TOPOLOGICAL K-THEORY, EXERCISE SHEET 3, 19.02.2015

Exercise 1. Let $\xi\colon E \to X$ be a complex vector bundle over X of rank n. Define the *frame bundle* $F(E)$ of ξ to be the subspace of the Hom-bundle $\mathrm{Hom}(X \times \mathbb{C}^n, E)$ consisting of those linear maps $\mathbb{C}^n \to E_x$ that are linear isomorphisms. Let $\pi\colon F(E) \to X$ be the canonical projection to X.

(1) There is a canonical (fiberwise) right action of $\mathrm{GL}(n, \mathbb{C})$ on the frame bundle $F(E)$: an element $g \in \mathrm{GL}(n, \mathbb{C})$ acts on a linear isomorphism $f\colon \mathbb{C}^n \to E_x$ by precomposition. Show that this action turns the bundle $F(E) \to X$ into a principal $\mathrm{GL}(n, \mathbb{C})$-bundle.

(2) Conversely, let $\pi\colon P \to X$ be a principal $\mathrm{GL}(n, \mathbb{C})$-bundle. Let $P \times_{\mathrm{GL}(n,\mathbb{C})} \mathbb{C}^n$ be the quotient of $P \times \mathbb{C}^n$ by the relation $(p \cdot g, v) \sim (p, g \cdot v)$. Show that the canonical map

$$P \times_{\mathrm{GL}(n,\mathbb{C})} \mathbb{C}^n \longrightarrow X; \qquad (p, v) \longmapsto \pi(p)$$

has the structure of a vector bundle over X.

Exercise 2.

(1) Let $E \to X$ be a (real) vector bundle over a compact Hausdorff space. Show that there exists a vector bundle $F \to X$ with the property that $E \oplus F$ is a trivial vector bundle over X.

(2) We say a (real) vector bundle over a space X admits a (stable) n-framing if it can be realized as a subbundle of the trivial bundle $X \times \mathbb{R}^n \to X$. If a vector bundle $E \to X$ of rank k admits an n-framing, show that there is a map $f\colon X \to \mathrm{Gr}_k(\mathbb{R}^n)$ such that E is isomorphic to the pullback along f of the canonical k-plane bundle over $\mathrm{Gr}_k(\mathbb{R}^n)$.

Exercise 3. The infinite Stiefel manifold $V_k(\mathbb{R}^\infty)$ is the union of the sequence of inclusions

$$V_k(\mathbb{R}^k) \longrightarrow V_k(\mathbb{R}^{k+1}) \longrightarrow V_k(\mathbb{R}^{k+2}) \longrightarrow \cdots$$

equipped with the weak topology.

(1) Show that $V_k(\mathbb{R}^\infty)$ is contractible. Proceed as follows: consider the map $\mathbb{R}^\infty \to \mathbb{R}^\infty$ sending a vector $(v_1, v_2, ...)$ to the vector $(0, 0, ..., 0, v_1, v_2, ...)$ starting with k zeros. Show that this induces a homotopy equivalence between the identity on $V_k(\mathbb{R}^\infty)$ and the map f that sends a k-tuple of linearly independent vectors to the k-tuple of linearly independent vectors with k zeros added as first coordinates.

Next, show that the map f is homotopic to the constant map with value $(e_1, ..., e_k)$ (where e_i has 1 as its i-th coordinate and zeros as all other coordinates).

(2) Recall from the course on homotopy theory that each quotient map $V_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$ is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle, where $\mathrm{GL}(n, \mathbb{R})$. Conclude from this that $V_k(\mathbb{R}^\infty) \to \mathrm{Gr}_k(\mathbb{R}^\infty)$ is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle.

(3) Show that $\pi_0(\mathrm{Gr}_k(\mathbb{R}^\infty))$ is trivial and that

$$\pi_i(\mathrm{Gr}_k(\mathbb{R}^\infty)) \simeq \pi_{i-1}(\mathrm{GL}(n, \mathbb{R}))$$

for all $i \geq 1$.