# LECTURE 8: THE COMPLEX $K$-THEORY SPECTRUM

In this lecture we will show how to define the spectrum associated to the cohomology theory given by $\widetilde{K}^*$. This allows to formulate yet another version of Bott periodicity. In the second part we will discuss the Hopf invariant one problem.

## 8.1. Reduced cohomology theories and spectra

**8.1.1.** Let $\widetilde{h}^*$ be a reduced cohomology theory (see 6.1.2 for a precise definition). We will restrict to cohomology theories defined on pointed CW-complexes and we will also assume that they are additive, that is, they satisfy the *wedge axiom*:

$$\widetilde{h}^n(\vee_{i\in I}X_i) \xrightarrow{\cong} \Pi_{i\in I}\widetilde{h}^n(X_i).$$

**8.1.2.** For every $n \in \mathbb{Z}$ the functor $\widetilde{h}^n$ satisfies the conditions of the Brown representability theorem (we need to be a bit careful here because Brown representability applies to functors from pointed *connected* CW-complexes, so we have to restrict to those spaces). So there is a (unique up to homotopy) pointed *connected* CW-complex $L_n$ and a natural equivalence

$$\widetilde{h}^n(X) \xrightarrow{\cong} [X, L_n]_*$$

for each pointed *connected* CW-complex $X$ (recall that $[-,-]_*$ denotes pointed homotopy classes of maps).

**8.1.3.** Let $E_n = \Omega L_{n+1}$ ,where $\Omega$ denotes the loop space functor, right adjoint to the suspension functor $\Sigma$. For *any* $X$ the suspension $\Sigma X$ is connected, so

$$\widetilde{h}^{n+1}(\Sigma X) \cong [\Sigma X, L_{n+1}]_*.$$

Since $\widetilde{h}^*$ is a reduced cohomology theory $\widetilde{h}^{n+1}(\Sigma X) \cong \widetilde{h}^n(X)$, so

$$\widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(\Sigma) \cong [\Sigma X, L_{n+1}]_* \cong [X, \Omega L_{n+1}] = [X, E_n]_*,$$

where the third isomorphism is given by the adjunction between $\Sigma$ and $\Omega$. Thus, we can associate to $\widetilde{h}^*$ the family of pointed CW-complexes $\{E_n\}_{n\in\mathbb{Z}}$ which satisfies that

$$[X, E_n]_* \cong \widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(\Sigma X) \cong [\Sigma X, E_{n+1}]_* \cong [X, \Omega E_{n+1}]_*$$

for *all* pointed CW-complexes $X$. This implies that there is a homotopy equivalence

$$E_n \xrightarrow{\simeq} \Omega E_{n+1}.$$

**Definition 8.1.4.** An $\Omega$-*spectrum* is a sequence of pointed CW-complexes $\{E_n\}_{n\in\mathbb{Z}}$ together with homotopy equivalences $\varepsilon_n \colon E_n \to \Omega E_{n+1}$ for every $n \in \mathbb{Z}$.

So we have proved the following

**Theorem 8.1.5.** *Every additive reduced cohomology theory $\widetilde{h}^*$ on pointed CW-complexes determines an $\Omega$-spectrum $\{E_n\}_{n\in\mathbb{Z}}$ such that $\widetilde{h}^n(X) = [X, E_n]_*$ for every $n \in \mathbb{Z}$.* □

**8.1.6.** The converse is also true. Let $\{E_n\}_{n\in\mathbb{Z}}$ be an $\Omega$-spectrum and define

$$\widetilde{E}^n(X) = [X, E_n]_*.$$

Then $\widetilde{E}^*$ is a reduced cohomology theory. The homotopy invariance is easy to check and the suspension isomorphism is given by

$$\widetilde{E}^{n+1}(\Sigma) = [\Sigma X, E_{n+1}]_* \cong [X, \Omega E_{n+1}]_* \xrightarrow{(\varepsilon_n)_*^{-1}} [X, E_n]_* = \widetilde{E}^n(X).$$

This also implies that $\widetilde{E}^n(-)$ takes values in abelian groups, since

$$\widetilde{E}^n(X) \cong \widetilde{E}^{n+2}(\Sigma^2 X) = [\Sigma^2 X, E_{n+2}]_*$$

and $[\Sigma^2(-), -]$ is always an abelian group. To prove exactness, consider a pair $(X, A)$ and the sequence $A \xrightarrow{i} X \xrightarrow{j} X \cup CA$. This gives an exact sequence

$$[X \cup CA, Z]_* \xrightarrow{j^*} [X, Z]_* \xrightarrow{i^*} [A, Z]_*$$

for every $Z$. Taking $Z = E_n$ gives the required exact sequence

$$\widetilde{E}^n(X \cup CA) \xrightarrow{j^*} \widetilde{E}^n(X) \xrightarrow{i^*} \widetilde{E}^n(A).$$

Note that $\widetilde{E}^*$ is also additive since

$$\widetilde{E}^n(\vee_{i\in I} X_i) = [\vee_{i\in I} X_i, E_n]_* \cong \Pi_{i\in I}[X_i, E_n]_* = \Pi_{i\in I}\widetilde{E}^n(X).$$

**Theorem 8.1.7.** *If $\{E_n\}_{n\in\mathbb{Z}}$ is an $\Omega$-spectrum, then the functors $\widetilde{E}^n$ defined as $\widetilde{E}^n(X) = [X, E_n]_*$ for every $n \in \mathbb{Z}$ form an additive reduced cohomology theory on pointed CW-complexes.* □

**8.1.8.** Let $G$ be an abelian group and let $K(G, n)$ be the associated Eilenberg–Mac Lane space. This space is characterized (up to homotopy) by the property that $\pi_k K(G, n) \cong G$ if $k = n$ and zero if $k \neq n$. There is a homotopy equivalence

$$K(G, n) \xrightarrow{\simeq} \Omega K(G, n + 1).$$

So the spaces $K(G, n)$ define an $\Omega$-spectrum $HG$ called the *Eilenberg–Mac Lane* spectrum associated to $G$. It is defined as $(HG)_n = K(G, n)$ For $n \geq 0$ and zero for $n < 0$. The cohomology theory that it describes

$$\widetilde{HG}^n(X) = [X, K(G, n)]_* \cong \widetilde{H}^n(X; G)$$

for $n \geq 0$, corresponds to singular cohomology with coefficients in $G$.

## 8.2. The spectrum $KU$

**8.2.1.** Recall from the first lectures that the Grassmannian $G_k(\mathbb{C}^n)$ consists of all $k$-dimensional linear subspaces of $\mathbb{C}^n$. The canonical inclusion $\mathbb{C}^n \to \mathbb{C}^{n+1}$ that sends $(v_1, \ldots, v_n)$ to $(v_1, \ldots, v_n, 0)$ induces maps

$$i_n : G_k(\mathbb{C}^n) \longrightarrow G_k(\mathbb{C}^{n+1}).$$

We define $BU_k = \operatorname{colim}_n\{G_k(\mathbb{C}^n), i_n\}$ as the colimit of the previous sequence. (Note that in the first lectures we were using the notation $G_k(\mathbb{C}^\infty)$ for $BU_k$.)

**8.2.2.** Recall also that we proved the existence of a 'universal' vector bundle $E_k(\mathbb{C}^\infty)$ over $BU_k$ and that every vector bundle is a pullback of this one:

**Theorem 8.2.3.** *Let $X \in \mathsf{Top}$. There is a natural bijection $[X, BU_k] \cong \mathrm{Vect}^k_{\mathbb{C}}(X)$ that sends $f$ to the pullback $f^*(E_k(\mathbb{C}^n))$.* □

If we apply the previous theorem with $k+1$ and $X = BU_k$, we obtain a bijection $[BU_k, BU_{k+1}] \cong \mathrm{Vect}_{\mathbb{C}}^{k+1}(BU_k)$. So taking on the right-hand side the vector bundle $E_k(\mathbb{C}^\infty) \oplus \tau_1$ over $BU_k$ gives a map

$$i_k \colon BU_k \longrightarrow BU_{k+1}$$

such that $i_k^*(E_{k+1}(\mathbb{C}^\infty) \cong E_k(\mathbb{C}^\infty) \oplus \tau_1$. We define $BU = \mathrm{colim}_k\{BU_k, i_k\}$ as the colimit of the sequence given by the maps $i_k$.

**8.2.4.** Let us see now how $BU$ is related to $K^0$ and $\widetilde{K}^0$. Let $d\colon \mathrm{Vect}_{\mathbb{C}}(X) \to [X, \mathbb{N}]$ be the function that assigns to a vector bundle $p\colon E \to X$ the function $d_E \colon X \to \mathbb{N}$ defined as $d_E(x) = \dim p^{-1}(x)$.

The set $[X, \mathbb{N}]$ has an abelian monoid structure (defined using the one on $\mathbb{N}$) such that $d$ is a map of abelian monoids. Consider the natural inclusion $[X, \mathbb{N}] \to [X, \mathbb{Z}]$ (which is in fact the group completion or Grothendieck construction of $[X, \mathbb{N}]$). By the universal property of the group completion there is a map $\overline{d}\colon K^0(X) \to [X, \mathbb{Z}]$ and a commutative square

$$\begin{array}{ccc} \mathrm{Vect}_{\mathbb{C}}(X) & \xrightarrow{d} & [X, \mathbb{N}] \\ \downarrow & & \downarrow \\ K^0(X) & \xrightarrow{\overline{d}} & [X, \mathbb{Z}]. \end{array}$$

We will denote $\widehat{K}(X) = \ker \overline{d}$.

**Proposition 8.2.5.** *There is a split short exact sequence*

$$0 \longrightarrow \widehat{K}(X) \longrightarrow K^0(X) \longrightarrow [X, \mathbb{Z}] \longrightarrow 0.$$

*In particular* $K^0(X) \cong \widehat{K}(X) \oplus [X, \mathbb{Z}]$.

*Proof.* To prove the result it is enough to find a section to the map on the right. Let $f\colon X \to \mathbb{N}$. Since $X$ is compact $f(X)$ is compact in $\mathbb{N}$ and hence finite. So suppose that $f(X) = \{n_1, \ldots, n_r\}$. Then $X = X_1 \coprod \cdots \coprod X_r$, where each $X_i = f^{-1}(n_i)$. We define a bundle over $X$ by taking trivial bundles $\tau_{n_i}$ at each $X_i$. This defines a map $\varphi\colon [X, \mathbb{N}] \to \mathrm{Vect}_{\mathbb{C}}(X)$ that satisfies $d \circ \phi = \mathrm{id}$

Now, using the universal property of the group completion there exists a map $\overline{\varphi}\colon [X, \mathbb{Z}] \to K^0(X)$ that satisfies $\overline{d} \circ \overline{\varphi} = \mathrm{id}$. The map $\overline{\varphi}$ is the required section. □

**Corollary 8.2.6.** *If $X \in \mathsf{Top}_*$ is connected, then $\widehat{K}(X) \cong \widetilde{K}^0(X)$.*

*Proof.* Consider the following commutative diagram of split short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{K}(X) & \longrightarrow & K^0(X) & \xrightarrow{\overline{d}} & [X, \mathbb{Z}] & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow{\scriptstyle i^*} & & \\ 0 & \longrightarrow & \widetilde{K}^0(X) & \longrightarrow & K^0(X) & \longrightarrow & [*, \mathbb{Z}] & \longrightarrow & 0 \end{array}$$

where $i\colon * \to X$ is the inclusion of the basepoint. If $X$ is connected, then $i^*$ is an isomorphism and hence $\widehat{K}(X) \cong \widetilde{K}^0(X)$. □

**8.2.7.** Consider the sets $\mathrm{Vect}_{\mathbb{C}}^k(X)$ and define for every $k \geq 0$ a function

$$t_k\colon \mathrm{Vect}_{\mathbb{C}}^k(X) \longrightarrow \mathrm{Vect}_{\mathbb{C}}^{k+1}(X)$$

sending $[E]$ to $[E \oplus \tau_1]$. Define $\mathrm{Vect}_s(X) = \mathrm{colim}_k\{\mathrm{Vect}^k_{\mathbb{C}}(X), t_k\}$ as the colimit of the sequence given by the maps $t_k$.

**Proposition 8.2.8.** *For every $X$ in $\mathsf{Top}$ we have that $\mathrm{Vect}_s(X) \cong \widehat{K}(X)$.*

*Proof.* For each $k \geq 0$ define a map $\varphi_k\colon \mathrm{Vect}^k_{\mathbb{C}}(X) \to \widehat{K}(X)$ as $\varphi_k([E]) = [E] - [\tau_k] \in \widehat{K}(X)$. Then $\varphi_{k+1}t_k([E]) = \varphi_k([E])$ for every $k$, so by the universal property of the colimit, there is a map $\varphi\colon \mathrm{Vect}_s(X) \to \widehat{K}(X)$ and a commutative triangle

$$\begin{array}{ccc} \mathrm{Vect}^k_{\mathbb{C}}(X) & \longrightarrow & \mathrm{Vect}_s(X) \\ {\scriptstyle \varphi_k}\downarrow & \swarrow{\scriptstyle \varphi} & \\ \widehat{K}(X) & & \end{array}$$

Now using the fact that for every bundle $E$ we can find another bundle $E'$ such that $E \oplus E' \cong \tau_n$ for some $n$, one checks that $\varphi$ is injective and surjective. □

**Proposition 8.2.9.** *For every $X$ in $\mathsf{Top}$ there is an isomorphism $\widehat{K}(X) \cong [X, BU]$.*

*Proof.* By Theorem 8.2.3, we know that $\mathrm{Vect}^k_{\mathbb{C}}(X) \cong [X, BU_k]$. The previously defined maps $t_k\colon \mathrm{Vect}^k_{\mathbb{C}}(X) \to \mathrm{Vect}^{k+1}_{\mathbb{C}}(X)$ and $i_k\colon BU_k \to BU_{k+1}$ are compatible with this isomorphism. So we have an isomorphism after taking colimits

$$\mathrm{colim}_k\, \mathrm{Vect}^k_{\mathbb{C}}(X) \cong \mathrm{colim}_k[X, BU_k]$$

The left-hand side is $\mathrm{Vect}_s(X)$, which by Proposition 8.2.8 is isomorphic to $\widehat{K}(X)$. Using the fact that $X$ is compact and that the maps $i_k$ are embeddings (see exercise sheet 8) the right-hand side is isomorphic to $[X, \mathrm{colim}_k\, BU_k] = [X, BU]$. □

**Corollary 8.2.10.** *If $X \in \mathsf{Top}$, then $K^0(X) \cong [X, BU \times \mathbb{Z}]$. If $X \in \mathsf{Top}_*$ and $X$ is connected, then $\widetilde{K}^0(X) \cong [X, BU]$.*

*Proof.* By Proposition 8.2.5 we have a splitting $K^0(X) \cong \widehat{K}(X) \oplus [X, \mathbb{Z}]$. This fact, together with Proposition 8.2.9 implies that $[X, BU \times \mathbb{Z}]$. The second part follows from Corollary 8.2.6 and Proposition 8.2.9. □

**Corollary 8.2.11.** *Let $X \in \mathsf{Top}_*$ such that the inclusion $i\colon * \to X$ is a cofibration (e.g., if $X$ is a CW-complex). Then $\widetilde{K}^0(X) \simeq [X, BU \times \mathbb{Z}]_*$.*

*Proof.* We need to show that $[X, BU \times \mathbb{Z}]_*$ is the kernel of the map

$$K^0(X) \cong [X, BU \times \mathbb{Z}] \xrightarrow{i^*} [*, BU \times \mathbb{Z}] \cong K^0(*)$$

Let $j\colon [X, BU \times \mathbb{Z}]_* \to [X, BU \times \mathbb{Z}]$ be the natural inclusion. If $f \in [X, BU \times \mathbb{Z}]_*$, then $i^*j(f)$ is zero in $[*, BU \times \mathbb{Z}]$. So $[X, BU \times \mathbb{Z}]_* \subseteq \ker i^*$.

To prove the converse, let $g \in [X, BU \times \mathbb{Z}]$ and suppose that $i^*j(g)$ is zero. Since $BU$ is connected, there is a homotopy between the basepoint of $BU$ and $g_1(x_0)$, where $x_0$ denotes the basepoint of $X$. So we can build a homotopy

$$\alpha\colon \{x_0\} \times I \longrightarrow BU \times \mathbb{Z}$$

between $(g_1(x_0), 0)$ and $(*, 0)$, where here $*$ is the basepoint of $BU$. Now consider the following diagram

$$\begin{array}{ccc} X \times \{0\} \cup \{x_0\} \times I & \xrightarrow{(g,\alpha)} & BU \times \mathbb{Z} \\ \downarrow & \overset{H}{\nearrow} & \\ X \times I. & & \end{array}$$

Since $* \to X$ is a cofibration, there is a lifting $H$, giving a homotopy between $g(x) = H(x, 0)$ and $H(x, 1)$ which is a *pointed* map, since $H(x_0, 1) = \alpha(x_0, 1) = (*, 0)$. □

**8.2.12.** The family of spaces $E_{2n} = BU \times \mathbb{Z}$ and $KU_{2n+1} = \Omega BU$ for $n \in \mathbb{Z}$ have the property that

$$KU_{2n-1} = \Omega BU = \Omega(BU \times \mathbb{Z}) = \Omega KU_{2n}.$$

By the standard form of Bott periodicity (see Theorem 7.3.2) we know that $\widetilde{K}^0(X) \cong \widetilde{K}^0(\Sigma^2 X)$, hence Corollary 8.2.11 shows that

$$[X, BU \times \mathbb{Z}]_* \cong \widetilde{K}^0(X) \cong \widetilde{K}^0(\Sigma^2 X) \cong [X, \Omega^2(BU \times \mathbb{Z}]_* = [X, \Omega^2 BU]_*$$

for every pointed (CW-complex) $X$. So $KU_{2n} = BU \times \mathbb{Z} \simeq \Omega^2 BU = \Omega KU_{2n+1}$.

**8.2.13.** Therefore the sequence $\{KU_n\}_{n\in\mathbb{Z}}$ defines an $\Omega$-spectrum called the *complex $K$-theory spectrum*, and hence a reduced cohomology theory, by Theorem 8.1.7. If $X$ is a pointed finite CW-complex, then $\widetilde{KU}^0(X) \cong \widetilde{K}^0(X)$.

**8.2.14.** The existence of a homotopy equivalence $BU \times \mathbb{Z} \cong \Omega^2 BU$ is equivalent to Bott periodicity. In fact, this equivalence will be proved in the last part of the course, by using simplicial methods.

**Theorem 8.2.15** (Bott periodicity —topological version)**.** *There is a homotopy equivalence $BU \times \mathbb{Z} \simeq \Omega^2 BU$.*

## 8.3. The Hopf invariant one problem

**8.3.1.** A *multiplication* for the sphere $S^n$ is a continuous map $\mu\colon S^n \times S^n \to S^n$ with a tow-sided unit $e \in S^n$ such that $\mu(x, e) = \mu(e, x) = x$ for all $x$ in $S^n$. For the values $n = 0, 1, 3$ and $7$ such a multiplication exists, and it is given by the multiplication in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, respectively. We would like to know if these are the only possible cases. Indeed, this is true.

**Theorem 8.3.2.** *$S^n$ admits a multiplication if and only if $n = 0, 1, 3$ or $7$.*

We will not prove this theorem, but we will show how it is implied by the existence of Hopf invariant one maps between certain spheres. This result is important because of the following

**Proposition 8.3.3.** *If $\mathbb{R}^n$ is a division algebra, then $S^{n-1}$ admits a multiplication.*

*Proof.* We can assume that the multiplication on $\mathbb{R}^n$ has an identity that is a unit vector (if this is not the case one can always modify the multiplication to obtain one with this property). The multiplication on $S^{n-1}$ is defined by sending $(x, y)$ to $xy$ divided by its norm. □

Hence, from Theorem 8.3.2, we deduce that $\mathbb{R}^n$ is a division algebra if and only if $n = 1, 2, 4$ or 8.

**8.3.4.** We can use the computations that we have made of the $K$-groups of the spheres to show that multiplications do not exist for spheres of even dimension. We will use that $K^0(S^{2n}) = \mathbb{Z} \oplus \mathbb{Z}$ (generated by the trivial bundle and $(H-1)^n$), but we will also need the ring structure of $K^0(S^{2n})$, which can be deduced from the following proposition. Recall that for $X \in \mathsf{Top}_*$ we have defined the external product $\widetilde{K}^0(X) \otimes \widetilde{K}^0(X) \to \widetilde{K}^0(X \wedge X)$. We can compose this with a map induced by the diagonal $\Delta\colon X \to X \times X$ to get a product map

$$\widetilde{K}^0(X) \otimes \widetilde{K}^0(X) \longrightarrow \widetilde{K}^0(X \wedge X) \longrightarrow \widetilde{K}^0(X)$$

**Proposition 8.3.5.** *Let $X \in \mathsf{Top}_*$ and let $X = A \cup B$, where $A$ and $B$ are closed contractible subspaces of $X$. Then the product map $\widetilde{K}^0(X) \otimes \widetilde{K}^0(X) \to \widetilde{K}^0(X)$ is trivial.*

*Proof.* Since $A$ and $B$ are contractible $\widetilde{K}^0(X) \cong \widetilde{K}^0(X/A)$ and $\widetilde{K}^0(X) \cong \widetilde{K}^0(X/B)$ by Corollary 6.1.6. The external product defines a map

$$\widetilde{K}^0(X/A) \otimes \widetilde{K}^0(X/B) \longrightarrow \widetilde{K}^0(X/A \wedge X/B)$$

and one can check that $X/A \wedge X/B \cong X \times X/(A \times X) \cup (X \times B) = W$ and that the diagonal induces a map $X/A \cup B \to W$. So, we have a commutative diagram

$$\begin{array}{ccccc}
\widetilde{K}^0(X/A) \otimes \widetilde{K}^0(X/B) & \longrightarrow & \widetilde{K}^0(W) & \longrightarrow & \widetilde{K}^0(X/A \cup B) = 0 \\
{\scriptstyle\cong}\downarrow & & \downarrow & & \downarrow \\
\widetilde{K}^0(X) \otimes \widetilde{K}^0(X) & \longrightarrow & \widetilde{K}^0(X \wedge X) & \longrightarrow & \widetilde{K}^0(X)
\end{array}$$

where the right up corner is zero (because $A \cup B = X$). Since the left map is an isomorphism the bottom composition (which is the product) is zero. □

**Corollary 8.3.6.** $K^0(S^{2n}) \cong \mathbb{Z}[\gamma]/(\gamma^2)$.

*Proof.* Let $\gamma = (H-1)^n$. Then $\gamma \in \widetilde{K}^0(S^{2n})$ and we know, by Proposition 8.3.5, that $\gamma^2 = 0$ (by taking $A$ the closed upper hemisphere and $B$ the closed lower hemisphere of $S^{2n}$). □

**Proposition 8.3.7.** *The sphere $S^{2n}$ does not admit a multiplication for $n \geq 1$.*

*Proof.* Recall that Bott periodicity for reduced $K$-theory (see Theorem 7.3.2) states that we have an isomorphism

$$\widetilde{K}^0(X) \otimes \widetilde{K}^0(S^2) \xrightarrow{\cong} \widetilde{K}^0(X \wedge S^2).$$

We can iterate this isomorphism several times, by replacing $X$ by $X \wedge S^2$, and we get an isomorphism

$$\widetilde{K}^0(X) \otimes \widetilde{K}^0(S^{2n}) \xrightarrow{\cong} \widetilde{K}^0(X \wedge S^{2n}).$$

for every $n \geq 1$, and a corresponding isomorphism for the unreduced case

$$K^0(X) \otimes K^0(S^{2n}) \xrightarrow{\cong} K^0(X \times S^{2n}).$$

Suppose now that $\mu\colon S^{2n} \times S^{2n} \to S^{2n}$ is a multiplication. Applying $K^0$ gives a map

$$\mathbb{Z}[\gamma]/(\gamma^2) \cong K^0(S^{2n}) \xrightarrow{\mu^*} K^0(S^{2n} \times S^{2n}) \cong K^0(S^{2n}) \otimes K^0(S^{2n}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2).$$

Since $\mu$ has a two-sided unit (given by map $e\colon * \to S^{2n}$) we have a commutative diagram

$$\begin{array}{ccccc} S^{2n}\times\{*\} & \xrightarrow{1\times e} & S^{2n}\times S^{2n} & \xleftarrow{e\times 1} & \{*\}\times S^{2n} \\ & \searrow\!\!= & \downarrow \mu & =\!\!\swarrow & \\ & & S^{2n}, & & \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccccc} K^0(S^{2n}) & \longleftarrow & K^0(S^{2n}\times S^{2n}) & \longrightarrow & K^0(S^{2n}) \\ & =\!\!\nwarrow & \uparrow \mu^* & \nearrow\!\!= & \\ & & K^0(S^{2n}), & & \end{array}$$

where the map to the left sends $\alpha$ to $\gamma$ and $\beta$ to 0, and the map to the right sends $\alpha$ to 0 and $\beta$ to $\gamma$. This forces $\mu^*(\gamma)$ to be of the form $\alpha+\beta+t\alpha\beta$. Hence $0=\mu^*(\gamma^2)=\alpha^2+\beta^2+t^2\alpha^2\beta^2+2t\alpha^2\beta+2t\alpha\beta^2+2\alpha\beta=2\alpha\beta$. But $2\alpha\beta$ cannot be zero, so $\mu$ cannot be a multiplication. □

**8.3.8.** Things became more involved for spheres of odd dimension. Here is where the Hopf invariant appears. Suppose that we have a map $g\colon S^{n-1}\to S^{n-1}\to S^{n-1}$ for $n$ an even number. We can decompose the sphere $S^{2n-1}$ as follows

$$S^{2n-1}=\partial(D^{2n})=\partial(D^n\times D^n)=S^{n-1}\times D^n\cup D^n\times S^{n-1},$$

where $D^n$ denotes the $n$-dimensional disk in $\mathbb{R}^n$. We can define maps

$$\begin{array}{ccc} S^{n-1}\times D^n & \longrightarrow & D^n_+ \\ (x,y) & \longmapsto & |y|g(x,y/|y|) \end{array} \qquad \begin{array}{ccc} D^n\times S^{n-1} & \longrightarrow & D^n_- \\ (x,y) & \longmapsto & |x|g(x/|x|,y). \end{array}$$

They coincide in the intersection of the domains so the give a map

$$\widehat{g}\colon S^{2n-1}=S^{n-1}\times D^n\cup D^n\times S^{n-1}\longrightarrow D^n_+\cup D^n_-=S^n.$$

The map $\widehat{g}$ is called the *Hopf construction* of the map $g$.

**8.3.9.** We are interested in odd spheres, so let now $g\colon S^{2n-1}\times S^{2n-1}\to S^{2n-1}$, build the corresponding $\widehat{g}\colon S^{4n-1}\to S^{2n}$ and take the mapping cone

$$S^{4n-1}\xrightarrow{\widehat{g}} S^{2n}\longrightarrow C(\widehat{g})$$

obtained by taking the cone on $S^{4n-1}$, which is $D^{4n}$, and gluing it to $S^{2n}$ via $\widehat{g}$. Then, applying $\widetilde{K}^0$ gives a split short exact sequence

$$0\longrightarrow \widetilde{K}^0(S^{4n})\longrightarrow \widetilde{K}^0(C(\widehat{g}))\longrightarrow \widetilde{K}^0(S^{2n})\longrightarrow 0$$

since $\widetilde{K}^0$ of odd degree spheres is zero. So the group in the middle is $\mathbb{Z}\oplus\mathbb{Z}$ generated by $\alpha$ and $\beta$, where $\alpha$ is the image of the generator of $\widetilde{K}^0(S^{4n})$ (hence $\alpha^2=0$) and $\beta$ is some class mapped to the generator of $\widetilde{K}^0(S^{2n})$ (hence $\beta^2$ is mapped to 0). By exactness, $\beta^2$ is in the image of the second map, so there is an integer $H(\widehat{g})$ such that $\beta^2=H(\widehat{g})\alpha$. The integer $H(\widehat{g})$ is called the *Hopf invariant*.

One has to be a bit careful here, because we have to check that $\beta$ is well-defined. We could have replaced $\beta$ by $\beta+t\alpha$. Then $\beta^2$ would be $\beta^2+2t\alpha\beta$. We claim that $\alpha\beta=0$. Indeed, $\alpha\beta$ is mapped to 0 in $\widetilde{K}^0(S^{2n})$, so $\alpha\beta=k\alpha$ for some $k$. But $\alpha\beta^2=k\alpha\beta=k^2\alpha$. On the other hand, $\alpha\beta^2=H(\widehat{g})\alpha^2=0$ so $k^2\alpha=0$ and hence $k=0$. So the Hopf invariant is independent of $\beta$.

Note, however, that the Hopf invariant is not independent of the choice of $\alpha$ since we could choose $-\alpha$ and this would change $H(\widehat{g})$ by $-H(\widehat{g})$. So, by convention, we will always assume that the Hopf invariant is non-negative, and hence independent of $\alpha$ and $\beta$.

**Proposition 8.3.10.** *If $g\colon S^{2n-1} \times S^{2n-1} \to S^{2n-1}$ is a multiplication, then $\widehat{g}\colon S^{4n-1} \to S^{2n}$ has Hopf invariant one.*

*Proof.* As we have seen, the map $\widehat{g}\colon S^{4n-1} \to S^{2n}$ is constructed by attaching the cone on $S^{4n-1}$ to $S^{2n}$ via $\widehat{g}$. So we have a map of pairs

$$(D^{4n}, S^{4n-1}) \longrightarrow (C(\widehat{g}), S^{2n}),$$

that can be seen as a map

$$\Phi\colon (D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n})) \longrightarrow (C(\widehat{g}), S^{2n}).$$

Recall that $\widehat{g}$, and therefore also $\Phi$, send $S^{2n-1} \times D^{2n}$ into $D^{2n}_+$ and $D^{2n} \times S^{2n-1}$ into $D^{2n}_-$. Consider the following diagram

$$\begin{array}{ccc}
\widetilde{K}^0(C(\widehat{g})) \otimes \widetilde{K}^0(C(\widehat{g})) & \longrightarrow & \widetilde{K}^0(C(\widehat{g})) \\
\uparrow \cong & & \uparrow \\
\widetilde{K}^0(C(\widehat{g})/D^{2n}_+) \otimes \widetilde{K}^0(C(\widehat{g})/D^{2n}_-) & \longrightarrow & \widetilde{K}^0(C(\widehat{g})/S^{2n}) \\
\downarrow \Phi^*\otimes\Phi^* & & \cong \downarrow \Phi^* \\
\widetilde{K}^0(\frac{D^{2n}\times D^{2n}}{S^{2n-1}\times D^{2n}}) \otimes \widetilde{K}^0(\frac{D^{2n}\times D^{2n}}{D^{2n}\times S^{2n-1}}) & \longrightarrow & \widetilde{K}^0(\frac{D^{2n}\times D^{2n}}{\partial(D^{2n}\times D^{2n})}) \\
\cong \downarrow & \nearrow \cong & \\
\widetilde{K}^0(\frac{D^{2n}\times *}{S^{2n-1}\times *}) \otimes \widetilde{K}^0(\frac{*\times D^{2n}}{*\times S^{2n-1}}), & &
\end{array}$$

where the diagonal map is an isomorphism by Bott periodicity. Take now the composite

$$\widetilde{K}^0(C(\widehat{g})/D^{2n}_+) \xrightarrow{\Phi^*} \widetilde{K}^0(D^{2n} \times D^{2n}/S^{2n-1} \times D^{2n}) \longrightarrow \widetilde{K}^0(D^{2n} \times */S^{2n-1} \times *).$$

Due to how we have defined the map $\widehat{g}$, the previous composite is induce by the map of pairs

$$(D^{2n} \times *, S^{2n-1} \times *) \cong (D^{2n}_-, S^{2n-1}) \longrightarrow (S^{2n}, D^{2n}_+) \longrightarrow (C(\widehat{g}), D^{2n}_+),$$

where the last two maps on the right are inclusions. But then $\beta$ in $\widetilde{K}^0(C(\widehat{g}))$ maps to a generator of $\widetilde{K}^0(S^{2n}/D^{2n}_+) \cong \widetilde{K}^0(D^{2n}_-, S^{2n-1})$. Similarly, one can show that $\beta \otimes \beta$ maps to a generator of the bottom left-hand group. Thus, $\beta \otimes \beta$ maps to a generator of the bottom right-hand group and then the Hopf invariant has to be, by our convention, equal to 1. □

**8.3.11.** By Proposition 8.3.10, to prove Theorem 8.3.2 it is enough to show that there is no map $S^{4n-1} \to S^{2n}$ of Hopf invariant 1 unless $n = 1, 2$ or 4. This is a non-trivial statement and uses *Adams operations* on $K$-theory and the *splitting principle*.