Transfer of algebras along derived Quillen adjunctions

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Outline of the talk

1. Framework
2. Transfer of algebra structures
3. Localization and colocalization of algebras
4. Application to motivic stable homotopy

Operads and algebras in enriched categories

- Let $\mathcal{V}$ be a closed symmetric monoidal category.
- $Coll_C(\mathcal{V})$ the category of C-**coloured collections** in $\mathcal{V}$.
- $Oper_C(\mathcal{V})$ the category of C-**coloured operads** in $\mathcal{V}$.
- Let $\mathcal{M}$ a monoidal $\mathcal{V}$-category, i.e., closed symmetric monoidal and equipped with a monoidal left adjoint

$$\mathcal{V} \longrightarrow \mathcal{M}$$

$\mathcal{M}$ is enriched, tensored and cotensored over $\mathcal{V}$ in a compatible way.

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Algebras in $\mathcal{M}^C = \prod_{c \in C} \mathcal{M}$

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Model structures for operads

Let $\mathcal{V}$ be a cofibrantly generated monoidal model category. There is a free-forgetful adjunction

$$F\colon \mathit{Coll}_C(\mathcal{V}) \rightleftarrows \mathit{Oper}_C(\mathcal{V})\colon U.$$

Transfer principle

If $\mathcal{V}$ has a cofibrant unit, a symmetric monoidal fibrant replacement functor and an interval with a coassociative and cocommutative multiplication, then the model structure on $\mathit{Coll}_C(\mathcal{V})$ can be transferred to a model structure on $\mathit{Oper}_C(\mathcal{V})$ along the free-forgetful adjunction.

In general, the transfer principle does not provide a full model structure but a ***semi model structure*** on $\mathit{Oper}_C(\mathcal{V})$.

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In general, the transfer principle does not provide a full model structure but a ***semi model structure*** on $\mathit{Oper}_C(\mathcal{V})$.

Model structures for operads

In a semi model structure on $\mathcal{V}$ one replaces the lifting and factorization axiom by the following:

- The class of fibrations has the right lifting property with respect to the class of trivial cofibrations **with cofibrant domain**. Similarly for the class of trivial fibrations.
- There exist functorial factorizations of any morphism **with cofibrant domain** into a cofibration followed by a trivial fibration and also into a trivial cofibration followed by a fibration.

One also assumes that:

- The initial object is cofibrant.
- The classes of fibrations and trivial fibrations are closed under pullbacks and (possibly transfinite) composites.

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Theorem

For any cofibrantly generated monoidal model category $\mathcal{V}$ the category $Oper_C(\mathcal{V})$ has a transferred semi model structure.

Examples

- Simplicial sets.
- Topological spaces.
- Chain complexes.
- Symmetric spectra.
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Fibrant and cofibrant replacement of algebras

Key Lemma

Let $\mathcal{P}$ be a cofibrant operad in $\mathcal{V}$ and $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ a morphism in $\mathcal{M}^C$.

- *If the natural map $End_{\mathcal{P}}(\mathbf{Y}) \to Hom_{\mathcal{P}}(\mathbf{X}, \mathbf{Y})$ is a trivial fibration of collections, then any $\mathcal{P}$-algebra structure on $\mathbf{X}$ extends to a homotopy unique $\mathcal{P}$-algebra structure on $\mathbf{Y}$ such that $\mathbf{f}$ is a map of $\mathcal{P}$-algebras.*
- *If the natural map $End_{\mathcal{P}}(\mathbf{X}) \to Hom_{\mathcal{P}}(\mathbf{X}, \mathbf{Y})$ is a trivial fibration of collections, then any $\mathcal{P}$-algebra structure on $\mathbf{Y}$ lifts to a homotopy unique $\mathcal{P}$-algebra structure on $\mathbf{X}$ such that $\mathbf{f}$ is a map of $\mathcal{P}$-algebras.*

- $Hom_{\mathcal{P}}(\mathbf{X}, \mathbf{Y})(c_1, \ldots, c_n; c) = Hom(X(c_1) \otimes \cdots \otimes X(c_n), Y(c))$ if $\mathcal{P}(c_1, \ldots, c_n; c) \neq 0$ and zero otherwise.
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Fibrant and cofibrant replacement of algebras

We denote by R a fibrant replacement functor and by Q a cofibrant replacement functor in $\mathcal{M}$.

Proposition

Let $\mathcal{P}$ be a cofibrant operad in $\mathcal{V}$ and $\mathbf{X}$ a $\mathcal{P}$-algebra such that $X(c)$ is cofibrant in $\mathcal{M}$ for every $c \in C$. Then $R\mathbf{X}$ admits a $\mathcal{P}$-algebra structure such that $\mathbf{X} \to R\mathbf{X}$ is a map of $\mathcal{P}$-algebras

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Theorem

Let $\mathcal{M}$ and $\mathcal{N}$ be monoidal $\mathcal{V}$-model categories and $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon U$ a monoidal Quillen $\mathcal{V}$-adjunction. Let $\mathcal{P}$ be a cofibrant operad in $\mathcal{V}$. Let

$$\mathbb{L}F\colon Ho(\mathcal{M}^C) \rightleftarrows Ho(\mathcal{N}^C)\colon \mathbb{R}U$$

denote the derived adjunction. If $\mathbf{X}$ is a $\mathcal{P}$-algebra in $\mathcal{M}$ such that $X(c)$ is cofibrant for every $c \in C$, then $\mathbb{R}U\mathbb{L}F\mathbf{X}$ admits a homotopy unique $\mathcal{P}$-algebra structure such that the unit of the derived adjunction

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is a trivial fibration in $\mathcal{V}$, then $\mathbb{L}F\mathbb{R}U\mathbf{X}$ admits a homotopy unique $\mathcal{P}$-algebra structure such that the counit of the derived adjunction

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Proof.

If $\mathbf{X}$ is a $\mathcal{P}$-algebra, so is $U\mathbf{X}$ since U is a (lax) monoidal functor, and the counit of the adjunction

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$$i_{U\mathbf{X}} \colon Q(U\mathbf{X}) \longrightarrow U\mathbf{X}$$

has a homotopy unique $\mathcal{P}$-algebra structure such that $i_{U\mathbf{X}}$ is a map of $\mathcal{P}$-algebras. The functor F sends $\mathcal{P}$-algebras to $\mathcal{P}$-algebras and maps of $\mathcal{P}$-algebras to maps of $\mathcal{P}$-algebras, hence we have that

$$\mathbb{L}F\mathbb{R}U\mathbf{X} = \mathbb{L}F(U\mathbf{X}) = FQ(U\mathbf{X}) \longrightarrow FU\mathbf{X}$$

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Enriched localizations and colocalizations

Let $\mathcal{M}$ be a monoidal $\mathcal{V}$-model category. Let $\mathcal{L}$ be a set of objects of $\mathcal{M}$ and $\mathcal{K}$ a set of morphisms of $\mathcal{M}$.

We denote by $\mathcal{M}_{\mathcal{L}}$ the **enriched localized model structure** and by $\mathcal{M}^{\mathcal{K}}$ the **enriched colocalized model structure**.

Local and colocal objects and morphisms are defined using an **enriched homotopy function complex**, e.g., $Hom(Q(-), R(-))$.

Fibrant replacement in $\mathcal{M}_{\mathcal{L}}$ models the $\mathcal{L}$-localization functor in $\mathcal{M}$.
Cofibrant replacement in $\mathcal{M}^{\mathcal{K}}$ models the $\mathcal{K}$-colocalization functor in $\mathcal{M}$.

The identity functors

$$Id\colon \mathcal{M} \leftrightarrows \mathcal{M}_{\mathcal{L}} : Id \qquad Id\colon \mathcal{M}^{\mathcal{K}} \leftrightarrows \mathcal{M} : Id$$

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Local and colocal objects and morphisms are defined using an **enriched homotopy function complex**, e.g., $Hom(Q(-), R(-))$.

Fibrant replacement in $\mathcal{M}_{\mathcal{L}}$ models the $\mathcal{L}$-localization functor in $\mathcal{M}$.
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The identity functors

$$Id\colon \mathcal{M} \leftrightarrows \mathcal{M}_{\mathcal{L}} \colon Id \qquad Id\colon \mathcal{M}^{\mathcal{K}} \leftrightarrows \mathcal{M} \colon Id$$

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Ideals and coideals

Let $\mathcal{P}$ be a C-coloured operad. A subset $J \subseteq C$ is called an **ideal relative to** $\mathcal{P}$ if $\mathcal{P}(c_1, \ldots, c_n; c) = 0$ whenever $n \geq 1$, $c \in J$, and $c_i \notin J$ for some $i \in \{1, \ldots, n\}$. A subset $I \subseteq C$ is a **coideal relative to** $\mathcal{P}$ if $I = C \setminus J$ for some ideal J.

Example

The operad for modules over monoids *Mod* has as set of colours $C = \{r, m\}$. The only nonzero terms are $Mod(r, \ldots, r; r)$ and $Mod(r, \ldots, m, \ldots, r; m)$. In this case, $J = \emptyset$, C, $\{r\}$ are ideals, and $I = \emptyset$, C, $\{m\}$ are coideals.

Let F be an endofunctor on $\mathcal{M}$ and $J \subseteq C$. The **extension of** F **over** $\mathcal{M}^C$ **away from** J is the endofunctor on $\mathcal{M}^C$ defined as

$$F\mathbf{X} = (F_c X(c))_{c \in C},$$

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Localization of $\mathcal{P}$-algebras

Theorem

Let $\mathcal{P}$ be a cofibrant C-coloured operad in $\mathcal{V}$ and consider the extension of a localization functor (L, η) over $\mathcal{M}^C$ away from an ideal $J \subseteq C$.

If $\mathbf{X}$ is a $\mathcal{P}$-algebra in $\mathcal{M}$ such that $X(c)$ is cofibrant for every $c \in C$ and the morphism

$$(\eta_{\mathbf{X}})_{c_1} \otimes \cdots \otimes (\eta_{\mathbf{X}})_{c_n} : X(c_1) \otimes \cdots \otimes X(c_n) \longrightarrow L_{c_1}X(c_1) \otimes \cdots \otimes L_{c_n}X(c_n)$$

is an $\mathcal{L}$-local equivalence for every $n \geq 0$ whenever $\mathcal{P}(c_1, \ldots, c_n; c)$ is nonzero, then $\mathbf{LX}$ admits a homotopy unique $\mathcal{P}$-algebra structure such that $\eta_{\mathbf{X}}$ is a map of $\mathcal{P}$-algebras.

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Colocalizaton of $\mathcal{P}$-algebras

Theorem

Let $\mathcal{P}$ be a cofibrant C-coloured operad in $\mathcal{V}$ and consider the extension of a colocalization functor (K, ε) over $\mathcal{M}^C$ away from a coideal $C \setminus J$.

If $\mathbf{X}$ is a $\mathcal{P}$-algebra in $\mathcal{M}$ such that $X(c)$ is fibrant for every $c \in C$, the unit I is $\mathcal{K}$-colocal if $\mathcal{P}(\ , c) \neq 0$ for some $c \in C$, and for every $n \geq 1$

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are $\mathcal{K}$-colocal whenever $\mathcal{P}(c_1, \ldots, c_n; c)$ is nonempty and $c \notin C \setminus J$, then $K\mathbf{X}$ admits a homotopy unique $\mathcal{P}$-algebra structure such that $\varepsilon_{\mathbf{X}}$ is a map of $\mathcal{P}$-algebras.

The theorem remains true for an ideal J under the same assumptions, by additionally imposing that $X(c)$ is also cofibrant for every $c \in J$.

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Postnikov sections and connective covers

(joint work with M. Spitzweck, O. Röndigs and P. A. Østvær)

Let S be a base scheme and $\mathcal{M}$ the category $Spt_T^{\Sigma}(S)$ of **motivic symmetric spectra** with the stable model structure. This is a combinatorial simplicial symmetric monoidal proper model category.

The functor c_q is the cofibrant replacement functor of the colocalized model structure of $Spt_T^{\Sigma}(S)$ with respect to

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Slices of rings and modules

The q-th slice functor s_q is obtained by first colocalizing with respect to $\mathcal{K}(q)$ and second localizing with respect to $\mathcal{L}(q+1)$.

Theorem

- *If E is a motivic E_∞-ring spectrum, then c_0E and s_0E are motivic E_∞-ring spectra.*
- *If M is a module over a motivic E_∞-ring spectrum E, its $(q-1)$-connective cover c_qM is an c_0E-module and the slice s_qM is an s_0E-module.*

Examples

Slices of the motivic K-theory spectrum KGL, Hermitian K-theory KQ, and algebraic cobordism MGL.

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