt{1 - x_{n+1}^2}$. In particular, Sf restricts to f on the 'meridian' $x_{n+1} = 0$.

Proposition 4. *For any $f\colon S^n \to S^n$ and the associated $Sf\colon S^{n+1} \to S^{n+1}$, we have an equality of degrees* $\deg(Sf) = \deg(f)$.

Proof. Write $S^{n+1} = S(S^n) = A \cup B$ for the northern and southern hemispheres A and B (given by $x_{n+1} \geq 0$, resp. $x_{n+1} \leq 0$). Then $A \cap B = S^n$ is the meridian. Since A and B are contractible, the Mayer-Vietoris sequence (for slight extensions to open neighborhoods of A and B which are homotopy equivalent to A and B) gives:

$$\begin{array}{ccc} H_{n+1}(S^{n+1}) & \xrightarrow[\cong]{\Delta} & H_n(S^n) \\ \downarrow{\scriptstyle (Sf)_*} & & \downarrow{\scriptstyle f_*} \\ H_{n+1}(S^{n+1}) & \xrightarrow[\Delta]{\cong} & H_n(S^n) \end{array}$$

The square commutes by *naturality* of the Mayer-Vietoris sequence. From this, the statement follows by tracing the generator $[S^{n+1}]$ through this diagram. In more detail, since Δ is an isomorphism, we have $\Delta([S^{n+1}]) = \epsilon \cdot [S^n]$ with $\epsilon \in \{-1, +1\}$, and hence

$$(f_* \circ \Delta)([S^{n+1}]) = f_*(\epsilon \cdot [S^n]) = \epsilon \cdot f_*([S^n]) = \epsilon \cdot \deg(f) \cdot [S^n].$$

Similarly, if we trace $[S^{n+1}]$ through the lower left corner, we calculate

$$(\Delta \circ (Sf)_*)([S^{n+1}]) = \Delta(\deg(Sf) \cdot [S^{n+1}]) = \deg(Sf) \cdot \Delta([S^{n+1}]) = \deg(Sf) \cdot \epsilon \cdot [S^n].$$

Comparing these two expressions concludes the proof. □

This proof is a further instance of a calculation showing that *naturality* of certain long exact sequences is not just a technical issue but actually useful in calculations. With this preparation we now obtain the following higher-dimensional versions of Example 3.

Proposition 5.

1. *The degree of the map $f\colon S^n \to S^n\colon (x_0, x_1, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)$ is -1. More generally, the degree of the any reflection at an arbitrary hyperplane through the origin is -1.*
2. *Let $f\colon S^n \to S^n$ be given by $f(x_0, x_1, \ldots, x_n) = (\epsilon_0 x_0, \epsilon_1 x_1, \ldots, \epsilon_n x_n)$ for some signs ϵ_i. Then* $\deg(f) = \epsilon_0 \cdot \ldots \cdot \epsilon_n$. *In particular, if $f = a$ is the* antipodal map *(thus all ϵ_i are -1) then* $\deg(a) = (-1)^{n+1}$.
3. *Let $f\colon S^n \to S^n$ be given by $f(x_0, x_1, \ldots, x_n) = (x_1, x_0, x_2, x_3, \ldots, x_n)$. Then* $\deg(f) = -1$.

(4) *Let* $f\colon S^n \to S^n\colon (x_0,\ldots,x_n) \mapsto (x_{\tau(0)},\ldots,x_{\tau(n)})$ *for some permutation* $\tau \in \Sigma_{n+1}$. *Then* $\deg(f) = \mathrm{sg}(f)$ *is the signature of the permutation* τ.

Proof. Part (1) follows Example 3.(1) and Proposition 4 together with the observation that all reflections at hyperplanes are homotopic. The remaining parts are an immediate consequence of (1) and the multiplicativity of the degree (to obtain (4), write an arbitrary permutation as a sequence of transpositions). □

Lemma 6. *Let* $f, g\colon X \to S^n \subseteq \mathbb{R}^{n+1}$. *If* $f(x) \neq -g(x)$ *for all* $x \in X$ *then* $f \simeq g$.

Proof. Let $H\colon X \times [0,1] \to \mathbb{R}^{n+1}$ be given by $H(x,t) = (1-t)f(x) + tg(x)$. Then for a fixed x, the partial map $H(x,-)\colon [0,1] \to \mathbb{R}^{n+1}$ is the line from $f(x)$ to $g(x)$. By assumption, this line does not pass through the origin, so we can normalize H to obtain the map

$$K\colon X \times [0,1] \to S^n\colon (x,t) \mapsto H(x,t)/||H(x,t)||,$$

a well-defined homotopy from f to g. □

Corollary 7. *Let* $f\colon S^n \to S^n$.

(1) *If* f *has no fixed point, then* $\deg(f) = (-1)^{n+1}$.
(2) *If* f *has no antipodal point (a point* x *with* $f(x) = -x$*), then* $\deg(f) = 1$.

Proof. For the first statement, if $f(x) \neq x$ for all x, then f is homotopic to the antipodal map a defined by $a(x) = -x$, according to the lemma. But since the degree is homotopy-invariant we can conclude by Proposition 5.(2). The proof of the second statement is similar, since by the lemma, $f(x) \neq -x$ for all x implies that f is homotopic to the identity. □

Corollary 8. *If* n *is even, then any* $f\colon S^n \to S^n$ *has a fixed point or an antipodal point.*

Proof. If f has neither a fixed point nor an antipodal point, then, by the previous corollary, the degree of f has to be -1 and 1 which is impossible. □

Corollary 9. *Let* $n \in \mathbb{N}$ *be even, then any vector field* v *on* S^n *has a zero.*

Proof. Such a vector field assigns to any $x \in S^n$ a vector $v(x)$ based at x and lying in the hyperplane tangent to $S^n \subseteq \mathbb{R}^{n+1}$ at x. If $v(x) \neq 0$ for all x, then the map $\bar{v} = v/||v||\colon S^n \to S^n$ (obtained by normalizing the vector field and considering the vectors as attached to the origin of $\mathbb{R}^{n+1}$) is such that each $\bar{v}(x)$ is a unit vector parallel to the hyperplane tangent to S^n at x. But this contradicts the previous corollary, since $\bar{v}$ would have neither a fixed point nor an antipodal point. □

For $n = 2$ this is sometimes referred to as the **hairy ball theorem**: you cannot comb a hairy ball without a parting. The corollary tells us that there are no no-where vanishing vector fields on even-dimensional spheres. Thus one might wonder what happens for odd-dimensional spheres. It is easy to construct a no-where vanishing vector field on S^{2n+1},

$$v(x_0, x_1, \ldots x_{2n}, x_{2n+1}) = (-x_1, x_0, -x_2, x_3, \ldots, -x_{2n+1}, x_{2n}).$$

For a long time it was an open problem in algebraic topology to determine the maximal number of everywhere linearly independent vector field on spheres, which was finally solved by Adams using fairly advanced techniques. Using singular homology we managed to obtain first partial results in that direction.

2. The cellular boundary operator

If we want to be able to calculate the cellular homology of CW complexes which have cells in subsequent dimensions, then it is helpful to have a more geometric description of the cellular boundary homomorphism. Such a description can be obtained by means of the homological degrees of self-maps of the spheres, as discussed in Section 1. Let us recall that the cellular chain groups are free abelian groups, i.e., we have isomorphisms $\bigoplus_{J_n}\mathbb{Z}\cong C_n^{\mathrm{cell}}(X)$ where J_n denotes the index set for the n-cells of X. Under these isomorphisms, the cellular boundary maps hence correspond to homomorphisms

$$\bigoplus_{J_n}\mathbb{Z}\cong C_n^{\mathrm{cell}}(X)\stackrel{\partial_n^{\mathrm{cell}}}{\to} C_{n-1}^{\mathrm{cell}}(X)\cong\bigoplus_{J_{n-1}}\mathbb{Z}.$$

This composition sends every n-cell σ of X to a sum

$$\sigma\mapsto\Sigma_{\tau\in J_{n-1}}z_{\sigma,\tau}\tau$$

for suitable integer coefficients $z_{\sigma,\tau}$. To conclude the description of this assignment we thus have to specify these coefficients. For that purpose, let us fix an n-cell σ and an $(n-1)$-cell τ. The n-cell σ comes with an attaching map

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\chi_\sigma} & X^{(n-1)} \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & X^{(n)}. \end{array}$$

Now, associated to this attaching map we can consider the following composition

$$f_{\sigma,\tau}\colon S^{n-1}\stackrel{\chi_\sigma}{\to}X^{(n-1)}\to X^{(n-1)}/X^{(n-2)}\cong\bigvee_{J_{n-1}}S^{n-1}\to S^{n-1}$$

in which the last arrow maps all copies of the spheres constantly to the base point except the one belonging to the index $\tau\in J_{n-1}$ on which the map is the identity. Thus, for each such pair of cells we obtain a pointed self-map $f_{\sigma,\tau}$ of S^{n-1} and its degree turns out to coincide with $z_{\sigma,\tau}$. Note that since S^{n-1} is compact it follows that for any n-cell σ there are only finitely many $(n-1)$-cells τ such that $f_{\sigma,\tau}$ is not the constant map. Thus, the sums in the next proposition are well-defined. Working out the details, by using the long exact sequence of the homology of the pair, one can easily verify the following result:

Proposition 10. *Under the above isomorphisms the cellular boundary homomorphism is given by the map*

$$\bigoplus_{J_n}\mathbb{Z}\to\bigoplus_{J_{n-1}}\mathbb{Z}\colon\quad\sigma\mapsto\Sigma_{\tau\in J_{n-1}}\deg(f_{\sigma,\tau})\tau.$$

Hence in the context of a specific CW complex in which we happen to be able to calculate all the degrees showing up in the proposition, the problem of calculating the homology of the CW complex is reduced to a purely algebraic problem.

Let us give a brief discussion the example of the real projective spaces $\mathbb{R}P^n, n\geq 0$. We begin by recalling that $\mathbb{R}P^n$ is obtained from S^n by identifying antipodal points. Hence, there are quotient maps $p=p_n\colon S^n\to\mathbb{R}P^n$. The real projective space $\mathbb{R}P^n$ can be endowed with a CW structure such that there is a unique k-cell in each dimension $0\leq k\leq n$. One can check that the cellular boundary homomorphism $\partial\colon C_k^{\mathrm{cell}}(\mathbb{R}P^n)\to C_{k-1}^{\mathrm{cell}}(\mathbb{R}P^n), 0<k\leq n$ is zero if k is odd and multiplication by 2 if k is even. From this one can derive the following calculation.

Example 11. The homology of an even-dimensional real projective space is given by

$$H_k(\mathbb{R}P^{2m}) \cong \begin{cases} \mathbb{Z} & , \quad k=0, \\ \mathbb{Z}/2\mathbb{Z} & , \quad \text{k odd}, \ 0<k<2m \\ 0 & , \quad \text{otherwise.} \end{cases}$$

In particular, the top-dimensional homology group $H_{2m}(\mathbb{R}P^{2m})$ is zero. The homology of odd-dimensional real projective spaces looks differently and is given by

$$H_k(\mathbb{R}P^{2m+1}) \cong \begin{cases} \mathbb{Z} & , \quad k=0,2m+1 \\ \mathbb{Z}/2\mathbb{Z} & , \quad \text{k odd}, \ 0<k<2m+1 \\ 0 & , \quad \text{otherwise.} \end{cases}$$

In this case, the top-dimensional homology group is again simply a copy of the integers. Any generator of this group is called **fundamental class** of $\mathbb{R}P^{2m+1}$.

Note that these are our first examples of spaces in which the homology groups have non-trivial torsion elements. This should not be considered as something exotic but instead it is a general phenomenon. We conclude this lecture with a short outlook. There is an axiomatic approach to homology which is due to Eilenberg and Steenrod. By definition a **homology theory** consists of functors $h_n, n \geq 0$, from the category of pairs of topological spaces to abelian groups together with natural transformations (called *connecting homomorphisms*)

$$\delta\colon h_n(X,A) \to h_{n-1}(A,\emptyset), \quad n \geq 1.$$

This data has to satisfy the *long exact sequence axiom*, the *homotopy axiom*, the *excision axiom*, and the *dimension axiom*. We let you guess the precise form of the first three axioms, but we want to be specific about the dimension axiom. It asks that $h_k(*,\emptyset)$ is trivial in positive dimensions. Thus, the only possibly non-trivial homology group of the point sits in degree zero and that group $h_0(*,\emptyset)$ is referred to as the **group of coefficients** of the homology theory. So, parts of this course can be summarized by saying that singular homology theory defines a homology theory in the sense of Eilenberg-Steenrod with integral coefficients. In the sequel to this course we study closely related algebraic invariants of spaces, namely *homology groups with coefficients* and *cohomology groups*.