LECTURE 5: HIGHER K-THEORY GROUPS

In this lecture we are going to define the higher (negative) K-groups of a space and see a few properties of them. We have defined $K(X)$ as the group completion of the abelian monoid of isomorphism classes of vector bundles over X. In fact, $K(X)$ is $K^0(X)$ in an infinite sequence of abelian groups $K^n(X)$ for $n \in \mathbb{Z}$. Our aim is to show see that this sequence defines a cohomology theory in the sense of Eilenberg–Steenrod. In order to define the higher K-theory groups we need to introduce first some notation and topological constructions.

5.1. Notation and basic constructions

5.1.1. Let Top denote the category of compact Hausdorff spaces and Top_* the category of *pointed* compact Hausdorff spaces. By Top^2 we denote the category of *compact pairs*, that is, the objects are pairs of spaces (X, A), where X is compact Hausdorff and $A \subseteq X$ is closed. There are functors

$$\begin{array}{ccccccc} \mathsf{Top} & \longrightarrow & \mathsf{Top}^2 & \text{and} & \mathsf{Top}^2 & \longrightarrow & \mathsf{Top}_* \\ X & \longmapsto & (X, \emptyset) & & (X, A) & \longmapsto & X/A, \end{array}$$

where the basepoint in the quotient X/A is A/A. If $A = \emptyset$, then $X/\emptyset = X_+$ is the space X with a disjoint basepoint.

5.1.2. In what follows, we will consider complex vector bundles although most of the theory works the same in the real case. Recall that for a space X in Top we denote by $K(X)$ the group completion of $\mathrm{Vect}_{\mathbb{C}}(X)$. For a pointed space X in Top_*, the *reduced* K-theory group $\widetilde{K}(X)$ is the kernel of $i^*\colon K(X) \to K(x_0) = \mathbb{Z}$, where i^* is the map induced by the inclusion of the basepoint $i\colon x_0 \to X$. There is a short exact sequence

$$0 \to \ker i^* = \widetilde{K}(X) \longrightarrow K(X) \xrightarrow{i^*} K(x_0) \longrightarrow 0$$

which has a section c^* induced by the unique map $c\colon X \to x_0$. So it gives a natural splitting $K(X) \cong \widetilde{K}(X) \oplus K(x_0)$. We also have that $K(X) = \widetilde{K}(X_+)$ for every X in Top. Hence $\widetilde{K}$ defines a contravariant functor from Top_* to abelian groups.

For a compact pair (X, A), we define $K(X, A) = \widetilde{K}(X/A)$. So $K(-,-)$ is a contravariant functor from Top^2 to abelian groups.

5.1.3. Recall that the *smash product* of two pointed spaces is defined as the quotient $X \wedge Y = X \times Y / X \vee Y$, where $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$ is the wedge of X and Y, that is, the disjoint union glued by the basepoints.

We will use as a model for the nth sphere S^n in Top_* the space $I^n/\partial I^n$, where $I = [0, 1]$. There is a homeomorphism $S^n \cong S^1 \wedge \cdots \wedge S^1$.

For a pointed space X in Top_*, we define the *reduced suspension* ΣX as $S^1 \wedge X$. The nth reduced suspension of X is then $\Sigma^n X = S^n \wedge X$.

5.2. Negative K-groups

We can use the reduced suspension to define the negative K-groups for spaces, pointed spaces and pairs of spaces.

Definition 5.2.1. Let $n \geq 0$. For X in Top_*, we define $\widetilde{K}^{-n}(X) = \widetilde{K}(\Sigma^n X)$. If (X, A) is in Top^2, then we define $K^{-n}(X, A) = \widetilde{K}^{-n}(X/A) = \widetilde{K}(\Sigma^n(X/A)$. Finally, for X in Top, we define $K^{-n}(X) = K^{-n}(X, \emptyset) = \widetilde{K}^{-n}(X_+) = \widetilde{K}(\Sigma^n(X_+))$.

Thus, $\widetilde{K}^{-n}(-)$, $K^{-n}(-,-)$ and $K^{-n}(-)$ are contravariant functors for every $n \geq 0$ from Top_*, Top^2 and Top, respectively, to abelian groups.

5.2.2. Another useful construction is the cone on a space. Given X in Top, we define the *cone on* X as the quotient $CX = X \times I / X \times \{0\}$. The cone CX has a natural basepoint given by $X \times \{0\}$ and thus defines a functor $C \colon \mathsf{Top} \to \mathsf{Top}_*$. The quotient CX/X is called the *unreduced suspension* of X.

If X is a pointed space, then we have an inclusion $Cx_0/x_0 \cong I \to CX/X$ and the quotient space is precisely the reduced suspension ΣX. Since I is a closed contractible subspace of CX/X we have that $\mathrm{Vect}_{\mathbb{C}}(CX/X) \cong \mathrm{Vect}_{\mathbb{C}}((CX/X)/I)$. Hence, $K(CX/X) \cong K(\Sigma X)$ and $K(CX, X) = \widetilde{K}(CX/X) \cong \widetilde{K}(\Sigma X)$.

5.2.3. For a compact pair (X, A) we define $X \cup CA$ to be the space obtained by identifying $A \subseteq X$ with $A \times \{1\}$ in CA. There is a natural homeomorphism $X \cup CA/X \cong CA/A$. Thus, if A is a pointed space we have that

$$K(X \cup CA, X) = \widetilde{K}(CA, A) \cong \widetilde{K}(\Sigma A) = \widetilde{K}^{-1}(A).$$

5.3. Exact sequences of K-groups

We want to relate the K-groups of a pair (X, A) with the K-groups of X and A. We are going to need the following result about "collapsing" vector bundles that we recall from a previous lecture.

Lemma 5.3.1. *If $A \subseteq X$ is a closed subspace, then any trivialization $\alpha \colon E|A \cong \tau_n$ on A of a vector bundle $E \to X$ defines a vector bundle $E/\alpha \to X/A$ on the quotient X/A.* □

Lemma 5.3.2. *Let (X, A) be a compact pair in Top^2 and let $i \colon A \to X$ and $j \colon (X, \emptyset) \to (X, A)$ be the canonical inclusions. Then there is an exact sequence*

$$K^0(X, A) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(A).$$

Proof. The composition $(A, \emptyset) \xrightarrow{i} (X, \emptyset) \xrightarrow{j} (X, A)$ factors through (A, A). Applying K^0 yields a commutative diagram

$$\begin{array}{ccc} K^0(X, A) & \xrightarrow{\quad i^* j^* \quad} & K^0(A) \\ & \searrow \qquad \nearrow & \\ & K^0(A, A) = \widetilde{K}^0(A/A) = 0, . & \end{array}$$

So, $i^* j^* = 0$ and hence $\operatorname{Im} j^* \subseteq \ker i^*$.

To prove the converse, let ξ be any element in $\ker i^*$. We can represent ξ as a difference $[E] - [\tau_n]$, where E is a vector bundle over X and τ_n is the trivial bundle of rank n over X. By assumption $i^*(\xi) = 0$, which means that $i^*(\xi) = [E|A] - [\tau_n] = 0$.

So, $[E|A] = [\tau_n]$ in $K^0(A)$. This means that these two bundles become isomorphic after we sum with a trivial bundle of certain dimension. More precisely, there is an $m \geq 0$ such that

$$\alpha\colon (E \oplus \tau_m)|A \cong \tau_n \oplus \tau_m.$$

So, we have found a vector bundle that is trivial in A. By Lemma 5.3.1 we have a vector bundle $(E \otimes \tau_m)/\alpha$ over X/A. Take now $\eta = [(E \oplus \tau_m)/\alpha] - [\tau_n \oplus \tau_m]$ and observe that η lies in $\widetilde{K}^0(X/A)$ since the rank of $(E \oplus \tau_m)/\alpha$ in the component of the basepoint is $n + m$. Finally,

$$j^*(\eta) = [E \oplus \tau_m] - [\tau_n \oplus \tau_m] = [E] - [\tau_n] = \xi,$$

so $\ker i^* \subseteq \operatorname{Im} j^*$. □

Corollary 5.3.3. *Let (X, A) be a compact pair in Top^2 and A in Top_*. Then, there is an exact sequence*

$$K^0(X, A) \xrightarrow{j^*} \widetilde{K}^0(X) \xrightarrow{i^*} \widetilde{K}^0(A).$$

Proof. We have natural isomorphisms $K^0(X) \cong \widetilde{K}^0(X) \oplus K^0(*)$ and $K^0(A) \cong \widetilde{K}^0(A) \oplus K^0(*)$ and thus the following commutative diagram

$$\begin{array}{ccccc}
 & & \widetilde{K}^0(X) & \longrightarrow & \widetilde{K}^0(A) \\
 & \nearrow & \downarrow & & \downarrow \\
K^0(X, A) & \longrightarrow & \widetilde{K}^0(X) \oplus K^0(*) & \longrightarrow & \widetilde{K}^0(A) \oplus K^0(*) \\
 & & \downarrow & \swarrow & \\
 & & K^0(*), & &
\end{array}$$

where the central row and the columns are exact. Now, any element in $K^0(X, A)$ goes to zero in $K^0(*)$ so there is a map $K^0(X, A) \to \widetilde{K}^0(X)$ that makes the diagram commutative. From this it is straightforward to check that the required sequence is exact. □

Proposition 5.3.4. *Let (X, A) be a compact pair of spaces and A in Top_*. Then there is a natural exact sequence of five terms*

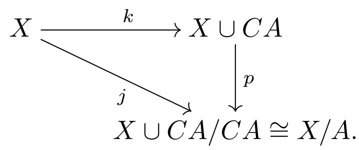
$$\widetilde{K}^{-1}(X) \xrightarrow{i^*} \widetilde{K}^{-1}(A) \xrightarrow{\delta} K^0(X, A) \xrightarrow{j^*} \widetilde{K}^0(X) \xrightarrow{i^*} \widetilde{K}^0(A).$$

Proof. We need to check exactness of the three subsequences of three terms. Exactness of $K^0(X, A) \to \widetilde{K}^0(X) \to \widetilde{K}^0(A)$ is given by Corollary 5.3.3.

To prove exactness at $\widetilde{K}^{-1}(A) \to K^0(X, A) \to \widetilde{K}^0(X)$ we consider the pair of spaces $(X \cup CA, X)$. Applying Corollary 5.3.3 we get an exact sequence

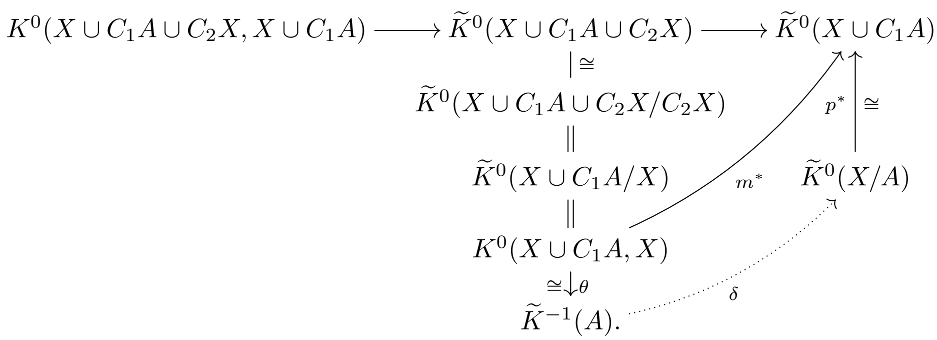
$$\begin{array}{ccccc}
K^0(X \cup CA, X) & \xrightarrow{m^*} & \widetilde{K}^0(X \cup CA) & \xrightarrow{k^*} & \widetilde{K}^0(X) \\
\theta \downarrow \cong & & p^* \uparrow \cong & \nearrow j^* & \\
\widetilde{K}^{-1}(A) & \xrightarrow{\delta} & \widetilde{K}^0(X/A). & &
\end{array}$$

Since CA is contractible, the quotient map $p\colon X \cup CA \to X/A$ induces isomorphism on $\widetilde{K}^0$ and moreover $k^* p^* = j^*$, which follows directly from the commutativity of the diagram



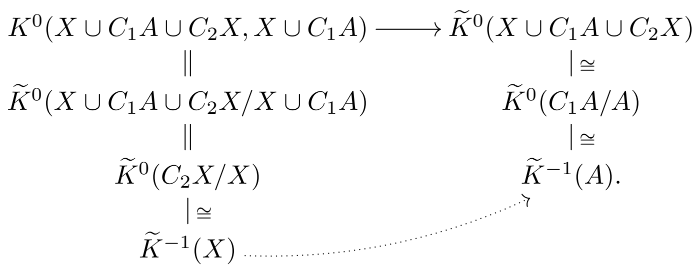
We define the *connecting* homomorphisms $\delta = (p^*)^{-1} m^* \theta^{-1}$, where the morphism $\theta\colon K^0(X \cup CA, X) \to \widetilde{K}^{-1}(A)$ is the isomorphism described in 5.2.3.

Exactness at $\widetilde{K}^{-1}(X) \to \widetilde{K}^{-1}(A) \to K^0(X, A)$ is a bit more involved. First, we apply Corollary 5.3.3 to the pair $(X \cup C_1A \cup C_2X, X \cup C_1A)$, where we used the notation C_1 and C_2 to distinguish between the two cones. This gives an exact sequence

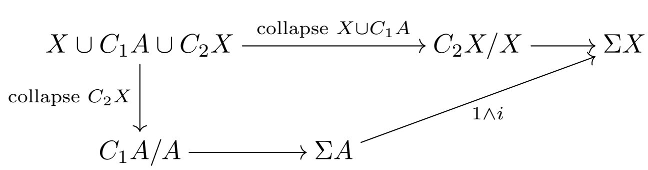


By using the definition of δ given in the previous step, we can check that the composition in the square on the right is indeed δ, as required. For the left part of the diagram we have a square as follows

(5.3.1)

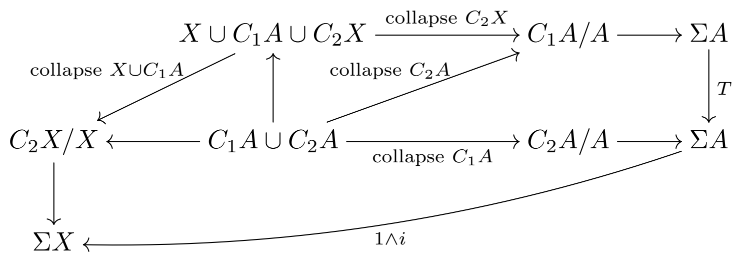


Now, we would like the dotted arrow that makes the diagram commutative to be i^* to conclude the proof. We can see that this is not going to be the case since at the level of spaces the diagram

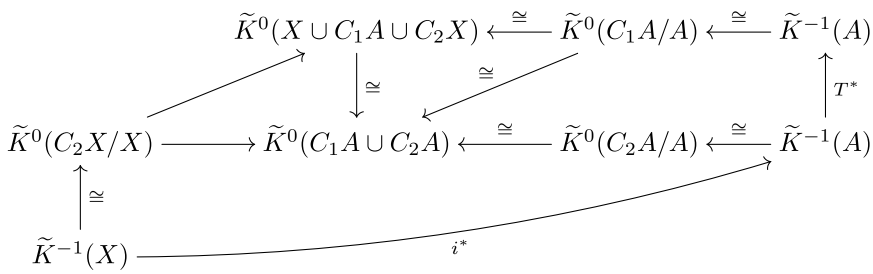


does not commute. We could try to replace $1 \wedge i$ by $(1 \wedge i) \circ T$, where $T\colon \Sigma A \to \Sigma A$ is the map that sends (a, t) to $(a, 1 - t)$ but the diagram would not commute either.

However, the diagram with $(1 \wedge i) \circ T$ commutes *up to homotopy*, which is enough for our purposes, since then it will *strictly commute* when we take $\widetilde{K}^0$. So we have the following diagram



which induces the following *commutative* diagram after applying $\widetilde{K}^0$



By "inserting" this diagram into diagram (5.3.1), we can check that the latter commutes if the dotted arrow is $T^* \circ i^*$. In the exercises we will prove that the map $T^* \colon \widetilde{K}^{-1}(A) \to \widetilde{K}^{-1}(A)$ sends every element to its inverse. So in the end, we get an exact sequence

$$\widetilde{K}^{-1}(X) \xrightarrow{-i^*} \widetilde{K}^{-1}(A) \xrightarrow{\delta} K^0(X, A).$$

But since $-i^*$ and i^* have both the same kernel and the same image, we can replace $-i^*$ by i^* and we still have an exact sequence. This completes the proof. □

Corollary 5.3.5. *Let (X, A) be a compact pair and A in Top_*. Then there is a long exact sequence*

$$\cdots \longrightarrow \widetilde{K}^{-2}(X) \xrightarrow{i^*} \widetilde{K}^{-2}(A) \xrightarrow{\delta} K^{-1}(X, A) \xrightarrow{j^*} \widetilde{K}^{-1}(X) \xrightarrow{i^*}$$
$$\xrightarrow{i^*} \widetilde{K}^{-1}(A) \xrightarrow{\delta} K^0(X, A) \xrightarrow{j^*} \widetilde{K}^0(X) \xrightarrow{i^*} \widetilde{K}^0(A).$$

Proof. Replace in the exact sequence of Proposition 5.3.4 the compact pair (X, A) by $(\Sigma^n X, \Sigma^n A)$ for $n = 1, 2, \ldots$ □

Corollary 5.3.6. *Let (X, A) be a compact pair. Then there is a long exact sequence*

$$\cdots \longrightarrow K^{-2}(X) \xrightarrow{i^*} K^{-2}(A) \xrightarrow{\delta} K^{-1}(X, A) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*}$$
$$\xrightarrow{i^*} K^{-1}(A) \xrightarrow{\delta} K^0(X, A) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(A).$$

Proof. The result follows directly by applying Corollary 5.3.5 to the pair (X_+, A_+) and using that $\widetilde{K}^{-n}(X_+) = K^{-n}(X)$. □