LECTURE 7: K-THEORY GROUPS OF THE SPHERES

In the previous lecture we proved that $K^0(S^1) \cong \mathbb{Z}$. The aim of this lecture is to compute the K-theory groups of all spheres and to state in a precise way the Bott periodicity theorem, that we used to prove that K-theory is a generalized cohomology theory.

7.1. Clutching functions for vector bundles over S^2

7.1.1. Recall that vector bundles over S^k are determined by clutching functions $S^{k-1} \to GL_n(\mathbb{C})$. Thus, for vector bundles over the sphere S^2, we have to study functions $S^1 \to GL_n(\mathbb{C})$.

Recall also that for $\mathbb{C}P^1 = S^2$ we have the canonical line bundle, that we denote by H,

$$\{(\ell, v) \mid \ell \in \mathbb{C}P^1,\, v \in \ell\} \longrightarrow \mathbb{C}P^1 = S^2$$

that sends (ℓ, v) to ℓ. This vector bundle has clutching function $f\colon S^1 \to GL_n(\mathbb{C})$ defined by $f(z) = z$, that is, $f(z)(v) = zv$ for every $z \in S^1$ and $v \in \mathbb{C}$.

The clutching function of the sum of two vector bundles is the block sum of the clutching functions of the two bundles. Thus, the bundle $H \oplus H$ has clutching function given by

$$f(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}.$$

The tensor product $H \otimes H$ has clutching function z^2 and so $(H \otimes H) \oplus \tau_1$ has clutching function

$$g(z) = \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

7.1.2. Although it does not seem so, the clutching functions f and g give rise to isomorphic bundles over S^2. Indeed, we can construct an explicit homotopy between g and f as follows. Let $H\colon S^1 \times [0,1] \to GL_2(\mathbb{C})$ be the map that sends (z,t) to the product

$$\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \pi t/2 & -\sin \pi t/2 \\ \sin \pi t/2 & \cos \pi t/2 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \pi t/2 & \sin \pi t/2 \\ -\sin \pi t/2 & \cos \pi t/2 \end{pmatrix}.$$

For every $z \in S^1$ and $t \in [0,1]$ the matrix $H(z,t)$ is invertible and $H(z,0) = g(z)$ and $H(z,1) = f(z)$. So, we have proved the following

Proposition 7.1.3. *Let H be the canonical line bundle over S^2. Then we have that $H \oplus H \cong (H \otimes H) \oplus \tau_1$ as bundles of rank 2 over S^2.* □

7.1.4. Observe that the isomorphism of Proposition 7.1.3 shows that in $K^0(S^2)$ we have that

$$([H] - [\tau_1]) \otimes ([H] - [\tau_1]) = [H \otimes H] - [H \oplus H] + [\tau_1] = 0.$$

This means that we have a well-defined ring homomorphism

$$\mu\colon \mathbb{Z}[H]/(H-1)^2 \longrightarrow K^0(S^2)$$

that sends H to $[H]$ and 1 to $[\tau_1]$ and which allows to formulate the Bott periodicity theorem:

Theorem 7.1.5 (Bott periodicity theorem —product version)**.** *For every space X in* Top*, the morphism*

$$\widetilde{\mu}\colon K^0(X)\otimes\mathbb{Z}[H]/(H-1)^2 \xrightarrow{1\otimes\mu} K^0(X)\otimes K^0(S^2) \longrightarrow K^0(X\times S^2),$$

where the second map is the external product, is an isomorphism.

We will not prove the theorem in its full generality, but will restrict to the proof in the case $X=*$, which will give a computation of $K^0(S^2)$.

Theorem 7.1.6. *The map $\mu\colon \mathbb{Z}[H]/(H-1)^2\to K^0(S^2)$ is an isomorphism.*

7.2. The proof of Theorem 7.1.6

7.2.1. We will divide the proof in several parts. Since every vector bundle is determined by a clutching function we will proceed according to the following steps:

(i) Prove that if f is linear clutching function, then the associated bundle is isomorphic to a linear combination of H and τ_1 (that is, it is in the image of μ).
(ii) Extend the previous result to polynomial clutching functions.
(iii) Extend the previous result to Laurent polynomial clutching functions.
(iv) Prove that any clutching function is homotopic to a Laurent polynomial clutching function.

This will show that μ is surjective. And finally:

(v) Prove that if μ is injective.

7.2.2. We will start by considering simple functions of the form $f(z)=\mathrm{Id}\,z+B$, where Id denotes the identity matrix. Observe that f is a clutching function if and only if $\det(\mathrm{Id}\,z+B)\neq 0$ for all $z\in S^1$, or equivalently if and only if $(\mathrm{Id}\,z+B)(v)\neq 0$ for all $z\in S^1$ and all $v\neq 0$. This happens if and only if $Bv\neq -zv$ for all $z\in S^1$ and all $v\neq 0$, that is, when the matrix B has no eigenvalues in S^1.

Lemma 7.2.3. *Let $f(z)=\mathrm{Id}\,z+B$ be a clutching function. Then:*

(i) *The matrix B has all eigenvalues outside S^1 if and only if $H(z,t)=\mathrm{Id}\,tz+B$ is a homotopy of clutching functions between B and $f(z)$.*
(ii) *The matrix B has all eigenvalues inside S^1 if and only if $H(z,t)=\mathrm{Id}\,z+tB$ is a homotopy of clutching functions between $\mathrm{Id}\,z$ and $f(z)$.*

Proof. We prove only part (i). Part (ii) is proved similarly and is left as an exercise. The map $H(z,t)$ is a clutching function for every t if and only if $(\mathrm{Id}\,tz+B)(v)\neq 0$ for every $t\in[0,1]$, $z\in S^1$ and $v\neq 0$. This happens if and only if $-tz$ is not an eigenvalue of B for every $t\in[0,1]$ and $z\in S^1$, that is, if and only if all eigenvalues of B are outside S^1. □

Thus, if $f(z)=\mathrm{Id}\,z+B$ is a clutching function and

(i) B has all eigenvalues outside S^1, then $f(z)\simeq B$ and the bundle associated to f is isomorphic to $n\tau_1$ for some $n\geq 0$ (recall that $GL_n(\mathbb{C})$ is path connected and so B is connected by a path to the identity matrix);
(ii) B has all eigenvalues inside S^1, then $f(z)\simeq \mathrm{Id}\,z$ and the bundle associated to f is isomorphic to nH for some $n\geq 0$.

7.2.4. In general, B will have eigenvalues inside and outside S^1. To deal with this case, we will use the following result that we will not prove.

Lemma 7.2.5. *Let B be an $n \times n$ matrix with coefficients in $\mathbb{C}$ and no eigenvalues in S^1. Then there are subspaces V_+ and V_- of $\mathbb{C}^n$ such that*

(i) $\mathbb{C} = V_+ \oplus V_-$.
(ii) *V_+ and V_- are invariant under B.*
(iii) *The restriction B_+ of B to V_+ has all eigenvalues outside S^1 and the restriction B_- of B to V_- has all eigenvalues inside S^1.* □

This means that the matrix B is similar (that is, conjugate) to the block matrix

$$\begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}.$$

Hence, the matrix $\operatorname{Id} z + B$ is similar to the matrix

$$\begin{pmatrix} \operatorname{Id} z + B_+ & 0 \\ 0 & \operatorname{Id} z + B_- \end{pmatrix}.$$

By Lemma 7.2.3, the bundle associated to this last clutching function is isomorphic to $k\tau_1 \oplus (n-k)H$ for some $k \geq 0$. Thus, we have proved

Lemma 7.2.6. *Any vector bundle over S^2 with clutching function $f(z) = \operatorname{Id} z + B$ is a linear combination of τ_1 and H.* □

7.2.7. Now, we want to extend the previous lemma to *linear* clutching functions of the form $f(z) = Az + B$. The idea is to reduce to the case of a function of the form $\operatorname{Id} z + B'$. We could try to multiply $f(z)$ by A^{-1}, but A is not invertible in general. However, $f(z)$ will be homotopic to a linear map in which the first coefficient is invertible.

Lemma 7.2.8. *Any bundle over S^2 with clutching function $f(z) = Az + B$ is a linear combination of τ_1 and H.*

Proof. Consider the function $H(z,t) = (A + tB)z + (tA + B)$. We have that $H(z,0) = f(z)$, but $H(z,1)$ is not a clutching function, since $H(-1,1) = 0$. So $H(z,t)$ give a homotopy of clutching functions for $t < 1$. Indeed, to see that $H(z,t)$ is invertible for all $z \in S^1$ and $t < 1$ we write

$$H(z,t) = A(z+t) + B(1+tz) = (1+tz)\left(A\frac{z+t}{1+tz} + B\right),$$

and note that $(z+t)/(1+tz)$ is in S^1 for all $z \in S^1$ and $t < 1$, since

$$\frac{|z+t|}{|1+tz|} = \frac{|z\overline{z} + t\overline{z}|}{|1+tz|} = \frac{|1+t\overline{z}|}{|1+tz|} = \frac{|\overline{v}|}{|v|} = 1.$$

So, $A(z+t)/(1+tz) + B$ is invertible for all $z \in S^1$ and $t < 1$ and hence so is $H(z,t)$.

For $z = 1$, we have that $f(1) = A + B$ is invertible. The function $\det(A + tB)$ is continuous, so there is a neighborhood V of 1 such that $\det(A + tB) \neq 0$ for all $t \in V$. Thus, taking $t_0 \in V$, we have that $f(z) = Az + B$ is homotopic to $(A + t_0B)z + (t_0A + B)$. But now, we can divide by $A + t_0B$, so the clutching functions $(A + t_0B)z + (t_0A + B)$ and $\operatorname{Id} z + (t_0A + B)(A + t_0B)^{-1}$ give isomorphic bundles.

In the end, we have proved that $f(z) = Az + B$ is homotopic to $\operatorname{Id} z + B'$ for some matrix B'. The result now follows from Lemma 7.2.6. □

7.2.9. The next step is to consider polynomial clutching functions of the form $f(z) = A_n z^n + \cdots + A_1 z + A_0$. This case will be treated in the exercises. The idea is to show that if E_f denotes the vector bundle associated to the clutching function f, then $\tau_m \oplus E_f \cong E_{L^n(f)}$ for some $m \geq 0$, where $L^n(f)$ is a linear clutching function obtained from f. Then, in $K^0(S^2)$, we will have that $[E_f] = [E_{L^n(f)}] - [\tau_m]$.

7.2.10. Next we consider Laurent polynomial clutching functions of the form $f(z) = \sum_{|i| \leq n} A_i z^i$.

Lemma 7.2.11. *Every vector bundle over S^2 with Laurent polynomial clutching function is equivalent in $K^0(S^2)$ to a linear combination of τ_1 and H.*

Proof. If $f(z)$ is a Laurent polynomial clutching function, then $f(z) = z^{-m} g(z)$ for some $m \geq 0$ and a polynomial clutching function $g(z)$. Then $[E_f] = [E_g \otimes H^{-m}]$.

By 7.2.9, $[E_g] = [E_{L^n(g)}] - [\tau_k]$. To take care of the term H^{-m} we can use the relation $(H-1)^2 = 0$. An easy induction argument shows that $H^n = nH - (n-1)$ for every $n \in \mathbb{Z}$. □

7.2.12. The last step is to consider arbitrary clutching functions. For this we are going to need the following result from analysis that we state without proof.

Theorem 7.2.13. *Let $f \colon S^1 \to \mathbb{C}$ be a continuous functions. Then for every $\varepsilon > 0$ there is a Laurent polynomial g such that $|f(z) - g(z)| < \varepsilon$ for all $z \in S^1$.* □

Proposition 7.2.14. *Let $f \colon S^1 \to GL_n(\mathbb{C})$ be a clutching function. Then there is a Laurent polynomial clutching function $g \colon S^1 \to GL_n(\mathbb{C})$ such that $f \simeq g$ and the homotopy is through clutching functions.*

Proof. Consider the set of all functions $S^1 \to M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the $n \times n$ matrices with coefficients in $\mathbb{C}$. This is a vector space over $\mathbb{C}$ and we can define a norm

$$|f| = \sup_{z \in S^1,\, |v|=1} |f(z)(v)|.$$

Using this norm we can put a topology on the set of all functions from S^1 to $M_n(\mathbb{C})$ and we claim that in this topology the clutching functions from an open subset.

First note that if f is a clutching function with matrix components f_{ij} and $\varepsilon > 0$, then we have by Theorem 7.2.13 that there are Laurent polynomials g_{ij} such that $|f_{ij}(z) - g_{ij}(z)| < \varepsilon$. Then we can check that

$$\begin{aligned} |f - g| &= \sup_{z \in S^1,\, |v|=1} |(f(z) - g(z))(v)| \\ &= \sup_{z \in S^1,\, |v|=1} \Big|(\sum_{j=1}^{n} (f_{1j}(z) - g_{1j}(z) v_j, \ldots, \sum_{j=1}^{n} (f_{nj}(z) - g_{nj}(z) v_j\Big| < \varepsilon\sqrt{n}. \end{aligned}$$

So let f be a clutching function. We know that $f(z)(v) \neq 0$ for every $z \in S^1$ and $|v| = 1$. Therefore, there is an $\varepsilon > 0$ such that $|f(z)(v)| > \varepsilon$ for all $z \in S^1$ and $|v| = 1$. Consider now the ball $B(f, \varepsilon/2)$ of center f and radius $\varepsilon/2$ and let $g \in B(f, \varepsilon/2)$. So $|f(z)(v) - g(z)(v)| < \varepsilon/2$. But this implies that $|g(z)(v)| > \varepsilon/2$ for all $z \in S^1$ and $|v| = 1$. This means that g is a clutching function and thus the set of clutching functions is open.

Now, let f be a clutching function and let $\varepsilon > 0$ such that $B(f, \varepsilon)$ is contained in the clutching functions (we can choose such an ε because the set is open). By

Theorem 7.2.13, for each f_{ij} there is a Laurent polynomial g_{ij} such that

$$|f_{ij}(z) - g_{ij}(z)| < \varepsilon/\sqrt{n}.$$

Then $|f - g| < \varepsilon\sqrt{n}/\sqrt{n} = \varepsilon$. So g is a clutching function and the map

$$H(z,t) = tf(z) + (1-t)g(z)$$

is a homotopy via clutching functions from g to f (because $B(f,\varepsilon)$ is convex). □

7.2.15. So far, we have seen that any vector bundle in $K^0(S^2)$ is equivalent to a linear combination of τ_1 and H, and hence it is in the image of μ.

Proof of Theorem 7.1.6. The morphism μ in surjective by Proposition 7.2.14 and Lemma 7.2.11. To prove that μ is injective we build a map

$$\nu\colon K^0(S^2) \longrightarrow \mathbb{Z}[H]/(H-1)^2$$

such that $\mu \circ \nu = \mathrm{id}$. The map ν is constructed as follows: start with a vector bundle on $K^0(S^2)$, take its clutching function, find a homotopic Laurent polynomial clutching function and reduce to a linear one. Thus the initial bundle is equivalent in $K^0(S^2)$ to one of the form $[nH]+[m\tau_1]$ for some $n, m \in \mathbb{Z}$. So we set $\nu([H]) = H$ and $\nu([\tau_1]) = 1$. Thus map clearly satisfies that $\mu \circ \nu = \mathrm{id}$.

But to finish the proof we should check that ν is well-defined. In other words, we have to see that if we have two equivalent vector bundles on $K^0(S^2)$ and we do the previous 'linearization' procedure to reduce the corresponding clutching functions to linear ones, we get the same thing (somehow we need to check that it is independent of all choices). For instance, we will need to prove that homotopies between Laurent polynomial clutching functions can be replaced by homotopies that are a Laurent polynomial at each $t \in [0,1]$. We leave all the details as an exercise. □

7.3. Some consequences of Bott periodicity

Corollary 7.3.1. $\widetilde{K}^0(S^2) \cong \mathbb{Z}$ *generated by* $(H-1)$.

Proof. We have a split short exact sequence (see 5.1.2)

$$0 \longrightarrow \widetilde{K}^0(S^2) \longrightarrow K^0(S^2) \longrightarrow K^0(*) = \mathbb{Z} \longrightarrow 0.$$

By Theorem 7.1.6, the term in the middle is $\mathbb{Z}[H]/(H-1)^2$ and the third map sends $aH + b$ to $a + b$. So $\widetilde{K}^0(S^2)$, which is the kernel, is isomorphic to $\mathbb{Z}$ and generated by $(H-1)$. □

Theorem 7.3.2 (Bott periodicity —standard form)**.** *For every X in Top_*, the external product with $(H-1)$ induces an isomorphism*

$$\widetilde{K}^0(X) \xrightarrow{\cong} \widetilde{K}^0(\Sigma^2 X) = \widetilde{K}^{-2}(X).$$

Proof. Recall from 6.2.2 that we have a commutative diagram

$$\begin{array}{ccc}
K^0(X)\otimes K^0(S^2) \cong & \widetilde{K}^0(X)\otimes \widetilde{K}^0(S^2) & \oplus\, \widetilde{K}^0(X) \oplus \widetilde{K}^0(S^2) \oplus \mathbb{Z} \\
\downarrow & \downarrow & \| \\
K^0(X\times S^2) \quad \cong & \widetilde{K}^0(X\wedge S^2) & \oplus\, \widetilde{K}^0(X) \oplus \widetilde{K}^0(S^2) \oplus \mathbb{Z},
\end{array}$$

Now, Theorem 7.1.5 states that the left map is an isomorphism, so the map in the middle is also an isomorphism. But, $\widetilde{K}^0(S^2) \cong \mathbb{Z}$ by Corollary 7.3.1. □

Remark 7.3.3. As usual, we have an unreduced version of the previous statement by using X_+ for $X \in \mathsf{Top}$. This give an isomorphism $K^0(X) \cong K^{-2}(X)$ for every X in Top.

Corollary 7.3.4. *$\widetilde{K}^0(S^{2n}) \cong \mathbb{Z}$ generated by $(H-1)^n$ and $\widetilde{K}^0(S^{2n+1}) = 0$.*

Proof. It follows from Bott periodicity and the previous computations and is left as an exercise. □

7.3.5. We finish with the list of all K-groups of the spheres (the computations are left as an exercise). By Bott periodicity it is enough to describe K^0 and K^{-1}. So, for every $n \geq 0$ we have

$$\begin{aligned} \widetilde{K}^0(S^{2n}) &\cong \mathbb{Z}, & \widetilde{K}^{-1}(S^{2n}) &= 0, \\ \widetilde{K}^0(S^{2n+1}) &= 0, & \widetilde{K}^{-1}(S^{2n+1}) &\cong \mathbb{Z}. \end{aligned}$$

$$\begin{aligned} K^0(S^{2n}) &\cong \mathbb{Z} \oplus \mathbb{Z}, & K^{-1}(S^{2n}) &= 0, \\ K^0(S^{2n+1}) &\cong \mathbb{Z}, & K^{-1}(S^{2n+1}) &\cong \mathbb{Z}. \end{aligned}$$