Localization of Algebras over Coloured Operads

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(joint work with C. Casacuberta, I. Moerdijk and R. M. Vogt)

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Outline of the talk

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Motivation

- **Homotopical localizations preserve loop spaces** up to homotopy. In fact, Dror Farjoun proved that

$$L_f(\Omega X) \simeq \Omega L_{\Sigma f} X$$

for all spaces X and all maps f. [E. Dror Farjoun, *Cellular Spaces, Null Spaces and Localization*, Lecture Notes in Math. 1622, Springer, 1996.]

- A **homotopical localization** is a functor L together with a natural map $l_X \colon X \to LX$ for every X with the following universal property: for every map $g \colon X \to LY$ there is a map $h \colon LX \to LY$, unique up to homotopy, such that $h \circ l_X \simeq g$.

- **Examples:** Localization at primes; localization with respect to homology theories; Postnikov sections.

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- **Homotopical localizations preserve GEMs** up to homotopy.

A **GEM** (*generalized Eilenberg–Mac Lane space*) is a weak product of Eilenberg–Mac Lane spaces $\prod_{n=1}^{\infty} K(A_n, n)$, where A_1 is abelian.

The following categories are equivalent (Dold–Thom):

- The homotopy category of GEMs.
- The homotopy category of topological abelian groups.
- The homotopy category of simplicial abelian groups.
- The homotopy category of chain complexes of abelian groups which are zero in negative dimensions.

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- **Localizations in the category of abelian groups preserve rings and modules over a ring.** More precisely, if R is a ring, M is an R-module and L is a localization, then LM admits a unique R-module structure such that $l_M\colon M \to LM$ is a morphism of R-modules, and this R-module structure on LM can be lifted to an LR-structure in a unique way.

Proof: Functoriality and the universal property of L yield

$$\mathrm{Hom}(R, \mathrm{End}(M)) \to \mathrm{Hom}(R, \mathrm{End}(LM)) \cong \mathrm{Hom}(LR, \mathrm{End}(LM)).$$

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The stable homotopy category

A **spectrum** is a sequence of spaces $\{E_n\}_{n\in\mathbb{Z}}$ and structure maps $\varepsilon_n\colon \Sigma E_n \longrightarrow E_{n+1}$.

- Maps in Ho^s are homotopy classes of maps $[X, Y]$.
- The suspension functor Σ is invertible in Ho^s. We can suspend and desuspend any spectrum. If E is a spectrum and $k \in \mathbb{Z}$, then $(\Sigma^k E)_n = E_{n+k}$, and $\overline{\varepsilon}_n = \varepsilon_{n+k}$.
- The *wedge* of two spectra $(X \vee Y)_n = X_n \vee Y_n$ and $\overline{\varepsilon}_n = \varepsilon_n \vee \varepsilon'_n$.
- The homotopy groups of a spectrum are defined as

$$\pi_k(E) = \lim_{n\to\infty} \pi_{n+k}(E_n).$$

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- The *suspension spectrum* $\Sigma^\infty X$ of a space X is defined as $(\Sigma^\infty X)_n = \Sigma^n X$ for $n \geq 0$ and structure maps $\varepsilon_n = id$.

$$\pi_k(\Sigma^\infty X) = \lim_{n\to\infty} \pi_{n+k}(\Sigma^n X) = \pi_k^s(X).$$

- The *sphere spectrum* S is $\Sigma^\infty S^0$.
- Given any abelian group G, the *Eilenberg-Mac Lane spectrum* HG is defined as $(HG)_n = K(G,n)$ for $n \geq 0$. The structure maps are the adjoint maps to the equivalence maps $K(G,n) \to \Omega K(G,n+1)$. In this case, $\pi_k(HG) = G$ if $k = 0$ and zero if $k \neq 0$.
- Spectra arising from (co)homology theories. K (K-theory), MU (complex cobordism), $K(n)$ (Morava K-theory), $E(n)$ (Johnson-Wilson).

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Localization of homotopy rings and modules

- **Localizations commuting with suspension in the stable homotopy category preserve ring spectra and module spectra.** [C. Casacuberta and J. J. Gutiérrez, Homotopical localization of module spectra, *Trans. Amer. Math. Soc.* **357** (2005), 2753–2770.]

- A **ring spectrum** is a monoid in the stable homotopy category, i.e., a spectrum R equipped with a homotopy associative multiplication $R \wedge R \to R$ and a homotopy unit $S \to R$, where S denotes the sphere spectrum. A (left) **module** over a ring spectrum R is a spectrum M equipped with a map $R \wedge M \to M$ satisfying the relations of a module over a monoid (up to homotopy).

- **Warning:** Postnikov sections of ring spectra need not be ring spectra. However, modules over **connective** ring spectra are preserved by any homotopical localization L, even if L does not commute with suspension.

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Localization of homotopy rings and modules

The following categories are equivalent:

- The homotopy category of modules over the spectrum $H\mathbf{Z}$ of ordinary integral homology theory.
- The homotopy category of stable GEMs (wedges of Eilenberg–Mac Lane spectra $\Sigma^n HA_n$, $n \in \mathbf{Z}$).
- The homotopy category of chain complexes of abelian groups, i.e., the derived category of $\mathbf{Z}$.

Hence, stable GEMs are also preserved by homotopical localizations, since $H\mathbf{Z}$ is connective.

Localization of strict rings

- Localizations commuting with suspension in the stable homotopy category also preserve **strict** ring spectra up to homotopy. A **strict ring spectrum** is a monoid in a monoidal model category of spectra, e.g., symmetric spectra [M. Hovey, B. Shipley, J. H. Smith, Symmetric spectra, *J. Amer. Math. Soc.* **13** (2000), 149–208].

Proof: The homotopy category of strict ring spectra is equivalent to the homotopy category of A_∞ ring spectra. In fact, as we will next explain, **localizations preserve algebras over arbitrary cofibrant operads**, up to homotopy. This also explains why loop spaces are preserved by localizations, since the connected A_∞ H-spaces are the connected loop spaces.

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Outline of the talk

Model categories

- A **model category** is a complete and cocomplete category $\mathcal{E}$ with three distinguished classes of morphisms (*weak equivalences*, *fibrations*, and *cofibrations*) satisfying Quillen's axioms. The **homotopy category** $\operatorname{Ho}\mathcal{E}$ is obtained by formally inverting the weak equivalences.
- The homotopy category $\operatorname{Ho}\mathcal{E}$ has the same objects as $\mathcal{E}$. A morphism $X \to Y$ in $\operatorname{Ho}\mathcal{E}$ can be described as a homotopy class of maps $QX \to RY$ where QX is a cofibrant approximation to X and RY is a fibrant approximation to Y. An object QX is **cofibrant** if the map $\emptyset \to QX$ from the initial object of $\mathcal{E}$ is a cofibration, and an object RY is **fibrant** if the map $RY \to *$ to the final object of $\mathcal{E}$ is a fibration.

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Simplicial and monodial model categories

- A model category $\mathcal{E}$ is **monoidal** if it has an associative internal product $\otimes$ with a unit I and an internal hom $\mathrm{Hom}_{\mathcal{E}}(-,-)$, satisfying the *pushout-product axiom*, that is, if $f\colon X \to Y$ and $g\colon U \to V$ are cofibrations in $\mathcal{E}$, then the induced map
$$(X \otimes V) \coprod_{X \otimes U} (Y \otimes U) \longrightarrow Y \otimes V$$
is a cofibration which is a weak equivalence if f or g are.

- A model category $\mathcal{E}$ is **simplicial** if it is enriched, tensored and cotensored over simplicial sets in such a way that Quillen's *SM7 axiom* holds, namely, if $f\colon X \to Y$ is a cofibration and $g\colon U \to V$ is a fibration in $\mathcal{E}$, then the induced map
$$\mathrm{Map}(Y, U) \longrightarrow \mathrm{Map}(Y, V) \times_{\mathrm{Map}(X,V)} \mathrm{Map}(X, U)$$
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Simplicial and monodial model categories

The following are simplicial monoidal model categories:

- Compactly generated topological spaces with the k-product.
- Simplicial sets with the levelwise cartesian product.
- Spectra with the smash product

Outline of the talk

Coloured operads

- Let $\mathcal{E}$ be a cocomplete closed symmetric monoidal category. Let C be a set, whose elements will be called *colours*. A C-coloured collection is a set P of objects $P(c_1, \ldots, c_n; c)$ in $\mathcal{E}$ for every $n \geq 0$ and each tuple $(c_1, \ldots, c_n; c)$ of colours, together with maps

$$\sigma^* \colon P(c_1, \ldots, c_n; c) \longrightarrow P(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c)$$

  for all permutations $\sigma \in \Sigma_n$, yielding together a right action.

- A **C-coloured operad** is a C-coloured collection P equipped with a *unit* map $I \to P(c; c)$ and *composition product* maps

$$P(c_1, \ldots, c_n; c) \otimes P(a_{1,1}, \ldots, a_{1,k_1}; c_1) \otimes \cdots \otimes P(a_{n,1}, \ldots, a_{n,k_n}; c_n)$$
$$\longrightarrow P(a_{1,1}, \ldots, a_{1,k_1}, a_{2,1}, \ldots, a_{2,k_2}, \ldots, a_{n,1}, \ldots, a_{n,k_n}; c)$$

  compatible with the action of the symmetric groups and subject to associativity and unitary compatibility relations.

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Algebras over coloured operads

- If P is a C-coloured operad, a **P-algebra** is an object $X = (X(c))_{c \in C}$ in $\mathcal{E}^C$ together with a morphism of C-coloured operads
$$P \longrightarrow \mathrm{End}(X)$$
where the C-coloured operad $\mathrm{End}(X)$ is defined as
$$\mathrm{End}(X)(c_1, \ldots, c_n; c) = \mathrm{Hom}_{\mathcal{E}}(X(c_1) \otimes \cdots \otimes X(c_n), X(c)).$$
- **An operad is a coloured operad with only one colour.** Indeed, if P is an operad, we view it as a C-coloured operad P' with $C = \{c\}$ by defining
$$P'(c, \overset{(n)}{\ldots}, c; c) = P(n)$$
for all n. Then the P'-algebras are precisely the P-algebras.
- The *associative operad* A is defined as $A(n) = I[\Sigma_n]$ for all n, where $I[\Sigma_n]$ is a coproduct of copies of the unit I of $\mathcal{E}$ indexed by Σ_n. **The A-algebras are the monoids** in $\mathcal{E}$.

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Algebras over coloured operads

- Let $C = \{r, m\}$. Define a C-coloured operad Mod whose only nonzero terms are

$$\mathrm{Mod}(r, \overset{(n)}{\ldots}, r; r) = I[\Sigma_n]$$

and

$$\mathrm{Mod}(c_1, \ldots, c_n; m) = I[\Sigma_n]$$

when exactly one c_i is m and the rest (if any) are r. Then an algebra over Mod is a pair (R, M) where R is a monoid and M is an R-bimodule. By using non-symmetric operads, one obtains left R-modules and right R-modules similarly.

Hence, **modules can be viewed as algebras over coloured operads**.

Algebras over coloured operads

- Choose $C = \{r, 0, 1\}$ and let P be an operad. Define a C-coloured operad Mor whose value on $(c_1, \ldots, c_n; c)$ is

$$\begin{cases} P(n) & \text{if } c = r \text{ and } c_i = r \text{ for all } i, \text{ or } c = r \text{ and } n = 0; \\ P(n) & \text{if } c = 0, \text{ exactly one } c_i \text{ is } 0, \text{ and the rest are } r; \\ P(n) & \text{if } c = 1, \text{ exactly one } c_i \text{ is } 1, \text{ and the rest are } r; \\ P(n) & \text{if } c_i \neq r \text{ for all } i,\ c \neq r, \text{ and } \max(c_1, \ldots, c_n) \leq c; \\ 0 & \text{otherwise.} \end{cases}$$

Then an algebra over Mor consists of a triple $(X(r), X(0), X(1))$ where $X(r)$ is a P-algebra and $X(0)$ and $X(1)$ are modules over $X(r)$ together with a morphism $X(0) \to X(1)$ of modules.

Hence, **morphisms of modules can be viewed as algebras over coloured operads.**

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Localization of algebras

- A model structure on the category of C-coloured operads for any C was described in [C. Berger and I. Moerdijk, Axiomatic homotopy theory for operads, *Comment. Math. Helv.* **78** (2003), 805–831].

Theorem

Let L be a homotopical localization functor on a simplicial monoidal model category $\mathcal{E}$. Let C be any set and let P be a cofibrant C-coloured operad with values in simplicial sets. Let X be a P-algebra such that $X(c)$ is cofibrant for all $c \in C$. Suppose that the class of L-equivalences is closed under tensor products. Then LX admits a homotopy unique P-algebra structure such that $l_X \colon X \to LX$ is a map of P-algebras.

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Localization of algebras

Proof: For all $(c_1, \ldots, c_n; c)$, the map

$$X(c_1) \otimes \cdots \otimes X(c_n) \longrightarrow LX(c_1) \otimes \cdots \otimes LX(c_n)$$

is an L-equivalence by assumption, and it is also a cofibration since $X(c)$ is cofibrant for all c. Hence, the map

$$\mathrm{Map}(LX(c_1)\otimes\cdots\otimes LX(c_n), LX(c)) \longrightarrow \mathrm{Map}(X(c_1)\otimes\cdots\otimes X(c_n), LX(c))$$

is a fibration and a weak equivalence. By definition,

$$\mathrm{Map}(LX(c_1) \otimes \cdots \otimes LX(c_n), LX(c)) = \mathrm{End}(LX)(c_1, \ldots, c_n; c)$$

and

$$\mathrm{Map}(X(c_1) \otimes \cdots \otimes X(c_n), LX(c)) = \mathrm{Hom}(X, LX)(c_1, \ldots, c_n; c).$$

Localization of algebras

Define a C-coloured operad $\mathrm{End}(l_X)$ as the following pull-back:

$$\begin{array}{ccc}
\mathrm{End}(l_X) & \dashrightarrow & \mathrm{End}(LX) \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{End}(X) & \longrightarrow & \mathrm{Hom}(X, LX).
\end{array}$$

We just saw that the right-hand vertical arrow is a fibration and a weak equivalence. Hence, the left-hand vertical arrow is also a fibration and a weak equivalence.

Then, since P is cofibrant, the P-algebra structure map $P \to \mathrm{End}(X)$ admits a lift $P \to \mathrm{End}(l_X)$, which yields precisely a P-algebra structure on LX such that l_X is a map of P-algebras, and the uniqueness up to homotopy of this P-algebra structure follows from the homotopy uniqueness of the lift.

Consequences

If we choose a coloured operad P whose algebras are strict R-module spectra, and consider a cofibrant approximation $P_\infty \to P$, we find that homotopical localizations commuting with suspension preserve strict R-modules up to homotopy, and also module maps between them.

For arbitrary R, the homotopy category of R-module spectra in the classical sense is *not* equivalent to the homotopy category of strict R-module spectra. For example, the latter is equivalent to the derived category of A if A is a discrete ring and $R = HA$.