Topological K-theory, Lecture 3

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1 Applications of the classification theorem – continued

Let us see how the classification theorem can further be used.

Example 1. The bundle $\gamma_n : E_n(\mathbf{k}^\infty) \longrightarrow G_n(\mathbf{k}^\infty)$ admits an inner product, induced from an inner product on $\mathbf{k}^\infty$. Since every rank-n vector bundle is obtained as a pullback along γ_n, we deduce that any vector bundle admits an inner product – that obtained by pulling back the one on γ_n. This is a shortened proof to what you already showed in the exercise.

Example 2. Let us compute the Picard group of real projective spaces. By the classification theorem we have

$$\mathrm{VBun}^1_{\mathbf{R}}(\mathbf{R}P^n) \cong [\mathbf{R}P^n, G_1(\mathbf{R}^\infty)] = [\mathbf{R}P^n, \mathbf{R}P^\infty].$$

You have shown in the exercise that $V_1(\mathbf{R}^\infty) \longrightarrow G_1(\mathbf{R}^\infty)$ is a fiber bundle with fiber $\mathrm{GL}_1(\mathbf{R})$. By a theorem you proved in Algebraic topology I, any fiber bundle is a Serre fibration, so that we have a fibration sequence

$$\mathrm{GL}_1(\mathbf{R}) \longrightarrow V_1(\mathbf{R}^\infty) \longrightarrow G_1(\mathbf{R}^\infty). \tag{1}$$

You have seen in the exercise class that the spaces $V_n(\mathbf{k}^\infty)$ are contractible and it is easy to see that $\mathrm{GL}_1(\mathbf{R}) \simeq \mathbf{Z}/2$. Thus, the long exact sequence for the fibration sequence 1 implies that $G_1(\mathbf{R}^\infty) = \mathbf{R}P^\infty$ is a $K(\mathbf{Z}/2, 1)$. Now, an application of Brown's representability theorem (which you proved in Algebraic Topology I) implies that $[\mathbf{R}P^n, \mathbf{R}P^\infty] = [\mathbf{R}P^n, K(\mathbf{Z}/2, 1)] \cong H^1(\mathbf{R}P^n; \mathbf{Z}/2)$ – i.e. the first cohomology group of $\mathbf{R}P^n$ with coefficients in $\mathbf{Z}/2$. Using cellular cohomology (this is an elementary way of calculation, given in any first course in cohomology) we deduce that $\mathrm{Pic}(\mathbf{R}P^n) = \mathrm{VBun}^1_{\mathbf{R}}(\mathbf{R}P^n) \cong H^1(\mathbf{R}P^n; \mathbf{Z}/2) \cong \mathbf{Z}/2$. In fact, it follows from what you showed in the exercise that group structure is given by the tensor product. Thus, there is a line bundle ζ on $\mathbf{C}P^n$ (corresponding to $1 \in \mathbf{Z}$) such that $\zeta \otimes ... \otimes \zeta$ (n-times) correspond to $n \in \mathbf{Z}$ – this is the canonical line bundle introduced in Lecture 1!

Observe 1.1. Let B be a paracompact space. Then any n-dimensional bundle can be embedded in a trivial infinite bundle.

Proof. Write

$$\begin{array}{ccccccc} E & \xrightarrow{\cong} & f^*(E_n(\mathbf{k}^\infty)) & \xrightarrow{\tilde{f}} & E_n(\mathbf{k}^\infty)\mathbf{k} & \hookrightarrow & G_n(\mathbf{k}^\infty) \times \mathbf{k}^\infty \\ & \searrow^{p} & \downarrow & & \downarrow{\scriptstyle \gamma_n} & & \\ & & B & \xrightarrow{f} & G_n(\mathbf{k}^\infty). & & \end{array}$$

We saw last time that $\overline{f}$ is a linear injection on fibers and to the composite $E \longrightarrow G_n(\mathbf{k}^\infty) \times \mathbf{k}^\infty$ has the same property. □

Corollary 1.2. *If B is compact Hausdorff, any n-dimensional vector bundle can be embedded in a trivial (finite dimensional) bundle.*

Proof. For $d > n$,

$$G_n(\mathbf{k}^d) \subseteq G_n(\mathbf{k}^{d+1}) \subseteq \ldots \subseteq \bigcup_{d>n} G_n(\mathbf{k}^d) = G_n(\mathbf{k}^\infty).$$

Since B is compact, the classifying map $B \longrightarrow G_n(\mathbf{k}^\infty)$ factors as $B \xrightarrow{f'} G_n(\mathbf{k}^d) \xrightarrow{i} G_n(\mathbf{k}^\infty)$ and by the pasting lemma for pullbacks we get that $f^* E_n(\mathbf{k}^\infty) \cong f'^* E_n(\mathbf{k}^d)$. We thus get

$$\begin{array}{ccccccc} E & \xrightarrow{\cong} & f'^*(E_n(\mathbf{k}^d)) & \xrightarrow{\tilde{f}'} & E_n(\mathbf{k}^d) & \hookrightarrow & G_n(\mathbf{k}^d) \times \mathbf{k}^d \\ & \searrow^{p} & \downarrow & & \downarrow{\scriptstyle \gamma_n} & & \\ & & B & \xrightarrow{f'} & G_n(\mathbf{k}^d). & & \end{array}$$

i.e. an embedding of E in a trivial bundle.

□

2 K-theory

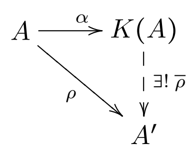
Corollary 1.2 is going to be crucial for us. We thus assume throughout that all our spaces are compact Hausdorff. This includes for example all finite CW-complexes.

Let B be a connected space. Denote by $\mathrm{VBun}^n_{\mathbf{k}}(B)$ the set of isomorphism classes of n-dimensional vector bundles over B. Set $\mathrm{VBun}^*_{\mathbf{k}}(B) = \bigoplus_{n \geq 0} \mathrm{VBun}^n_{\mathbf{k}}(B)$ where by convention $\mathrm{VBun}^0_{\mathbf{k}}(B) = *$. The direct sum of vector bundles induces an abelian monoid structure on $\mathrm{VBun}^*_{\mathbf{k}}(B)$. We can further extend this by setting, for a non-connected space $B \coprod_\alpha B_\alpha$ (a disjoint union of path components), $\mathrm{VBun}(B) = \prod_\alpha \mathrm{VBun}^*(B_\alpha)$ with the ordinary abelian monoid structure.

2.1 Group completion

We would like to turn the abelian monoid $\mathrm{VBun}^*_{\mathbf{k}}(B)$ (or $\mathrm{VBun}(B)$) into an abelian group so that we could apply group theoretic methods in calculations. Of course, we need some canonical way to do so, and we can obtain such by requiring a universal property. The following is a purely algebraic method.

Definition 2.1. Let A be an abelian monoid. A **group completion** of A is an abelian group $K(A)$ together with a map of (abelian) monoids $\alpha = \alpha_A : A \longrightarrow K(A)$ such that for any abelian group A' and any map of abelian monoids $\rho : A \longrightarrow A'$, there exists a unique map of abelian groups $\overline{\rho} : K(A) \longrightarrow A'$ rendering the following diagram commutative:

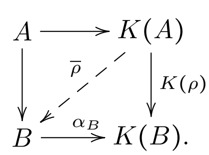


(2)

Remark 2.2. Clearly, if $K(A)$ exists, it is unique up to a unique isomorphism.

Let us construct $K(A)$ for an arbitrary $(A, \oplus)$. Let $F(A)$ be the free abelian group generated by the underlying set of A and let $E(A) \subseteq F(A)$ be the subgroup generated by elements of the form $a + a' - a \oplus a'$ where $+ = +_{F(A)}$. The quotient $K(A) \coloneqq F(A)/E(A)$ is clearly an abelian group which together with the obvious map $\alpha : A \longrightarrow K(A)$ satisfy the universal property of 2.

Alternatively we can define $K(A)$ as follows. Let $\Delta : A \longrightarrow A \times A$ be the diagonal. The quotient $K(A) = A \times A/\Delta(A)$ inherits an abelian monoid structure which has inverses since $[a, a] = 0$. We think of an element $[a, b]$ of $K(A)$ as a formal difference $a - b$ where $[a, b] = [a', b']$ iff $a \oplus b' = a' \oplus b$. We set $\alpha_A : A \longrightarrow K(A)$ by $a \mapsto [a, 0]$. Since $K(A)$ is functorial in A, we get for any map of abelian monoids $\rho : A \longrightarrow B$, a commutative square



If B is in fact an abelian group, α_B is an isomorphism so that $\overline{rho} \coloneqq \alpha_B^{-1} \circ K(\rho)$ satisfy the universal property.

Exercise 2.3.

- *Let* `AbGp` *and* `AbMon` *be the categories of abelian groups and abelian monoids respectively. Show that there is an adjunction*

  $$K : \mathtt{AbMon} \underset{}{\overset{\perp}{\rightleftarrows}} \mathtt{AbGp} : U$$

  where U is the forgetful functor.

- *Show that if A was a (commutative) semi-ring (i.e. admits a commutative 'multiplication' operation $\otimes$ which distributes over $\oplus$) then $K(A)$ is in fact a commutative ring.*

Recall that our assumption throughout was that our base space B is connected.

Definition 2.4. The K-groups of a connected space B are defined to be

$$K(B) = KU(B) := K(\mathrm{VBun}^*_{\mathbf{C}}(B), \oplus)$$

and

$$KO(B) = K(\mathrm{VBun}^*_{\mathbf{R}}(B)).$$

where the monoid structure is taken to be direct sum of vector bundles and the ring structure is induced from tensor product of bundles.

If $B = \coprod_\alpha B_\alpha$ a disjoint union of path components, we set

$$K(B) = K(\mathrm{VBun}(B))$$

and similarly for $KO(B)$.

From now on, we will focus on $K(B)$ but almost everything works equally well for $KO(B)$.

Let $\mathtt{CH}$ be the category of compact Hausdorff spaces. The assignment $B \mapsto K(B)$ defines a functor $K : \mathtt{CH}^{\mathrm{op}} \longrightarrow \mathtt{AbGp}$ by setting for a map $B' \longrightarrow B$, $K(f) := f^* : K(B) \longrightarrow K(B')$. The pasting lemma for pullbacks verifies $(f \circ f')^* = f'^* \circ f^*$.

Observe 2.5. Using our second construction of K, an element of $K(B)$ can be described as a formal difference $[E] - [F]$ of isomorphism classes of vector bundles. The expression is sometimes called a **virtual vector bundle**.

Let τ_n denote the trivial bundle of rank n.

Observe 2.6. If E is a vector bundle over B, there is $n \in \mathbf{N}$ and an embedding $E \longrightarrow \tau_n$ (i.e. a map of vector bundles which is linear injection on each fiber). Then we can take the orthogonal complement $E^\perp$ of E with respect to τ_n. This is done just as the other operations on vector bundles we talked about before – fiberwise. Strictly speaking, $(-)^\perp$ is not a functor on finite dimensional vector spaces but rather a (topological) functor on finite dimensional vector spaces, embedded in some ambient vector space. The induced functor on (suitable) vector bundles is constructed in the same way as before.

It follows that for any E there is an $n \in \mathbf{N}$ such that $E \oplus E^\perp \cong \tau_n$.

Suppose $[E] - [F] \in K(B)$ and let G be a vector bundle such that $F \oplus G$ is trivial. Then

$$[E] - [F] = [E] + [G] - ([G] - [F]) = [E \oplus G] - [\tau_n].$$

Thus every element in $K(B)$ is of the form $[H] - [\tau_n]$. Suppose $[E] = [F]$ in $K(B)$. Then $([E],[F]) = ([G],[G])$ for some G so that $E \oplus G \cong F \oplus G$. Let G' be such that $G \oplus G' \cong \tau_n$. Then $E \oplus \tau_n \cong F \oplus \tau_n$. We would like to view all trivial as one (trivial) element. We thus make the following

Definition 2.7. Two vector bundles E and F over B are said to be **stably equivalent** if there are $m, n \in \mathbf{N}$ such that $E \oplus \tau_n \cong F \oplus \tau_m$.

We denote by $\simeq_S$ the equivalence relation of stably equivalent vector bundles and let $\mathrm{SVBun}^*_{\mathbf{k}}(B) \coloneqq \mathrm{VBun}^*_{\mathbf{k}}(B)/\simeq_S$.

Suppose now B is pointed, namely equipped with a map $* \longrightarrow B$. We obtain an **augmentation** map $\epsilon : K(B) \longrightarrow K(*) \cong \mathbf{Z}$.

Definition 2.8. The **reduced K-theory** of a pointed space $(B, *)$ is defined to be $\tilde{K}(B) = \ker(\epsilon : K(B) \longrightarrow K(*))$.

The map $\epsilon : K(B) \longrightarrow \mathbf{Z}$ is given by $[E] \longrightarrow \dim E$. It follows that $\tilde{K}(B)$ consists of elements of the form $[E] - [F]$ with $\dim E = \dim F$.

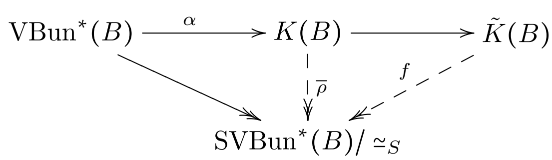
Observe 2.9. The map $B \longrightarrow *$ gives a natural splitting $K(B) \cong \tilde{K}(B) \oplus \mathbf{Z}$.

The following proposition shows that the algebraic description of Definition 2.8 is equivalent to the geometric description of Definition 2.7.

Proposition 2.10. *Let $(B, *)$ be a pointed (compact) space. Then* $\mathrm{SVBun}^*(B)$ *is an abelian group, and there is an isomorphism* $\mathrm{SVBun}^*(B) \cong \tilde{K}(B)$.

Proof. Clearly, $\mathrm{SVBun}^*(B)$ is an abelian monoid under direct sum and has inverses since the isomorphism $E \oplus E^\perp \cong \tau_n$ implies $[E]^{-1} \cong [E^\perp]$.

The natural surjection $\mathrm{VBun}^*(B) \longrightarrow \mathrm{SVBun}^*(B)$ is a map into an abelian group and the universal property of K implies the existence of the dashed arrow $\overline{\rho}$, which must also be a surjection:



Here, the map $K(B) \longrightarrow \tilde{K}(B)$ is given by $[E] \mapsto [E] - [\tau_{\dim E}]$ (recall that elements in $\tilde{K}(B)$ are of the form $[E] - [F]$ with $\dim_E = \dim_F$).

Since $\overline{\rho}(\tau_n) = 0$, we get a factorization of ρ through the map $f : \tilde{K}(B) \longrightarrow \mathrm{SVBun}^*(B)$ given by $[E]-[F] \mapsto [E]_S - [F]_S$. The map f is surjective since ρ is. To prove injectivity of f, we construct a left inverse. The map $\mathrm{VBun}^*(B) \longrightarrow K(B) \longrightarrow \tilde{K}(B)$ given by $[E] \mapsto [E] - [\tau_n]$ respects $\simeq_S$ and hence induces a map $j : \mathrm{SVBun}^*(B) \longrightarrow \tilde{K}(B)$. If $[E] - [F] \in \tilde{K}(B)$ then $j(f([E] - [F]) = [E] - [\tau_n] - ([F] - [\tau_n])$ since $\dim_E = \dim_F$. We see that $jf = \mathrm{id}$ so that f is injective and hence an isomorphism.

□

3 K-theory as a generalized cohomology theory

If $B = B' \coprod B'' \in \mathtt{CH}$ (note that it must be a finite disjoint union because of compactness), we have $\mathrm{VBun}*(B) = \mathrm{VBun}*(B') \oplus \mathrm{VBun}^*(B')$. Since $\oplus$ is the coproduct in both AbMon and AbGp and K is a left adjoint, $K(B) = K(B') \oplus K(B'')$. Let $(-)^+$ be the left adjoint to the forgetful functor $\mathtt{CH}_* \longrightarrow \mathtt{CH}$ from pointed compact Hausdorff spaces (and pointed maps) to compact Hausdorff spaces. It is given by $B^+ := B \coprod \{*\}$. We then have $\tilde{K}(B^+) = \ker(\epsilon : K(B) \oplus K(*) \longrightarrow K(*)) = K(B)$. For an inclusion $i : B' \longrightarrow B$ in CH we make a

Definition 3.1. The **relative K-groups** of a pair $B' \subseteq B \in \mathtt{CH}$ are $K(B, B') := \tilde{K}(B/B')$ where the base-point is taken to be B'/B'.

We have $K(B, \varnothing) = \tilde{K}(B^+) = K(B)$ so that our new definition specializes to the old one in the degenerate case. Our aim now is to establish an exact sequence of the form

$$K(B, B') \longrightarrow K(B) \longrightarrow K(B')$$

for any pair $B' \subseteq B \in \mathtt{CH}$. In order to do this, we need to be able to construct vector bundles on B/B' from vector bundles on B which are trivial on B'.

3.1 Construction of bundles over quotients

We assume that $B' \subseteq B \in \mathtt{CH}$ is a pair and denote by $q : B \longrightarrow B/B'$ the quotient map. Suppose $p : E \longrightarrow B$ is a vector bundle which is trivial over B'. Let $\alpha : E|_{B'} \xrightarrow{\cong} B' \times V$ be a trivialization and let $\pi : B' \times V \longrightarrow V$ be the projection. Define an equivalence relation on $E|_{B'}$ by setting $e\ e'$ iff $\pi(\alpha(e)) = \pi(\alpha(e'))$ and extend this relation by identity to E. Let $E/\alpha := E/$ be the quotient space and set $\overline{p} : E/\alpha \longrightarrow B/B'$ by $\overline{p}([e]) = q(p(e))$. Note that $\overline{p}$ is well-defined since if $e \neq e'$, $e\ e'$ only if $p(e), p(e') \in B'$. In fact, $e\ e'$ only if they are in a different fiber which means that we collapsed all the fibers parametrized by B' into a single fiber. Thus, $\overline{p} : E/\alpha \longrightarrow B/B'$ has a fiber isomorphic to V over every point. We would like to show that $\overline{p} : E/\alpha \longrightarrow B/B'$ is in fact a fiber bundle. For that will will need a lemma which you are requested to prove in the exercise.

Lemma 3.2. *If $E \longrightarrow B$ is trivial over a closed subspace $B' \subseteq B$ then there exists an open neighbourhood $B' \subseteq U \subseteq B$ over which E is still trivial.*

Take such an open $B' \subseteq U$ and a trivialization $(\varphi_1, \varphi_2) : E|_U \xrightarrow{\cong} U \times V$. Then this induces a trivialization $\varphi : (E/\alpha)|_U = (E|_U)/\alpha \longrightarrow (U/B') \times V$ given by $\varphi([e]) = (q\varphi_1(e), \varphi_2(e))$. This is a local trivialization of E/α around $B'/B' \in B/B'$. Around $b \in B - B'$ we have an open neighbourhood $U \subseteq B - B'$ so that we can use the same local trivializations of $E \longrightarrow B$ (restricted to U) to get a trivialization of $E/\alpha \longrightarrow B/B'$. We deduce that $E/\alpha \longrightarrow B/B'$ is a vector bundle.