TOPOLOGICAL K-THEORY, EXERCISE SHEET 6, 12.03.2015

Exercise 1.

(1) Let $f, g\colon (X,x) \to (Y,y)$ be two pointed homotopic maps between pointed compact spaces. Show that

$$f^* = g^*\colon \widetilde{K}^{-n}(Y) \to \widetilde{K}^{-n}(X)$$

for all $n \geq 0$. Conclude that the reduced K-groups define functors

$$\widetilde{K}^{-n}(-)\colon \mathsf{Ho}(\mathsf{Top}_*)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$$

where Top_* is the category of pointed compact Hausdorff spaces.
Hint: at some point you may need to use that the functor $(-)\times I$ preserves quotients, i.e. if $X/_\sim$ is a Hausdorff quotient of a compact Hausdorff space, then $X/_\sim \times I$ is homeomorphic to $X \times I/_\sim$, where we identify $(x,t) \sim (y,t)$ for all $x \sim y$.

(2) Similarly, show that two homotopic maps between compact spaces induce isomorphisms on K-groups (in all non-positive degrees). Conclude that the K-groups define functors

$$K^{-n}(-)\colon \mathsf{Ho}(\mathsf{Top})^{\mathrm{op}} \longrightarrow \mathsf{Ab}$$

where Top is the category of compact Hausdorff spaces.

(3) Finally, prove that the relative K-groups are invariant under homotopies of pairs and conclude that they define functors

$$K(-,-)\colon \mathsf{Ho}(\mathsf{Top}^2)^{\mathrm{op}} \longrightarrow \mathsf{Ab}$$

where Top^2 is the category of pairs (X,A) of spaces, where X is compact Hausdorff and A is closed in X.

Exercise 2. Recall that a clutching function $f\colon S^k \to \mathrm{GL}(n,\mathbb{C})$ determines a vector bundle E_f of rank n over the suspension $\Sigma S^k \simeq S^{k+1}$. Let $f\colon S^k \to \mathrm{GL}(n,\mathbb{C})$ and $g\colon S^k \to \mathrm{GL}(m,\mathbb{C})$ be two maps.

(1) Show that the direct sum $E_f \oplus E_g$ is isomorphic to the vector bundle associated to the clutching function

$$S^k \longrightarrow \mathrm{GL}(n+m,\mathbb{C});\quad x \longmapsto \begin{pmatrix} f(x) & 0 \\ 0 & g(x) \end{pmatrix}$$

(2) Suppose that $n=1$, so that the clutching function f takes values in $\mathrm{GL}(1,\mathbb{C}) = \mathbb{C} - \{0\}$. Show that the tensor product $E_f \otimes E_g$ is isomorphic to the rank m vector bundle associated to the clutching function

$$S^k \longrightarrow \mathrm{GL}(m,\mathbb{C});\quad x \longmapsto f(x)\cdot g(x)$$

Exercise 3 (Algebraic Mayer-Vietoris sequence)**.** Consider the following commuting diagram of abelian groups in which the rows are exact sequences and all the maps f'_n are isomorphisms:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & C''_{n-1} & \xrightarrow{\delta_{n-1}} & C'_n & \xrightarrow{p_n} & C_n & \xrightarrow{i_n} & C''_n & \xrightarrow{\delta_n} & C'_{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f''_{n-1}} & & \downarrow{\scriptstyle f'_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f''_n} & & \downarrow{\scriptstyle f'_{n+1}} & & \\
\cdots & \longrightarrow & D''_{n-1} & \xrightarrow[\delta'_{n-1}]{} & D'_n & \xrightarrow[q_n]{} & D_n & \xrightarrow[j_n]{} & D''_n & \xrightarrow[\delta'_n]{} & D'_{n+1} & \longrightarrow & \cdots
\end{array}$$

Show that there is an exact sequence

$$\cdots \longrightarrow C_n \xrightarrow{(i_n, f_n)} C''_n \oplus D_n \xrightarrow{f''_n - j_n} D''_n \xrightarrow{\Delta_n} C_{n+1} \longrightarrow \cdots$$

where $\Delta_n = p_{n+1} \circ {f'_{n+1}}^{-1} \circ \delta'_n$.

Exercise 4 (Mayer-Vietoris sequence for K-theory)**.** Let X be a compact space Hausdorff space and let $i_A \colon A \subseteq X$ and $i_B \colon B \subseteq X$ be two closed subspaces of X such that $X = A \cup B$. Show that there is a long exact sequence of the form

$$\cdots \longrightarrow K^{i-1}(A \cap B) \xrightarrow{\delta} K^i(X) \xrightarrow{(i_A^*, i_B^*)} K^i(A) \oplus K^i(B) \longrightarrow K^i(A \cap B) \longrightarrow K^{i+1}(X) \longrightarrow \cdots$$

where the map $K^i(A) \oplus K^i(B)$ sends (α, β) to $j_A^*\alpha - j_B^*\beta$, with $j_A \colon A \cap B \to A$ and $j_B \colon A \cap B \to B$ the natural inclusions.

Hint: use that the map $B/A \cap B \to X/A$ is a continuous bijection between compact Hausdorff spaces.