Localization and Preservation of Structures in Stable Homotopy

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Outline of the talk

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Introduction

Localization theory has been a common procedure in commutative algebra and algebraic geometry

- Construction of rings of fractions.
- Localization of rings and modules.

Axiomatization of the concept of localization in algebraic topology

- Precedents: Serre (1953), Adams (1961).
- Development: Quillen (1969), Sullivan (1970), Bousfield–Kan (1972).
- Localization of spaces at sets of primes. Homological localizations.
- Localization with respect to a map: Bousfield, Farjoun (1994–96).

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Localization theory has become a standard practice in modern homotopy theory

- The main problem of algebraic topology is the classification of homotopy types using algebraic invariants such as homotopy or homology groups.
- The idea behind localization is to consider a problem one prime at a time, solve it at each prime, and then put the solutions back together to obtain a full integral solution.
- This type of division into p-primary problems for each p can be carried out at the level of (co)homology groups, homotopy groups, and even at the level of spaces.

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Stable homotopy

- Studies stable phenomena in algebraic topology with respect to the suspension functor.
- The stable homotopy category has a *smash product* similar to the tensor product of modules, that is associative and commutative and has a unit.
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Outline of the talk

Localization functors on groups

Example (Abelianization)

Let G be any group and $[G, G]$ the commutator subgroup of G. The quotient $G/[G, G]$ is called the abelianization of G and denoted G_{ab}. We have a functor $L_1 : Grp \longrightarrow Grp$ defined as $L_1(G) = G_{ab}$.

There is a natural map $G \longrightarrow G_{ab}$ and $(G_{ab})_{ab} \cong G_{ab}$.

Example (P-localization)

Let P be a set of primes, A an abelian group and $\mathbb{Z}_P$ be the integers localized at P. We have a functor $L_2 : Ab \longrightarrow Ab$ defined as $L_2(A) = A \otimes \mathbb{Z}_P$.

There is a natural map $A \cong A \otimes \mathbb{Z} \xrightarrow{1 \otimes i} A \otimes \mathbb{Z}_P$ and $L_2 L_2 A \cong L_2 A$, since $\mathbb{Z}_P \otimes \mathbb{Z}_P \cong \mathbb{Z}_P$.

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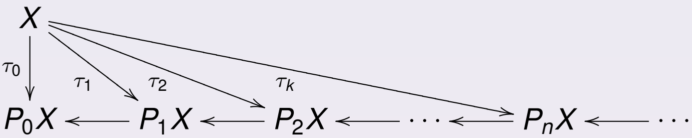
Localization functors on topological spaces

Let *Top* denote the category of topological spaces. Given a space X we can construct its Postnikov tower:



The map $(\tau_k)_*\colon \pi_i(X) \longrightarrow \pi_i(P_kX)$ is an isomorphism for $i \leq k$ and $\pi_i(P_kX) = 0$ for $i > k$. The Postnikov tower of a space is unique up to homotopy.

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Localization functors on topological spaces

Let Top^1 denote the category of simply connected spaces and let P be a set of primes. Given any simply connected space X we can construct another space X_P such that its homotopy groups are the P-localization of the homotopy groups of X, i.e., $\pi_k(X_P) \cong \pi_k(X) \otimes \mathbb{Z}_P$.

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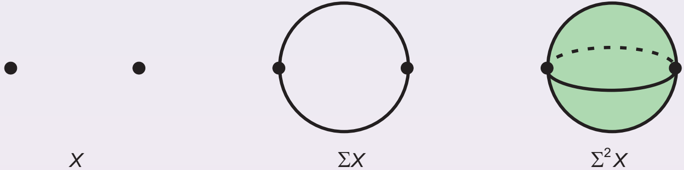
Stable homotopy theory studies stable phenomena in algebraic topology with respect to the suspension functor

$$\Sigma X = X \times [0,1]/(X \times \{0\}) \sqcup (X \times \{1\}).$$



Theorem (Freudenthal, 1937)

If X and Y are CW-complexes of finite dimension, then the sequence of maps induced by the suspension functor

$$[X,Y] \longrightarrow [\Sigma X, \Sigma Y] \longrightarrow [\Sigma^2 X, \Sigma^2 Y] \longrightarrow \cdots \text{ stabilizes}$$

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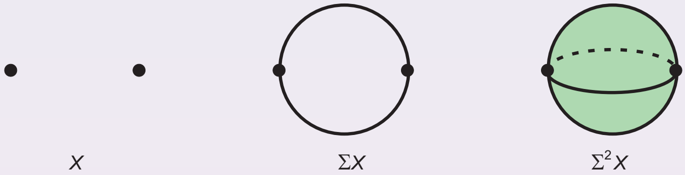

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Example (Homotopy groups of spheres)

The group homomorphism $\pi_{n+k}(S^n) \longrightarrow \pi_{n+k+1}(S^{n+1})$ is an isomorphism for $n > k + 1$.

The idea is to construct a category to isolate these problems. Roughly speaking the stable category is obtained from the category of topological spaces by introducing spheres of negative dimensions. The objects of the stable homotopy category Ho^s are called *spectra*.

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- Boardman (1964).
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Definition

A spectrum is a sequence of spaces $\{E_n\}_{n\in\mathbb{Z}}$ and structure maps $\varepsilon_n \colon \Sigma E_n \longrightarrow E_{n+1}$.

- Maps in Ho^s are homotopy classes of maps $[X, Y]$.
- The suspension functor Σ is invertible in Ho^s. We can suspend and desuspend any spectrum. If E is a spectrum and $k \in \mathbb{Z}$, then $(\Sigma^k E)_n = E_{n+k}$, and $\overline{\varepsilon}_n = \varepsilon_{n+k}$.
- The *wedge* of two spectra $(X \vee Y)_n = X_n \vee Y_n$ and $\overline{\varepsilon}_n = \varepsilon_n \vee \varepsilon'_n$.
- The homotopy groups of a spectrum are defined as

$$\pi_k(E) = \lim_{n\to\infty} \pi_{n+k}(E_n).$$

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- The *suspension spectrum* $\Sigma^\infty X$ of a space X is defined as $(\Sigma^\infty X)_n = \Sigma^n X$ for $n \geq 0$ and structure maps $\varepsilon_n = id$.

$$\pi_k(\Sigma^\infty X) = \lim_{n\to\infty} \pi_{n+k}(\Sigma^n X) = \pi_k^s(X).$$

- The *sphere spectrum* S is $\Sigma^\infty S^0$.
- Given any abelian group G, the *Eilenberg-Mac Lane spectrum* HG is defined as $(HG)_n = K(G,n)$ for $n \geq 0$. The structure maps are the adjoint maps to the equivalence maps $K(G,n) \to \Omega K(G,n+1)$ In this case, $\pi_k(HG) = G$ if $k = 0$ and zero if $k \neq 0$.
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Properties of the stable homotopy category

- We have homotopy, homology and cohomology groups in positive and negative dimensions.
- Additive ($[X, Y]$ is always an abelian group).
- Triangulated category (fiber sequences=cofiber sequences).
- There is a smash product $X \wedge Y$ analogous to the tensor product, that is associative, commutative and S is the unit. It has a right adjoint $F(X, Y)$ called the *function spectrum*

$$[X \wedge Y, Z] \cong [X, F(Y, Z)]$$

- Any spectrum E gives rise to a homology and a cohomology theory

$$E_k(X) = \pi_k(E \wedge X) \qquad E^k(X) = [X, \Sigma^k E].$$

- $(Ho^s, \wedge, S, F(-,-))$ is a closed symmetric monoidal category.

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$$\cdots \leftarrow [\Sigma^{-1}Z, E] \leftarrow [X, E] \leftarrow [Y, E] \leftarrow [Z, E] \leftarrow [\Sigma X, E] \leftarrow \cdots$$

$$\cdots \rightarrow [E, \Sigma^{-1}Z] \rightarrow [E, X] \rightarrow [E, Y] \rightarrow [E, Z] \rightarrow [E, \Sigma X] \rightarrow \cdots$$

- (Universal coefficients) For any spectrum E, any abelian group G and any $k \in \mathbb{Z}$, we have exact sequences:

$$0 \rightarrow Ext((H\mathbb{Z})_{n-1}(E), G) \rightarrow (HG)^n(E) \rightarrow Hom((H\mathbb{Z})_n(E), G) \rightarrow 0$$

$$0 \rightarrow (H\mathbb{Z})_n(E) \otimes G \rightarrow (HG)_n(E) \rightarrow Tor((H\mathbb{Z})_{n-1}(E), G) \rightarrow 0$$

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$$0 \rightarrow \bigoplus_{i+j=n} (H\mathbb{Z})_i(E) \otimes (H\mathbb{Z})_j(F) \rightarrow (H\mathbb{Z})_n(E \wedge F) \rightarrow \bigoplus_{i+j=n-1} Tor((H\mathbb{Z})_i(E), (H\mathbb{Z})_j(F)) \rightarrow 0$$

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Outline of the talk

Localization with respect to a map

Let $f\colon A \longrightarrow B$ a map of spectra in Ho^s.

- A spectrum X is *f-local* if the induced map $f^*\colon F^c(B,X) \longrightarrow F^c(A,X)$ is a homotopy equivalence.
- A map $g\colon X \longrightarrow Y$ is an *f-equivalence* if the induced map $g^*\colon F^c(Y,Z) \longrightarrow F^c(X,Z)$ is a homotopy equivalence for every f-local spectrum Z.

$F^c(-,-)$ denotes the connective cover of the function spectrum, i.e., $\pi_k F^c(X,Y) = \pi_k F(X,Y)$ for $k \geq 0$ and zero if $k < 0$.

Definition

An *f-localization* of X consists of an f-local spectrum $L_f X$ together with an f-equivalence $X \longrightarrow L_f X$.

- There is an f-localization functor $L_f\colon Ho^s \longrightarrow Ho^s$ which is unique up to homotopy.
- L_f is an idempotent functor on Ho^s.

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An *f-localization* of X consists of an f-local spectrum $L_f X$ together with an f-equivalence $X \longrightarrow L_f X$.

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f-localization and the suspension functor

For every spectrum X, there is a natural map

$$\Sigma L_f X \longrightarrow L_f \Sigma X.$$

If we use in the definition of f-localizations the full function spectrum $F(-,-)$ instead of $F^c(-,-)$, then this map is always an equivalence.

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We say that the functor L_f commutes with suspension if this map is an equivalence ,i.e., $\Sigma L_f X \simeq L_f \Sigma X$.

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Examples
Nullifications

Let $f\colon A \longrightarrow *$. The corresponding localization functor L_f is called *nullification* with respect to A, and it is denoted by P_A.

Example

If $A = \Sigma^{k+1}S$ for some $k \in \mathbb{Z}$, then P_A is the k-th Postnikov section functor. For every spectrum X and every $k \in \mathbb{Z}$, we have that $\pi_n(P_{\Sigma^{k+1}S}X) = \pi_n X$ if $n \leq k$ and zero if $n > k$.

- In general nullification functors do not commute with suspension.
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For any spectrum E there is a homological localization functor L_E with respect to E (Adams, Bousfield).

- A map of spectra $f\colon X \longrightarrow Y$ is an *E-equivalence* if $f_k\colon E_k(X) \longrightarrow E_k(Y)$ is an isomorphism for all $k \in \mathbb{Z}$.
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Localization at sets of primes

Let P be a set of primes. The P-localization of a spectrum X is the natural map $1 \wedge \eta\colon X \simeq X \wedge S \longrightarrow X \wedge M\mathbb{Z}_P = X_P$, where $M\mathbb{Z}_P$ is the Moore spectrum associated to $\mathbb{Z}_P$.

- P-localization P-localizes homotopy and homology groups.

$$\pi_k(X_P) \cong \pi_k(X) \otimes \mathbb{Z}_P, \quad E_k(X_P) \cong E_k(X) \otimes \mathbb{Z}_P.$$

- P-localization commutes with suspension and it is a nullification functor.

Theorem

If we define $g\colon \vee_{q \notin P} S \longrightarrow \vee_{q \notin P} S$ then $X_P \simeq P_C X$ for any X, where $C = cof(\vee_{k<0} \Sigma^k g)$.

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Outline of the talk

Structures on groups and spaces

Abelian groups

Let $L\colon Ab \longrightarrow Ab$ be a localization functor:

- If R is a (commutative) ring, then LR is a (commutative) ring and the localization map is a map of rings.
- If M is an R-module, then LM is an R-module and the localization map is a map of R-modules.

Topological spaces

Let $L\colon Ho(Top) \longrightarrow Ho(Top)$ be a localization functor:

- If X is an H-space, then LX homotopy equivalent to an H-space and the localization map is an H-map.
- If X is a loop space, then LX is homotopy equivalent to a loop space and the localization map is a loop map.

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Ring spectra and module spectra

There are examples of several structures preserved by localization functors in different categories:

- Abelian groups, nilpotent groups, rings.
- Generalized Eilenberg–Mac Lane spaces (GEMs), loop spaces, *H*-spaces.

Definition

A *ring spectrum* is a spectrum E together with two maps $\mu\colon E \wedge E \longrightarrow E$ and $\eta\colon S \longrightarrow E$ such that the following diagrams commute up to homotopy:

$$\begin{array}{ccc} E \wedge E \wedge E & \xrightarrow{1\wedge\mu} & E \wedge E \\ \downarrow{\scriptstyle \mu\wedge 1} & & \downarrow{\scriptstyle \mu} \\ E \wedge E & \xrightarrow[\mu]{} & E \end{array} \qquad \begin{array}{ccccc} E \wedge S & \xrightarrow{1\wedge\eta} & E \wedge E & \xleftarrow{\eta\wedge 1} & S \wedge E \\ & \searrow\!\!= & \downarrow{\scriptstyle \mu} & =\!\!\swarrow & \\ & & E. & & \end{array}$$

E is *commutative* if $\mu \circ \tau \simeq \mu$, where τ is the *twist* map.

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Given a ring spectrum E, we can also consider modules over E.

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Examples

- If R is a ring with unit and M is an R-module, then HR is a ring spectrum and HM is an HR-module.
- K, MU, $K(n)$ and $E(n)$ for every n.

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Localization of rings and modules

Theorem

If the localization functor L_f commutes with suspension, then the following hold:

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If L_f does not commute with suspension, then it does not preserve the structures of ring spectra and module spectra in general.

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Example (Rudyak)

Given $n \in \mathbb{N}$ and a prime p, let $K(n)$ be the ring spectrum corresponding to the n-th Morava K-theory referred to the prime p. Then $P_{\Sigma S}K(n)$ is not a ring spectrum.

If $P_{\Sigma S}K(n)$ is a ring spectrum, then so is $P_{\Sigma S}K(n) \wedge H\mathbb{Z}/p$. But the unit of this ring spectrum is null since $K(n) \wedge H\mathbb{Z}/p \simeq 0$. Therefore $P_{\Sigma S}K(n) \wedge H\mathbb{Z}/p = 0$. The cofiber sequence

$$\Sigma^d k(n) \longrightarrow K(n) \longrightarrow P_{\Sigma S}K(n),$$

where $d = 2(p^n - 1)$ and $k(n)$ is the connective cover of $K(n)$, and the fact that the mod p cohomology of $k(n)$ is a nonzero quotient of the Steenrod algebra, yield to a contradiction.

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The hypothesis that $L_f E$ be connective holds for example when E, A and B are all connective, where $f : A \longrightarrow B$.

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Localization of stable GEMs

Definition

Let R be a ring with unit. A spectrum E is a *stable R-GEM* if $E \simeq \vee_{k\in\mathbb{Z}}\Sigma^k HA_k$ where each A_k is an R-module. If $R = \mathbb{Z}$ we call E a *stable GEM*.

Any HR-module M is homotopy equivalent to $\vee_{k\in\mathbb{Z}}\Sigma^k HA_k$, where $A_k \cong \pi_k(M)$.

Theorem

If E is a stable R-GEM, then so is $L_f E$, and the localization map is a map of HR-modules.

Theorem

Let G be an abelian group and $k \in \mathbb{Z}$. Then $L_f\Sigma^k HG$ is equivalent to $\Sigma^k HG_1 \vee \Sigma^{k+1} HG_2$. If G is an R-module, then so are G_1 and G_2.

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Let G be an abelian group and $k \in \mathbb{Z}$. Then $L_f\Sigma^k HG$ is equivalent to $\Sigma^k HG_1 \vee \Sigma^{k+1} HG_2$. If G is an R-module, then so are G_1 and G_2.

Localization of stable GEMs

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Let R be a ring with unit. A spectrum E is a *stable R-GEM* if $E \simeq \vee_{k\in\mathbb{Z}}\Sigma^k HA_k$ where each A_k is an R-module. If $R = \mathbb{Z}$ we call E a *stable GEM*.

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Outline of the talk

Homological localization

If E and X are connective spectra, then $L_E X \simeq L_{MG} X$ (Bousfield), where $G = \mathbb{Z}_P$ or $\oplus_{p \in P} \mathbb{Z}/p$, for a set P of primes. If X or E fail to be connective, then $L_E X$ is more difficult to compute. $L_K S$ has infinitely many nontrivial homotopy groups in negative dimensions.

Bousfield's arithmetic square

For any spectrum E and X, we have the following arithmetic square

$$\begin{array}{ccc} L_E X & \longrightarrow & \prod_{p \in P} L_{E\mathbb{Z}/p} X \\ \downarrow & & \downarrow \\ L_{E\mathbb{Q}} X & \longrightarrow & L_{E\mathbb{Q}}(\prod_{p \in P} L_{E\mathbb{Z}/p} X), \end{array}$$

where $EG = E \wedge MG$ and P is the set of all primes.

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Theorem

Let $A_p = Ext(\mathbb{Z}/p^\infty, G)$, $B_p = Hom(\mathbb{Z}/p^\infty, G)$, and let P be the set of primes p such that $H\mathbb{Z}/p$ is not E-acyclic and G is not uniquely p-divisible. For any spectrum E and any abelian group G, we have the following:

- *If $H\mathbb{Q}$ is E-acyclic, then $L_E HG \simeq \prod_{p\in P}(HA_p \vee \Sigma HB_p)$.*
- *If $H\mathbb{Q}$ is not E-acyclic, then there is a cofiber sequence*

$$L_E HG \to H(G \otimes \mathbb{Q}) \vee \prod_{p\in P}(HA_p \vee \Sigma HB_p) \to M\mathbb{Q} \wedge \prod_{p\in P}(HA_p \vee \Sigma HB_p).$$

Some computations:

- For any HR-module X, $L_{K(n)}X = 0$ if $n \neq 0$ and $L_{K(0)}X \simeq X \wedge M\mathbb{Q}$.
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Computations of $L_E HG$

- Condition I. $E\mathbb{Q} = 0$ and $E \wedge H\mathbb{Z}/p = 0$ for every prime p.
- Condition II. $E\mathbb{Q} \neq 0$ and $E \wedge H\mathbb{Z}/p = 0$ for every prime p.
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- Condition IV. $E\mathbb{Q} \neq 0$ and $E \wedge H\mathbb{Z}/p \neq 0$ for every prime $p \in P$.

	Condition I	Condition II	Condition III	Condition IV
$L_E H\mathbb{Z}$	0	$H\mathbb{Q}$	$\prod_{p\in P} H\widehat{\mathbb{Z}}_p$	$H\mathbb{Z}_P$
$L_E H\mathbb{Z}/p^k$	0	0	$H\mathbb{Z}/p^k$	$H\mathbb{Z}/p^k$
$L_E H\mathbb{Q}$	0	$H\mathbb{Q}$	0	$H\mathbb{Q}$
$L_E H\mathbb{Z}_R$	0	$H\mathbb{Q}$	$\prod_{p\in P\cap R} H\widehat{\mathbb{Z}}_p$	$H\mathbb{Z}_{P\cap R}$
$L_E H\mathbb{Z}/p^\infty$	0	0	$\Sigma H\widehat{\mathbb{Z}}_p$	$H\mathbb{Z}/p^\infty$
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$L_E H\mathbb{Z}_R$	0	$H\mathbb{Q}$	$\prod_{p\in P\cap R} H\widehat{\mathbb{Z}}_p$	$H\mathbb{Z}_{P\cap R}$
$L_E H\mathbb{Z}/p^\infty$	0	0	$\Sigma H\widehat{\mathbb{Z}}_p$	$H\mathbb{Z}/p^\infty$
$L_E H\widehat{\mathbb{Z}}_p$	0	$H\widehat{\mathbb{Q}}_p$	$H\widehat{\mathbb{Z}}_p$	$H\widehat{\mathbb{Z}}_p$

Computations of $L_E HG$

Definition

An abelian group is reduced if it has no nontrivial divisible subgroups.

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Smashing localizations

- A localization functor L_f is *smashing* if, for every X, the natural map $1 \wedge l_S \colon X \longrightarrow X \wedge L_f S$ is an f-localization. (E is smashing if L_E is smashing.)
- Every smashing localization L_f is a homological localization, namely $L_f = L_E$, where $E = L_f S$.
- The spectra K and $E(n)$ are smashing.

Theorem

If L_f is smashing, then $(H\mathbb{Z})_k(L_f S) = 0$ if $k \neq 0$ and it is a subring of the rationals if $k = 0$.

Corollary

If L_f is smashing and $(H\mathbb{Z})_0(L_f S) \cong \mathbb{Q}$, then either $L_f X = L_{M\mathbb{Q}} X$ for every X or $L_f S$ has infinitely many nonzero homotopy groups in negative dimensions.

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Homotopical localizations of $H\mathbb{Z}$. Rigid rings

A ring with unit R is a *rigid ring* if the evaluation map $Hom(R, R) \longrightarrow R$ given by $\varphi \mapsto \varphi(1_R)$ is an isomorphism.

Examples

$\mathbb{Z}/n$, $\mathbb{Z}_P$, $\widehat{\mathbb{Z}}_p$, $\prod_{p\in P} \mathbb{Z}/p$, all solid rings in the sense of Bousfield–Kan. However $\mathbb{Z}/p^\infty$ is not a rigid ring.

Rigid rings were used to describe the f-localizations of the sphere S^1 (Casacuberta–Rodríguez–Tai).

Theorem

Given $k \in \mathbb{Z}$, we have that $L_f\Sigma^k H\mathbb{Z} \simeq \Sigma^k HA$, where A has a rigid ring structure. All rigid rings appear as f-localizations of $H\mathbb{Z}$, taking $f\colon S \longrightarrow MA$.

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There is a proper class of non-equivalent f-localization functors.

Theorem

If G is a reduced abelian group and $k \in \mathbb{Z}$, then

$$L_f \Sigma^k HG \simeq \Sigma^k HA$$

for some abelian group A.

Corollary

Let G be any abelian group and $k \in \mathbb{Z}$. If $\mathbb{Z}/p^\infty$ does not appear as a direct summand of G for any prime p, then $L_f \Sigma^k HG \simeq \Sigma^k HA$ for some abelian group A.

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