8

Jacobi polynomials and hypergeometric functions associated with root systems

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8.1 The Gauss hypergeometric function

The Gauss *hypergeometric equation* is the second order differential equation

$$[\theta(\theta + c - 1) - z(\theta + a)(\theta + b)]f = 0 \tag{8.1.1}$$

in the complex plane $\mathbb{C}$ with $\theta = zd/dz$ and a, b, c three complex parameters. It is regular outside $z = 0, 1$ and ∞. The singular points are regular singular with local exponents given by the Riemann scheme

$z = 0$	$z = 1$	$z = \infty$
0	0	a
$1 - c$	$c - (a + b)$	b

The first line contains the three singular points and the next two lines give the local exponents at these points. The Gauss *hypergeometric function*

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n = 1 + \frac{ab}{c1!} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + \cdots \tag{8.1.2}$$

is the holomorphic solution of the hypergeometric equation around $z = 0$ with exponent 0 and normalized by $F(a, b, c; 0) = 1$. It is well defined if $c \notin -\mathbb{N}$, is convergent for $|z| < 1$ and terminates if $a \in -\mathbb{N}$ or $b \in -\mathbb{N}$. The third way of defining the hypergeometric function is the *Euler integral*

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 t^{a-1}(1 - t)^{c-a-1}(1 - tz)^{-b} dt \tag{8.1.3}$$

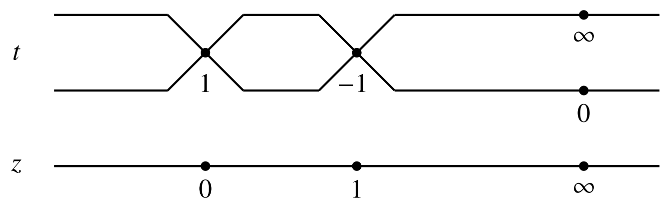
valid for $\Re c > \Re a > 0$. All three characterizations of the hypergeometric function, through the differential equation, the power series and the integral formula are in fact due to Euler.

In the next sections we shall sketch a multivariable generalization of the hypergeometric function in the context of root systems. It turns out that essentially all aspects of the one variable case have suitable generalizations with the exception of the Euler integral. In the general root system context integral representations still remain a mystery, apart from a hand full of isolated examples.

Consider the pull-back under the map $\mathbb{C}^\times \ni t \mapsto z \in \mathbb{C}$ given by

$$z = (\tfrac{1}{2} - \tfrac{1}{4}(t + 1/t)) = -\tfrac{1}{4}(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2$$

of the hypergeometric equation. The transformation $t \mapsto z$ has degree 2 and ramification points $t = 1, -1$ lying above $z = 0, 1$ respectively. It is the quotient map for the action of the group $S_2 = \{\pm 1\}$ acting by $t \mapsto t^{\pm 1}$.



The pull-back of the hypergeometric equation under $t \mapsto z$ takes the form

$$[\vartheta^2 + k_1 \frac{1 + t^{-1}}{1 - t^{-1}} \vartheta + 2k_2 \frac{1 + t^{-2}}{1 - t^{-2}} \vartheta + (\tfrac{1}{2}k_1 + k_2)^2 - \lambda^2] f = 0 \qquad (8.1.4)$$

(with $\vartheta = td/dt$ and) with the linear relations

$$a = \lambda + \tfrac{1}{2}k_1 + k_2 \ , \ \beta = -\lambda + \tfrac{1}{2}k_1 + k_2 \ , \ \gamma = \tfrac{1}{2} + k_1 + k_2$$

between the two parameter sets. Note the visible symmetry under $t \mapsto 1/t$. This equation has four regular singular points $t = 1, -1, 0, \infty$ with Riemann scheme

$t = 1$	$t = -1$	$t = 0$	$t = \infty$
0	0	a	a
$2 - 2c$	$2c - 2(a + b)$	b	b

as is clear from the ramification picture and the Riemann scheme of the Gauss hypergeometric equation.

The multiplicative group $\mathbb{C}^\times$ with the action of the group $S_2 = \{\pm 1\}$ by $z \mapsto z^{\pm 1}$ together with the pull-back of the hypergeometric equation has a natural generalization. Let T be a maximal torus in a simply connected complex simple Lie group with Weyl group W. Instead of $\mathbb{C}^\times$ with the action of S_2 we consider the complex torus $T \cong (\mathbb{C}^\times)^n$ with the action of W. It turns out that on the quotient space $W\backslash T$ there is an integrable system, in fact the eigenvalue system for a commutative algebra of linear partial differential operators, which can be viewed as a natural multivariable generalization of the Gauss hypergeometric equation.

Initial steps in this direction for rank 2 were taken by Koornwinder [44]. In general such a multivariable theory of hypergeometric functions associated with root systems was established by the authors [32],[28],[61],[62],[63],[64]. The original arguments used transcendental methods, but this all changed with the fundamental paper of Dunkl [17]. The extension of Dunkl operators from the rational to the trigonometric setting was obtained by Heckman [29] with further simplifications by Cherednik [7] and Opdam [65]. Dunkl operators are now the corner stone for obtaining the hypergeometric equations associated with root systems. Additional survey articles on this subject were written by Heckman [30] and Opdam [67].

8.2 Root systems

In this section we set up the notation, and give a brief exposition of the theory of root systems. Standard references are Bourbaki [5] and Humphreys [36].

Let V be a finite dimensional Euclidean vector space. The inner product of two vectors λ, μ in V will be denoted (λ, μ). For α a nonzero vector in V let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ be the covector of α, and denote by

$$s_\alpha : V \to V\ ,\ s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha$$

the orthogonal reflection with mirror the hyperplane V_α perpendicular to α. The transformation s_α is called the reflection with root α.

Definition 8.2.1 *A* root system *R in V is a finite subset of nonzero vectors spanning V, such that $s_\alpha(\beta) \in R$ and $(\beta, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in R$. The group W generated by the reflections s_α for $\alpha \in R$ is called the* Weyl group. *The second property is called the crystallographic condition.*

In Chapter 7 by Dunkl the concept of root system is used without the crystallographic condition. However, in this chapter a root system will always be crystallographic, as customary in semisimple Lie theory.

We do not require that $\mathbb{Q}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$, and so R need not be reduced. It is obvious that $R^\vee = \{\alpha^\vee; \alpha \in R\}$ is again a root system, called the coroot system. The lattice $Q = \mathbb{Z}R$ is called the root lattice of R, and the lattice P dual to the coroot lattice $Q^\vee = \mathbb{Z}R^\vee$ is called the weight lattice of R. Vectors in the weight lattice P are called weights. The root lattice is contained in the weight lattice by the crystallographic condition, and both lattices are invariant under W. It is easy to see that for $\alpha \in R$ one has either $\mathbb{Q}\alpha \cap P = \mathbb{Z}\alpha$ or $\mathbb{Q}\alpha \cap P = \mathbb{Z}\frac{1}{2}\alpha$, and in the latter case we say that the root α is twice a weight.

Let $T = \mathrm{Hom}(P, \mathbb{C}^\times)$ be the complex torus with character lattice P. We have the polar decomposition

$$T = T_v T_u\ ,\ T_v = \mathrm{Hom}(P, \mathbb{R}_{>0})\ ,\ T_u = \mathrm{Hom}(P, \{z \in \mathbb{C}^\times; |z| = 1\}) \tag{8.2.1}$$

with T_v the real vector subgroup and T_u the real compact torus. Since the weight lattice P is equal to $\mathrm{Hom}(T, \mathbb{C}^\times)$ the group algebra $\mathbb{C}[P]$ gets identified with the algebra $\mathbb{C}[T]$ of regular functions (or Laurent polynomials) on T. This defines T as a complex algebraic torus.

As a complex manifold $T = \mathfrak{t}/2\pi i Q^\vee$ with $\mathfrak{t} = \mathbb{C} \otimes V$ the Lie algebra of T. Extend the inner product $(\cdot,\cdot)$ on V to a symmetric bilinear form on $\mathfrak{t}$. For $\mu \in P$ the regular function t^μ on T, defined by

$$t^\mu = e^{(\mu,\log t)}$$

with $\log t$ a representative in $\mathfrak{t}$ for $t \in \mathfrak{t}/2\pi i Q^\vee$, is called a Laurent monomial. Addition on $\mathfrak{t}$ induces an Abelian group structure on T, and so T becomes a complex torus. For a root α the submanifold $T_\alpha = \{t \in T; t^\alpha = 1\}$ is a subgroup of T, called a toric mirror. It consists of one or possibly two connected components, depending on whether α is not or is twice a weight.

Fix once and for all a decomposition $R = R_+ \cup R_-$ in positive and negative roots. The cone of dominant weights $P_+ = \{\mu \in P; (\mu,\alpha^\vee) \in \mathbb{N}, \forall \alpha \in R_+\}$ has a basis over $\mathbb{N}$ of fundamental weights $\varpi_1,\cdots,\varpi_n$, which is just dual to the basis of simple coroots $\alpha_1^\vee,\cdots,\alpha_n^\vee$ of $R_+^\vee$. The corresponding simple roots $\alpha_1,\cdots,\alpha_n$ are a basis of simple roots for the root subsystem $R^0 = \{\alpha \in R; 2\alpha \notin R\}$ of unmultipliable roots. The corresponding simple reflections $s_1,\cdots,s_n$ generate the Weyl group W as a Coxeter group.

For $\mu \in P_+$ the regular function

$$m_\mu(t) = \sum_{\nu \in W\mu} t^\nu$$

is called the monomial invariant function with highest weight μ. Define a partial ordering $\leq$ on P by $\nu \leq \mu$ if $(\mu - \nu) \in \mathbb{N}R_+$, which explains the term highest weight μ for m_μ. It is easy to show that

$$m_\mu m_\nu = m_{\mu+\nu} + \cdots$$

with $\cdots$ denoting a linear combination of m_λ with $\lambda \in P_+$ and $\lambda < \mu + \nu$. Since the monomial invariant functions are a basis of $\mathbb{C}[T]^W$ one can derive that $\mathbb{C}[T]^W$ is equal to the polynomial algebra $\mathbb{C}[z_1,\cdots,z_n]$ with z_j the fundamental monomial invariant function with highest weight ϖ_j. In turn this implies that $W\backslash T$ is isomorphic to the linear space $\mathbb{C}^n$. The quotient map $T \to W\backslash T$ for the action of W on T has degree equal to the order of the Weyl group, and is ramified along the toric mirror arrangement $\cup_\alpha T_\alpha$. The hypergeometric system is an integrable system of linear partial differential equations with polynomial coefficients on $W\backslash T \cong \mathbb{C}^n$. Although in the rank one case of the previous section, with W of order 2 acting on $\mathbb{C}^\times$ by $t \mapsto 1/t$ and with quotient map $\mathbb{C}^\times \to \mathbb{C}, t \mapsto (t + 1/t)$ a degree two covering, this might seem odd, for higher rank the only sensible approach is never to work on the quotient space $W\backslash T$, but perform slick constructions on T with suitable equivariance under the Weyl group W.

8.3 The hypergeometric system

The complex torus T with character lattice equal to the weight lattice P of our given root system R has Lie algebra $\mathfrak{t}$ with the trivial Lie bracket. We have natural isomorphisms $S\mathfrak{t} \cong P\mathfrak{t}^* \cong U\mathfrak{t}$ of the symmetric algebra $S\mathfrak{t}$ of $\mathfrak{t}$, the algebra $P\mathfrak{t}^*$ of polynomial functions on $\mathfrak{t}^*$ and the universal enveloping algebra $U\mathfrak{t}$ of invariant linear differential operators on T. For p a polynomial function on $\mathfrak{t}^*$ we denote by $\partial(p)$ the corresponding invariant linear differential operator on T. The characters $t \mapsto t^\mu = e^{\mu(\log t)}$ are eigenfunctions for $U\mathfrak{t}$, so

$$\partial(p)t^\mu = p(\mu)t^\mu \tag{8.3.1}$$

for all $\mu \in P$. The root system R, the root lattice Q and the weight lattice P are naturally considered as subsets of the dual space $\mathfrak{t}^*$.

Let us denote by $T_{\text{reg}} = T - \cup T_\alpha$ the complement of the toric mirror arrangement, and by $\mathbb{C}[T_{\text{reg}}]$ the algebra of regular functions on T_{reg} generated by $\mathbb{C}[T]$ and the functions $t \mapsto 1/(1 - t^{-\alpha})$ for $\alpha \in R$. Denote by $\mathbb{D}(T_{\text{reg}}) = \mathbb{C}[T_{\text{reg}}] \otimes U\mathfrak{t}$ the corresponding algebra of linear differential operators on T_{reg}. Clearly $\mathbb{C}[T_{\text{reg}}]$ is a natural left module for $\mathbb{D}[T_{\text{reg}}]$. The Weyl group algebra $\mathbb{C}[W]$ acts on $\mathbb{C}[T_{\text{reg}}]$ by left multiplication (so $w(e^\mu) = e^{w\mu}$ for all $\mu \in P$) and on $\mathbb{D}[T_{\text{reg}}]$ by conjugation in a compatible way. There is a unique associative algebra structure on $\mathbb{D}[T_{\text{reg}}] \otimes \mathbb{C}[W]$ turning $\mathbb{C}[T_{\text{reg}}]$ into a left module for $\mathbb{D}[T_{\text{reg}}] \otimes \mathbb{C}[W]$.

Lemma 8.3.1 *The natural map* $\mathbb{D}[T_{\text{reg}}] \otimes \mathbb{C}[W] \to \operatorname{Hom}(\mathbb{C}[T], \mathbb{C}[T_{\text{reg}}])$ *is an injection.*

Definition 8.3.2 *Let us call the linear space*

$$\mathcal{K} = \{k \in \mathbb{C}^R; k = (k_\alpha), k_{w\alpha} = k_\alpha \forall w \in W, \alpha \in R\} \tag{8.3.2}$$

the space of multiplicity (or coupling) parameters for R. For $\xi \in \mathfrak{t}$ and $k \in \mathcal{K}$ the expression

$$T(\xi, k) = \partial(\xi) - \rho(k)(\xi) + \sum_{\alpha>0} k_\alpha \alpha(\xi)(1 - t^{-\alpha})^{-1} \otimes (1 - s_\alpha) \tag{8.3.3}$$

(viewed as element of $\mathbb{D}[T_{\text{reg}}] \otimes \mathbb{C}[W]$) is called the Dunkl–Cherednik operator or just the (trigonometric) Dunkl operator *with*

$$\rho(k) = \tfrac{1}{2} \sum_{\alpha>0} k_\alpha \alpha \in \mathfrak{t}^* \tag{8.3.4}$$

the Weyl vector for the multiplicity parameter $k \in \mathcal{K}$.

The Dunkl operator acts as a linear operator on $\mathbb{C}[T_{\text{reg}}]$ leaving the linear subspace $\mathbb{C}[T] \hookrightarrow \mathbb{C}[T_{\text{reg}}]$ invariant. Indeed for $\mu \in P$ with $\mu(\alpha^\vee) = m \in \mathbb{Z}$ we have

$$\frac{t^\mu - t^{s_\alpha \mu}}{1 - t^{-\alpha}} = t^\mu \frac{1 - t^{-m\alpha}}{1 - t^{-\alpha}} = \begin{cases} t^\mu(1 + e^{-\alpha} + \cdots + e^{-(m-1)\alpha}) & \text{if } m > 0 \\ 0 & \text{if } m = 0 \\ -t^\mu(e^\alpha + \cdots + e^{-m\alpha}) & \text{if } m < 0 \end{cases}$$

which in turn implies that $T(\xi, k) : \mathbb{C}[T] \to \mathbb{C}[T]$ for all $k \in \mathcal{K}$.

Lemma 8.3.3 *In case $k_\alpha \geq 0$ for all $\alpha \in R$ (denoted $k \in \mathcal{K}_+$) we can define a Hermitian inner product $\langle \cdot, \cdot \rangle_k$ on $\mathbb{C}[T]$ by*

$$\langle f, g \rangle_k = |W|^{-1} \int_{T_u} f(t)\overline{g(t)} \prod_{\alpha>0} |t^{\frac{1}{2}\alpha} - t^{-\frac{1}{2}\alpha}|^{2k_\alpha} d_u t \tag{8.3.5}$$

with $d_u t$ the normalized Haar measure on the compact torus T_u. Moreover the Dunkl operator satisfies

$$\langle T(\xi, k) f, g \rangle_k = \langle f, T(\overline{\xi}, k) g \rangle_k \tag{8.3.6}$$

with the bar for complex conjugation on $\mathfrak{t}$ *with respect to the real form* $\mathfrak{t}_v$. *In particular* $T(\xi, k)$ *is selfadjoint on* $\mathbb{C}[T]$ *with respect to* $\langle \cdot, \cdot \rangle_k$ *for all* $\xi \in \mathfrak{t}_v$.

If $f(t) = \sum c_\mu t^\mu$ (sum over μ in P) is a Laurent polynomial on T then the constant term c_0 is equal to $\int f(t) d_u t$ (integration over t in T_u). So the Hermitian inner product $\langle \cdot, \cdot \rangle_k$ is defined in algebraic terms for $k \in \mathcal{K} \cap \mathbb{N}^R$, since

$$\delta(k; t) := \prod_{\alpha > 0} |t^{\frac{1}{2}\alpha} - t^{-\frac{1}{2}\alpha}|^{2k_\alpha} = \prod_{\alpha > 0} (2 - t^\alpha - t^{-\alpha})^{k_\alpha} \in \mathbb{C}[T]$$

for $t \in T_u$. In turn this implies that our proof of the theorem below on the commutativity of the Dunkl operators is algebraic.

Lemma 8.3.4 *Recall the standard partial ordering* $\leq$ *on* P *defined by* $\nu \leq \mu$ *if* $\mu - \nu \in \mathbb{N}R_+$. *For* $\mu \in P$ *let* $\mu_+ \in P_+$ *be the unique dominant weight in the orbit* $W\mu$. *Define a new partial ordering* $\trianglelefteq$ *on* P *by*

$$\nu \trianglelefteq \mu \text{ if either } \nu_+ < \mu_+ \text{ or } \nu_+ = \mu_+ \wedge \mu \leq \nu. \tag{8.3.7}$$

So μ_+ *is the smallest and* $w_0\mu_+$ *is the largest element in the orbit* $W\mu$ *in this new ordering* $\trianglelefteq$. *Here* $w_0 \in W$ *is the longest element. Then the Dunkl operators are upper triangular with respect to the basis* t^μ *of* $\mathbb{C}[T]$ *partially ordered by* $\trianglelefteq$. *More precisely, writing dots for lower order terms with respect to* $\trianglelefteq$, *we have*

$$T(\xi, k) t^\mu = \tilde{\mu}(\xi) t^\mu + \cdots$$

for all $\mu \in P$, *with*

$$\tilde{\mu} = \mu + \tfrac{1}{2} \sum_{\alpha > 0} k_\alpha \epsilon(\mu(\alpha^\vee)) \alpha \tag{8.3.8}$$

and $\epsilon : \mathbb{R} \to \{\pm 1\}$ *defined by* $\epsilon(x) = +1$ *if* $x > 0$ *and* $\epsilon(x) = -1$ *if* $x \leq 0$.

For $k \in \mathcal{K}_+ = \{k \in \mathcal{K}; k_\alpha \geq 0 \ \forall \ \alpha\}$ define a new basis $E(\mu, k)$ for $\mu \in P$ of $\mathbb{C}[T]$ by the conditions

$$E(\mu, k) = t^\mu + \cdots \ , \ \langle E(\mu, k), t^\nu \rangle_k = 0$$

for all $\nu \in P$ with $\nu \triangleleft \mu$. This new basis is obtained from the original monomial basis by an upper unitriangular transformation, so that the inverse transformation is again upper unitriangular. Therefore the Dunkl operators are also upper triangular with respect to the new basis $E(\mu, k)$ for $\mu \in P$. Since $\langle E(\mu, k), E(\nu, k) \rangle_k = 0$ for all $\nu \in P$ with $\nu \triangleleft \mu$ it follows from Lemma 8.3.3 that the Dunkl operators with fixed multiplicity parameter $k \in \mathcal{K}$ are simultaneously diagonalized by the basis $E(\mu, k)$. Hence Dunkl operators commute, and we have proven the following theorem.

Theorem 8.3.5 *We have* $T(\xi, k) T(\eta, k) = T(\eta, k) T(\xi, k)$ *for all* $\xi, \eta \in \mathfrak{t}$ *and all* $k \in \mathcal{K}$.

The equality in the theorem is polynomial in $k \in \mathcal{K}$, and so it follows for all $k \in \mathcal{K}$ once it is known on the Zariski dense subsets $\mathcal{K} \cap \mathbb{N}^R \subset \mathcal{K}_+$ of $\mathcal{K}$. Due to the commutativety of the

Dunkl operators we can extend the linear map $\mathfrak{t} \to \mathbb{D}[T_{\text{reg}}] \otimes \mathbb{C}[W],\ \xi \mapsto T(\xi, k)$ to an algebra homomorphism

$$p \mapsto T(p, k)$$

from the symmetric algebra $S\mathfrak{t}$ into $\mathbb{D}[T_{\text{reg}}] \otimes \mathbb{C}[W]$, such that the induced natural action of $S\mathfrak{t}$ on $\mathbb{C}[T_{\text{reg}}]$ via higher order Dunkl operators preserves the linear subspace $\mathbb{C}[T]$. It is clear that

$$T(p, k)E(\mu, k) = p(\tilde{\mu})E(\mu, k) \tag{8.3.9}$$

for all $\mu \in P$ and all $k \in \mathcal{K}_+$. The following definition goes back to Drinfeld [16] and Lusztig [50].

Definition 8.3.6 *The* degenerate affine Hecke algebra $\mathbb{H} = \mathbb{H}(R_+, k)$ *is the unique associative algebra satisfying*

- $\mathbb{H} = S\mathfrak{t} \otimes \mathbb{C}[W]$ *as a vector space over* $\mathbb{C}$,
- $S\mathfrak{t} \to \mathbb{H},\ p \mapsto p \otimes 1$ *and* $\mathbb{C}[W] \to \mathbb{H},\ w \mapsto 1 \otimes w$ *are algebra homomorphisms, and so we will identify* $S\mathfrak{t}$ *and* $\mathbb{C}[W]$ *with their images in* $\mathbb{H}$ *via these maps,*
- $p \cdot w = p \otimes w$ *with* $\cdot$ *denoting the algebra multiplication in* $\mathbb{H}$,
- $s_i \cdot p - s_i(p) \cdot s_i = -k_i(p - s_i(p))/\alpha_i^\vee$ *with* $w(p)$ *the natural transform of* $p \in S\mathfrak{t}$ *under* $w \in W$, *and* $k_i = \frac{1}{2}k_{\alpha_i/2} + k_{\alpha_i}$.

Note that the last item of this definition holds if and only if the item holds for all $p = \xi \in \mathfrak{t}$ homogeneous of degree one, in which case it boils down to

$$s_i \cdot \xi - s_i(\xi) \cdot s_i = -k_i\alpha_i(\xi) \tag{8.3.10}$$

for all $\xi \in \mathfrak{t}$. If $W \ni w = s_{i_1} \cdots s_{i_p}$ is written as a shortest word in the simple reflections then $p = l(w)$ is called the length of this Weyl group element. By induction on the length $l(w)$ one can show that

$$w \cdot \xi \cdot w^{-1} = w(\xi) + \sum_{\alpha \in R_+ \cap wR_-} k_\alpha \alpha(w(\xi))s_\alpha \tag{8.3.11}$$

for all $\xi \in \mathfrak{t}$ and $w \in W$. In turn this implies that the centralizer of $\mathfrak{t}$ in $\mathbb{H}$ is equal to $S\mathfrak{t}$. Using the last item of the above definition it is straightforward to describe the center of the degenerate affine Hecke algebra $\mathbb{H}$.

Proposition 8.3.7 *The center* $Z(\mathbb{H})$ *of* $\mathbb{H}$ *is equal to* $S\mathfrak{t}^W$.

It is easy to check from the definition of the Dunkl operator (just a rank one computation) that

$$s_i T(\xi, k) - T(s_i\xi, k)s_i = -k_i\alpha_i(\xi) \tag{8.3.12}$$

for all $\xi \in \mathfrak{t}$. Therefore the conclusion is that the action of the Weyl group and the Dunkl operators on $\mathbb{C}[T]$ define a representation of the degenerate affine Hecke algebra $\mathbb{H}(R_+, k)$ on the function space $\mathbb{C}[T]$.

Definition 8.3.8 *The representation via Dunkl operators*

$$p \mapsto T(p,k), w \mapsto w : \mathbb{H}(R_+,k) \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[T])$$

is called the Dunkl representation of the degenerate affine Hecke algebra.

Hence for $p \in S\mathfrak{t}^W$ the Dunkl operator

$$T(p,k) = \sum_w D(w,p,k) \otimes w \in \mathbb{D}[T_{\mathrm{reg}}] \otimes \mathbb{C}[W]$$

commutes with all elements from W, and therefore the linear differential operator

$$D(p,k) := \sum_w D(w,p,k) \tag{8.3.13}$$

lies in $\mathbb{D}[T_{\mathrm{reg}}]^W$. It is also clear that

$$D(p,k)D(q,k) = D(pq,k) \quad \forall p,q \in S\mathfrak{t}^W \tag{8.3.14}$$

and so $\{D(p,k); p \in S\mathfrak{t}^W\}$ is a commutative algebra of differential operators on T_{reg}. By definition $D(p,k)$ is the unique element of $\mathbb{D}[T_{\mathrm{reg}}]^W$ which has the same restriction to $\mathbb{C}[T]^W$ as the Dunkl operator $T(p,k)$. In particular $D(p,k)$ preserves the space $\mathbb{C}[T]^W$.

Definition 8.3.9 *Fix $k \in \mathcal{K}$ and $\lambda \in \mathfrak{t}^*$. The system of differential equations*

$$D(p,k)f = p(\lambda)f \quad \forall p \in S\mathfrak{t}^W \tag{8.3.15}$$

on $T_{\mathrm{reg}} \subset T$ is called the hypergeometric system *associated with the root system R with multiplicity parameter $k \in \mathcal{K}$ and spectral parameter $\lambda \in \mathfrak{t}^*$.*

An explicit expression for the linear differential operator $D(p,k)$ is only manageable for p equal to the quadratic invariant.

Theorem 8.3.10 *If $\xi_1, \cdots, \xi_n$ is a real orthornormal basis of $\mathfrak{t}$ then*

$$D(\sum_i \xi_i^2, k) = \sum_i \partial(\xi_i)^2 + \sum_{\alpha>0} k_\alpha \frac{1+t^{-\alpha}}{1-t^{-\alpha}} \partial(\alpha) + (\rho(k), \rho(k)) \tag{8.3.16}$$

with $\partial(p)t^\mu = p(\mu)t^\mu$ for $p \in S\mathfrak{t}$ and $\mu \in P \subset \mathfrak{t}^*$.

Proof For $p \in S\mathfrak{t}^W$ homogeneous the leading symbol of $D(p,k)$ is equal to $\partial(p)$ while the constant term equals $p(\rho(k))$. The intermediate linear terms require a small computation. □

Example 8.3.11 *In case $R = \{\pm 1, \pm 2\} \subset \mathbb{R}$ is a rank one root system with $\mathbb{C}[T] = \mathbb{C}[t, t^{-1}]$ and $\vartheta = td/dt$ the natural basis vector of $\mathfrak{t}$ the hypergeometric equation associated with R becomes*

$$[\vartheta^2 + k_1 \frac{1+t^{-1}}{1-t^{-1}}\vartheta + 2k_2 \frac{1+t^{-2}}{1-t^{-2}}\vartheta + (\tfrac{1}{2}k_1 + k_2)^2 - \lambda^2]f = 0$$

and after elimination of the symmetry $t \mapsto t^{\pm 1}$ reduces in the new coordinate $z = \frac{1}{4} - \frac{1}{2}(t + t^{-1})$ to the Gauss hypergeometric equation, as has been discussed in Section 8.1.

The algebra of invariants $\mathbb{C}[T]^W$ is a polynomial algebra $\mathbb{C}[z_1, \cdots, z_n]$ in the monomial invariant functions z_i with highest weight the fundamental weight $\varpi_i \in P_+$. Hence we have constructed for each $k \in \mathcal{K}$ a commutative subalgebra of the Weyl algebra $\mathbb{C}[z_1, \cdots, z_n, \partial_1, \cdots, \partial_n]$ of maximal rank n. In these algebraic coordinates the hypergeometric system associated with a rank one root system becomes the Gauss hypergeometric equation. However in higher rank the algebraic coordinates become intractible [44], [61], and it is best to work on the torus T in an equivariant way for W.

8.4 Jacobi polynomials

Throughout this section we will assume that $k \in \mathcal{K}_+$, which implies that $\langle \cdot, \cdot \rangle_k$ is a Hermitian inner product on $\mathbb{C}[T]$. The monomial invariant functions m_μ for $\mu \in P_+$ form a basis of the vector space $\mathbb{C}[T]^W$.

Definition 8.4.1 *The* Jacobi polynomials $P(\mu, k)$ *for* $\mu \in P_+$ *form a basis of* $\mathbb{C}[T]^W$ *satisfying*

$$P(\mu, k) = m_\mu + \cdots \ , \ \langle P(\mu, k), m_\nu \rangle_k = 0$$

for all $\nu \in P_+$ *with* $\nu < \mu$. *Here the dots denote lower order terms in the standard partial ordering* $\leq$ *on* P_+.

This Gram–Schmidt type definition is similar to that of the basis $E(\mu, k)$ for $\mu \in P$ of $\mathbb{C}[T]$. From this definition it follows that the Jacobi polynomials $P(\mu, k)$ are simultaneous eigenfunctions for the commutative algebra $\{T(p, k); p \in S\mathfrak{t}^W\}$. The $E(\mu, k) \in \mathbb{C}[T]$ are generally referred to as the *non-symmetric Jacobi polynomials*. It is clear that $P(\mu, k) = E(w_0\mu, k) + \cdots$ with $w_0 \in W$ the unique element interchanging positive and negative roots, and the dots denote lower order terms for the partial ordering $\trianglelefteq$ on P relative to the basis $E(\nu, k)$ of $\mathbb{C}[T]$. Hence

$$D(p, k)P(\mu, k) = p(\mu + \rho(k))P(\mu, k) \quad \forall p \in S\mathfrak{t}^W \ , \ \forall \mu \in P_+ \tag{8.4.1}$$

with $\rho(k) = \frac{1}{2} \sum k_\alpha \alpha$ the Weyl vector for the multiplicity parameter k as in (8.3.4). Since real Dunkl operators acting on $\mathbb{C}[T]$ are selfadjoint with respect to $\langle \cdot, \cdot \rangle_k$ and the algebra $S\mathfrak{t}^W$ separates the points of the real locus $P_+ + \rho(k)$ we find that

$$\langle P(\mu, k), P(\nu, k) \rangle_k = 0$$

for all $\mu, \nu \in P_+$ with $\mu \neq \nu$. The conclusion is that the Jacobi polynomials are a set of orthogonal polynomials for $\mathbb{C}[T]^W$ with respect to $\langle \cdot, \cdot \rangle_k$. Our normalization of the Jacobi polynomials is by leading coefficient at infinity equal to 1.

8.4.1 Jacobi polynomials and zonal spherical functions

For special values of the multiplicity parameters the Jacobi polynomials have a group theoretical meaning. Let G be a noncompact real semisimple group with Cartan subgroup A and Cartan decomposition $G = KAK$. Let $A_\mathbb{C} \subset G_\mathbb{C}$ be the complexification of A inside $G_\mathbb{C}$, with polar decomposition $A_{\mathbb{C}} = AA_u$, and let $U \subset G_{\mathbb{C}}$ be the corresponding compact real form of $G_{\mathbb{C}}$. The compact dual of the noncompact real Riemannian globally symmetric space $X = G/K$ is denoted $X_u = U/K$, and X and X_u are both real forms of the same complex symmetric space $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$. We shall assume that $X_{\mathbb{C}}$ is both connected and simply connected.

If $T = A_{\mathbb{C}}K_{\mathbb{C}} \subset X_{\mathbb{C}}$ then $T \simeq A_{\mathbb{C}}/F$ where $F \subset A_{\mathbb{C}}$ is the 2-torsion subgroup. If $T = T_v T_u$ is the polar decomposition then $T_v \simeq AK \subset G/K$ is a maximal flat subspace of X and $T_u \simeq A_u K \subset U/K$ is a maximal flat subspace of the compact dual X_u. The map $A_{\mathbb{C}}/F \ni aF \xrightarrow{\sim} a^2 \in A_{\mathbb{C}}$ identifies $A_{\mathbb{C}}/F \subset X_{\mathbb{C}}$ with $A_{\mathbb{C}}$. Let $\Sigma \subset \mathfrak{a}^* = \mathrm{Lie}(A)$ be the restricted root system of G, and let $R \subset \mathfrak{a}^* = 2\Sigma$ be the corresponding set of characters of $T = A_{\mathbb{C}}/F$ via the above identification map. Choose $k_{2\alpha} \in \frac{1}{2}\mathbb{Z}_+$ such that for all $\alpha \in \Sigma$, $m_\alpha = 2k_{2\alpha}$ is equal to the multiplicity of the restricted root α of G. Then the density function $\delta_u(k)$ of the Weyl measure on T_u is the density function of the defining orthogonality measure of the Jacobi polynomials on T_u as in Lemma 8.3.3.

It follows that there exists a close relationship between the Jacobi polynomials $P(\mu, k)$ with the above parameters k_α and the zonal spherical functions of the Gelfand pair (U, K) on the compact Riemannian symmetric space X_u. The zonal spherical functions of X_u form a complete set of orthogonal polynomials on $L^2(K\backslash U/K) \simeq L^2(W\backslash T_u, \delta_u(k)d_u t)$ (cf. [51], [34]). Therefore the restriction of a zonal spherical function on X_u to T_u is a Jacobi polynomial $P(\mu, k)$, up to normalization, and all Jacobi polynomials are obtained in this way. The leading coefficient at infinity of the zonal spherical function is by definition equal to corresponding the Harish-Chandra c-function. More precisely, the restrictions to T_u of the zonal spherical functions on X_u are the Laurent polynomials of the form (with $\mu \in P^+$):

$$\phi(\mu, k)|_{T_u} = c(\mu + \rho(k), k)P(\mu, k) \tag{8.4.2}$$

It is a basic property of zonal spherical functions that their evaluation at eK is equal to 1. Hence we find

$$P(\mu, k; e) = \frac{1}{c(\mu + \rho(k), k)} \tag{8.4.3}$$

for the normalization at the identity element of T.

Let $d(\mu, k)$ denote the dimension of the irreducible K-spherical representation of U associated to the zonal spherical function $\phi(\mu, k)$. Another basic aspect of the theory of zonal spherical functions is the following formula for their square L^2-norm:

$$(\phi(\mu, k), \phi(\mu, k)) = d(\mu, k)^{-1}\mathrm{Vol}(X_u) \tag{8.4.4}$$

According to a remarkable formula of Vretare we can also express $d(\mu, k)$ in terms of the Harish-Chandra c-function:

$$d(\mu, k) = \lim_{\epsilon \to 0} \frac{c(-\rho(k) + \epsilon, k)}{c(\mu + \rho(k), k)c(-\mu - \rho(k) + \epsilon, k)} \tag{8.4.5}$$

In addition, if we normalize the volume of T_u to be equal to 1, then the volume $\mathrm{Vol}(X_u)$ can also be expressed in terms of the close relatives of the Harish-Chandra c-function. By Weyl's integration formula we have

$$\mathrm{Vol}(X_u) = |W|^{-1} \int_{T_u} \delta_u(k) d_u t = |W|^{-1} \int_{T_u} \prod_{\alpha>0} |e^{\alpha/2} - e^{-\alpha/2}|^{2k_\alpha} d_u t \tag{8.4.6}$$

A closed formula for this integral was conjectured by I.G. Macdonald, not only for k_α equal to the half the restricted root multiplicities of a Riemannian symmetric space, but for arbitrary complex parameters k_α with $\mathrm{Re}(k_\alpha) > 0$ depending only on the length of the root α. This "constant term conjecture" of Macdonald stimulated much of the research on hypergeometric functions for root systems and double affine Hecke algebras. More generally the evaluation and norm formulas for zonal spherical functions on X_u as discussed above extend holomorphically to arbitrary complex parameters k_α.

All these closed formulae are expressible in terms of generalizations of the Harish-Chandra c-function, as we will see in the next subsection.

8.4.2 Norm and evaluation formulas

The c-functions for the zonal spherical functions on a Riemannian symmetric space are expressible, by a famous formula of Gindikin and Karpelevich [24], as a product over the positive roots of rank one c-functions. This product formula is used to define the generalized c-functions for arbitrary complex parameters k_α. In this section we will compute the square norms $\langle P(\mu, k), P(\mu, k)\rangle_k$ and their evaluation $P(\mu, k; e)$ at the identity element e of T, in terms of these generalized c-functions. In particular, these numbers are explicitly computable as a product taken over the positive roots of quotients of Γ-factors. This is one of the remarkable features of the theory of zonal spherical functions on semisimple symmetric spaces which generalizes to the theory of hypergeometric functions. In fact this feature even holds true for the more general non-symmetric Jacobi polynomials. Since the formulations as well as the proofs become somewhat easier when we exploit the extra freedom this generalization offers, we will discuss these results at this level of generality.

For $w \in W$ we define a function $\delta_w : R_+ \to \{0, 1\}$ by

$$\begin{aligned} \delta_w(\alpha) &= 0 \text{ if } w(\alpha) > 0 \\ &= 1 \text{ if } w(\alpha) < 0 \end{aligned}$$

and we define

$$c_w^*(\lambda, k) = \prod_{\alpha \in R_+} \frac{\Gamma\left(-\lambda(\alpha^\vee) - \frac{1}{2}k_{\alpha/2} - k_\alpha + \delta_w(\alpha)\right)}{\Gamma\left(-\lambda(\alpha^\vee) - \frac{1}{2}k_{\alpha/2} + \delta_w(\alpha)\right)}$$

and

$$\tilde{c}_w(\lambda, k) = \prod_{\alpha \in R_+} \frac{\Gamma\left(\lambda(\alpha^\vee) + \frac{1}{2}k_{\alpha/2} + \delta_w(\alpha)\right)}{\Gamma\left(\lambda(\alpha^\vee) + \frac{1}{2}k_{\alpha/2} + k_\alpha + \delta_w(\alpha)\right)} \tag{8.4.7}$$

For brevity we will write

$$\tilde{c}(\lambda, k) := \tilde{c}_e(\lambda, k), \; c^*(\lambda, k) := c_{w_0}^*(\lambda, k) \tag{8.4.8}$$

The Harish-Chandra *c-function* is defined by

$$c(\lambda, k) = \frac{\tilde{c}(\lambda, k)}{\tilde{c}(\rho(k), k)} \tag{8.4.9}$$

which in the Riemannian symmetric space case is just the celebrated Gindikin-Karpelevich [24] formula.

Theorem 8.4.2 ([63], [65]) *Let $\lambda \in P_+$. We denote by W_λ the isotropy subgroup of λ in W, and by w_λ the longest element of W_λ. Let W^λ be the set of shortest length representatives for the left cosets of W_λ in W, and let $w \in W^\lambda$. Then*

(i).

$$||E(w\lambda, k)||_k^2 = \frac{c^*_{ww_\lambda}(-(\lambda + \rho(k)), k)}{\tilde{c}_{ww_\lambda}(\lambda + \rho(k), k)}$$

(ii).

$$E(w\lambda, k; e) = \frac{\tilde{c}_{w_0}(\rho(k), k)}{\tilde{c}_{ww_\lambda}(\lambda + \rho(k), k)}$$

(iii).

$$||P(\lambda, k)||_k^2 = |W| \frac{c^*(-(\lambda + \rho(k)), k)}{\tilde{c}(\lambda + \rho(k), k)}$$

(iv).

$$P(\lambda, k; e) = \frac{1}{c(\lambda + \rho(k), k)}$$

Proof It is clearly enough to prove these assertions for a Zariski-dense subset of the parameter space. Therefore, without loss of generality, we may assume that all k_α are nonnegative integers. We may also assume without loss of generality that R is irreducible, since the general case easily reduces to this case.

Consider the subspace $\mathcal{E}(\lambda, k) \subset L^2(T_u, \delta(k)d_u t)$ spanned by the functions $E(w\lambda, k)$ with $w \in W$. Recall from Proposition 8.3.7 that the center $Z(\mathbb{H}) = S\mathfrak{t}^W$ of the degenerate affine Hecke algebra $\mathbb{H} := \mathbb{H}(R_+, k)$ acts on $\mathbb{C}[T]$ via the operators $T(p, k)$ (with $p \in S\mathfrak{t}^W$). By Lemma 8.3.4 we see that $\mathcal{E}(\lambda, k)$ is the $Z(\mathbb{H})$-eigenspace of the central character $S\mathfrak{t}^W \ni p \to p(\lambda + \rho(k))$. By Lemma 8.3.3 it follows that the subspaces $\mathcal{E}(\lambda, k)$ are mutually orthogonal $\mathbb{H}$-submodules of $\mathbb{C}[T]$. Let $\mathbb{H}_\lambda \subset \mathbb{H}$ be the "parabolic subalgebra" $\mathbb{H}_\lambda := S\mathfrak{t} \otimes \mathbb{C}[W_\lambda] \subset \mathbb{H}$. The algebra $\mathbb{H}_\lambda$ has a one dimensional trivial representation $\mathbb{C}_{\tilde{\lambda}}$ given by $\mathfrak{t} \ni \xi \to \tilde{\lambda}(\xi)$ and $W_\lambda \ni w \to \mathrm{id}_{\mathbb{C}}$. Let $V_{\lambda,k}$ be the induced $\mathbb{H}$-module

$$V_{\lambda,k} := \mathrm{Ind}_{\mathbb{H}_\lambda}^{\mathbb{H}} \mathbb{C}_{\tilde{\lambda}} = \mathbb{H} \otimes_{\mathbb{H}_\lambda} \mathbb{C}_{\tilde{\lambda}}$$

We show by induction on the length $l(w)$ of $w \in W^\lambda$ that $V_{\lambda,k}$ contains a nonzero $S\mathfrak{t}$ eigenvector v_w with eigenvalue $w\tilde{\lambda}$. The induction proces starts with the eigenvector $v_e := 1 \otimes 1$ with eigenvalue $\tilde{\lambda}$. Now let $w \in W^\lambda$ and let s_i be a simple reflection such that $l(s_i w) < l(w)$. Then

$s_i w \in W^\lambda$. By induction we may assume that there exists a nonzero eigenvector $v_{s_i w} \in V_{\lambda,k}$ with eigenvalue $s_i w\tilde{\lambda}$. Then it is easy to see that

$$v_w := \frac{s_i w\tilde{\lambda}(\alpha_i^\vee)}{s_i w\tilde{\lambda}(\alpha_i^\vee) + k_i}\left(s_i + \frac{k_i}{s_i w\tilde{\lambda}(\alpha_i^\vee)}\right) v_{s_i w} \in V_{\lambda,k}$$

is a *nonzero* eigenvector with eigenvalue $w\tilde{\lambda}$ for the action of $S\mathfrak{t}$. Since $V_{\lambda,k}$ obviously has dimension $|W^\lambda|$ it follows that $V_{\lambda,k}$ has a one-dimensional $S\mathfrak{t}$-eigenspace with eigenvalue $w\tilde{\lambda}$ for every $w \in W^\lambda$. In particular, by Frobenius reciprocity, it is clear that $V_{\lambda,k}$ is irreducible. By Frobenius reciprocity there exists a unique nonzero $\mathbb{H}$-module homomorphism

$$j = j_{\lambda,k} : V_{\lambda,k} \to L^2(T_u, \delta(k) d_u t)$$

such that $j(v_e) = E(\lambda, k)$. In particular, $V_{\lambda,k}$ admits a nondegenerate Hermitean form which turns $V_{\lambda,k}$ into a $*$-representation for $\mathbb{H}$ when we equip $\mathbb{H}$ with the $*$-structure $\mathfrak{t} \ni \xi \to \xi^* := \overline{\xi}$ and $w^* := w^{-1}$. By the irreducibility of $V_{\lambda,k}$, this Hermitean inner product is unique up to normalization, and the basis $\{v_w\}_{w\in W^\lambda}$ of eigenvectors we constructed is orthogonal. It is clear that such a form is definite (since it comes from the inner product on $L^2(T_u, \delta(k) d_u t)$). It is an easy matter to prove that such Hermitean inner product must be of the form

$$(v_w, v_{w'}) = a(\lambda, k)\delta_{w,w'} \prod_{\alpha\in R_+^0} \left(1 - \frac{k_\alpha + \frac{1}{2}k_{\alpha/2}}{w\tilde{\lambda}(\alpha^\vee)}\right)^{-1} \tag{8.4.10}$$

for some $a(\lambda, k) \neq 0$, where R^0 denotes the set of roots $\alpha \in R$ such that $2\alpha \notin R$. The eigenvector v_w is mapped via j to a multiple of $E(w\lambda, k)$. The constant of proportionality is easily determined inductively by comparing the normalization in the $E(\mu, k)$-polynomials and the basis v_w; we find

$$\begin{aligned} j(v_w) &= \prod_{\alpha\in R_+\cap w^{-1}R_-} \left(\frac{\tilde{\lambda}(\alpha^\vee) + \frac{1}{2}k_{\alpha/2} + k_\alpha}{\tilde{\lambda}(\alpha^\vee) + \frac{1}{2}k_{\alpha/2}}\right) E(w\lambda, k) \\ &= \prod_{\alpha\in R_+^0\cap w^{-1}R_-^0} \left(1 + \frac{k_\alpha + \frac{1}{2}k_{\alpha/2}}{\tilde{\lambda}(\alpha^\vee)}\right) E(w\lambda, k) \end{aligned}$$

Therefore the proof of (i) reduces to the determination of the constant $a(\lambda, k)$ such that j becomes an isometry. Recall the following well known formula of Macdonald expressing $|W_\lambda|$ in terms of the heights of the roots of $R^0_{\lambda,+}$:

$$|W_\lambda| = \prod_{\alpha\in R_{\lambda,+}} \left(\frac{\rho(k)(\alpha^\vee) + k_\alpha + \frac{1}{2}k_{\alpha/2}}{\rho(k)(\alpha^\vee) + \frac{1}{2}k_{\alpha/2}}\right) = \prod_{\alpha\in R^0_{\lambda,+}} \left(1 + \frac{k_\alpha + \frac{1}{2}k_{\alpha/2}}{\rho(k)(\alpha^\vee)}\right) \tag{8.4.11}$$

An elementary calculation using (8.4.11) shows that (i) is equivalent to proving that

$$a(\lambda, k) = |W_\lambda|^2 \frac{c^*\left(-(\lambda + \rho(k)), k\right)}{\tilde{c}(\lambda + \rho(k), k)} \tag{8.4.12}$$

The proof of (8.4.12) is an inductive argument on the parameter k, where the induction step is based on a generalization of Weyl's character formula. First we observe that (*i*) (hence (8.4.12)) holds if $k = 0$. Before discussing the induction step, let us point out a remarkable and extremely useful property of the polynomials $E(\lambda, k)$. Suppose that $R' \subset R$ is a subsystem of roots such that $k_\alpha = 0$ for $\alpha \in R\backslash R'$. It follows directly from the definitions that the Dunkl operators $T(\xi, k)$ for R and R' are the same for such parameters. As a consequence the polynomials $E(\lambda, k)$ for R and R' are equal in this situation. This allows us to delete the roots from R on which k is zero. Therefore to prove (8.4.12) it suffices to show that if (8.4.12) is true for a parameter k it is also true for $k + 1$, where 1 denotes the characteristic function of the set $R^0 \subset R$. Let Δ be the Weyl denominator, i.e. $\Delta = e^\delta \prod_{\alpha\in R^0_+}(1 - e^{-\alpha})$ where δ is the Weyl vector of R^0_+. If $\lambda \in P_+$ is regular and $t \in T$ then we define

$$P^-(\lambda, k; t) := \sum_{w\in W} \epsilon(w) E(\lambda, k; wt)$$

where ϵ denotes the sign character of W. Then $P^-(\lambda, k)$ spans the subspace of W-skew invariant polynomials in $\mathcal{E}(\lambda, k)$. From the Gram-Schmidt type of definition of the $E(\mu, k)$ we obtain the following generalization of Weyl's character formula:

$$P(\lambda, k + 1) = \Delta^{-1} P^-(\lambda + \delta, k) \tag{8.4.13}$$

for all $\lambda \in P_+$. In particular, we have

$$||P(\lambda, k + 1)||^2_{k+1} = ||P^-(\lambda + \delta, k)||^2_k \tag{8.4.14}$$

On the other hand, via j the Jacobi polynomial $P(\lambda, k+1)$ corresponds to the vector $|W_\lambda|^{-1} \sum_{w\in W} wv_e \in V_{\lambda,k+1}$. It is not very difficult to show that this vector has square norm equal to $|W||W_\lambda|^{-2} a(\lambda, k+1)$. Similarly, $P^-(\lambda+\delta, k)$ corresponds via j to the vector $\sum_{w\in W} \epsilon(w) wv_e \in V_{\lambda+\delta,k}$, and the square norm of this expression can be shown to be

$$|W| a(\lambda + \delta, k) \prod_{\alpha\in R^0_+} \left(\frac{(\lambda + \rho(k) + \delta)(\alpha^\vee) + k_\alpha + \frac{1}{2}k_{\alpha/2}}{(\lambda + \rho(k) + \delta)(\alpha^\vee) - k_\alpha - \frac{1}{2}k_{\alpha/2}} \right)$$

Hence (8.4.14) implies that

$$a(\lambda, k + 1) = a(\lambda + \delta, k)|W_\lambda|^2 \prod_{\alpha\in R^0_+} \left(\frac{(\lambda + \rho(k) + \delta)(\alpha^\vee) + k_\alpha + \frac{1}{2}k_{\alpha/2}}{(\lambda + \rho(k) + \delta)(\alpha^\vee) - k_\alpha - \frac{1}{2}k_{\alpha/2}} \right) \tag{8.4.15}$$

Using the induction hypothesis and easy manipulations we see that the right hand side of (8.4.15) is equal to the right hand side of (8.4.12) (with k replaced by $k + 1$) as desired. This finishes the proof of (*i*) and (*iii*).

The proof of (*ii*) and (*iv*) uses a similar type of inductive argument but we will skip the details. □

8.4.3 Hypergeometric shift operators

Let $\epsilon^+ \in \mathbb{C}[W]$ denote the central idempotent of the trivial character and $\epsilon^- \in \mathbb{C}[W]$ the central idempotent of the sign character. The relations of the degenerate affine Hecke algebra $\mathbb{H}$ (see

Definition 8.3.6) easily imply that the elements

$$\pi^{\pm}(k) := \prod_{\alpha\in R_+^0}(\alpha^{\vee} \pm (k_\alpha + \frac{1}{2}k_{\alpha/2})) \tag{8.4.16}$$

satisfy the relations $\pi^{\pm}\epsilon^{\pm} = \epsilon^{\mp}\pi^{\pm}$ in $\mathbb{H}$. The Dunkl representation yields operators

$$T(\pi^{\pm}(k), k) = \sum_{w} D^{\pm}(w, k) \otimes w \in \mathbb{D}[T_{\mathrm{reg}}] \otimes \mathbb{C}[W]$$

on the algebra $\mathbb{C}[T]$. Similar to the construction of the W-invariant differential operators $D(p, k)$ (with $p \in S\mathfrak{t}^W$) we introduce the associated differential operators

$$D^{\pm}(\pi^{\pm}(k), k) := \sum_{w\in W}(\pm 1)^{l(w)} D^{\pm}(w, k) \in \mathbb{D}[T_{\mathrm{reg}}]^{-W}$$

Here $\mathbb{D}[T_{\mathrm{reg}}]^{-W}$ denotes the space of W-skew invariant algebraic linear partial differential operators on T_{reg}. This gives rise to the following two W-invariant linear partial differential operators on T_{reg}:

$$G^{+}(k) : = \Delta^{-1} D^{+}(\pi^{+}(k), k)$$
$$G^{-}(k + 1) : = D^{-}(\pi^{-}(k), k)\Delta$$

where, as in (8.4.13), 1 denotes the multiplicity parameter which is equal to 1 on R^0 and equal to 0 on $R\backslash R^0$. We have for $\lambda \in P_+$:

$$G^{+}(k)P(\lambda + \delta, k) : = \prod_{\alpha\in R_+^0}((k_\alpha + \frac{1}{2}k_{\alpha/2}) - (\lambda + \rho(k + 1))(\alpha^{\vee}))P(\lambda, k + 1) \tag{8.4.17}$$
$$G^{-}(k + 1)P(\lambda, k + 1) : = \prod_{\alpha\in R_+^0}((k_\alpha + \frac{1}{2}k_{\alpha/2}) + (\lambda + \rho(k + 1))(\alpha^{\vee}))P(\lambda + \delta, k)$$

Indeed, the generalized Weyl character formula (8.4.13) and the skew W-invariance of the $D^{\pm}(\pi^{\pm}(k), k)$ imply that these formulas hold up to some constant factor. The upper unitriangularity of the action of the Dunkl operators on monomials t^{λ} of $\mathbb{C}[T]$ implies that we may easily determine the coefficients of the monomial $t^{w_0\lambda}$ in both the left and the right hand side, which determines the precise form of the constant. Because the Jacobi polynomials form a basis of the complex vector space $\mathbb{C}[T]^W$ we obtain:

Corollary 8.4.3 *The differential operator $G^{\pm}(k)$ is the pull-back of a* polynomial *differential operator, i.e. an element (also denoted by $G^{\pm}(k)$) of the Weyl algebra of the affine space $W\backslash T$.*

From (8.4.1) and (8.4.17) we see that these operators satisfy for all $p \in S\mathfrak{t}^W$:

$$G^{+}(k)D(p, k) = D(p, k + 1)G^{+}(k) \tag{8.4.18}$$
$$G^{-}(k)D(p, k) = D(p, k - 1)G^{-}(k)$$

The hypergeometric *shift operators* $G^{\pm}(k)$ derive their name from these relations.

These translations in the multiplicity parameters of the hypergeometric system along integer multiples of $1 \in \mathcal{K}$ can be further generalized to include the translations in $\mathcal{K}$ by vectors in the full lattice $\mathcal{K}^{\mathbb{Z}} \subset \mathcal{K}$ consisting of the elements $l \in \mathcal{K}$ such that $l_\alpha \in \mathbb{Z}$ for all $\alpha \in R$ and $l_{\alpha/2} \in 2\mathbb{Z}$ for all $\alpha \in R$. The translations in these lattice vectors can all be realized by similar hypergeometric shift operators.

Apart from these translations there exist fundamental reflection symmetries in the parameter space of the hypergeometric system. We again only discuss the principal instance of such a symmetry here. This symmetric originates from the following formula: for all $p \in S\mathfrak{t}^W$ one has:

$$D(p, 1-k) = \delta(k-\frac{1}{2}) \circ D(p,k) \circ \delta(\frac{1}{2}-k) \qquad (8.4.19)$$

This fundamental formula can be established by direct computation for $p = p_2$ equal to the quadratic invariant of W, using Theorem 8.3.10 (checking this formula for p_2 involves computations similar to those worked out in the proof of Theorem 8.5.1); it then follows for arbitrary p because the commutation relation

$$D(p_2,k)D(p,k) - D(p,k)D(p_2,k) = 0 \qquad (8.4.20)$$

together with the assertion $D(p,k) = \partial(p) + \sum_{\mu<0} e^\mu \partial(p_\mu)$ yields a recurrence relation on the p_μ which determines $D(p,k)$ completely. Similarly one proves the symmetry relation

$$G^+(-1/2-k) \circ \delta(k+1) = \delta(k) \circ G^-(3/2+k) \qquad (8.4.21)$$

Let us introduce the rational Dunkl operator $T^{\mathrm{rat}}(\xi,k) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[T])$ by the formula

$$T^{\mathrm{rat}}(\xi,k) = \partial(\xi) + \sum_{\alpha\in R_+^0} (k_\alpha + k_{\alpha/2}) \frac{\alpha(\xi)}{\alpha}(1-s_\alpha) \qquad (8.4.22)$$

which are discussed in Chapter 7 by Dunkl. These operators have homogeneous degree -1. If f denotes a holomorphic germ at $e \in T$ of vanishing order at least $v \geq 0$ let us write $f = f_v(1 + \mathcal{O}(1))$. Then the relation between the trigonometric and rational Dunkl operators is expressed by the formula

$$(T(\xi,k)(f))_{v-1} = T^{\mathrm{rat}}(\xi,k)(f_v) \qquad (8.4.23)$$

Observe that this implies that the rational Dunkl operators $T^{\mathrm{rat}}(\xi,k)$ are mutually commutative. Hence there exists a unique unital $\mathbb{C}$-algebra homomorphism $p \to T^{\mathrm{rat}}(p,k)$ from $S\mathfrak{t} \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[T])$ extending the map $\xi \to T^{\mathrm{rat}}(\xi,k)$. We see that for any holomorphic germ f at $e \in T$ we have:

$$G^-(k)(f)(e) = f(e)T^{\mathrm{rat}}(\pi^\vee,k)(\pi) \qquad (8.4.24)$$

where $\pi^\vee := \prod_{\alpha\in R_+^0} \alpha^\vee \in S\mathfrak{t}$ and $\pi := \prod_{\alpha\in R_+^0} \alpha \in S\mathfrak{t}$. Combining this formula with Theorem 8.4.2(iv) and equation (8.4.17) one deduces that

Corollary 8.4.4

$$T^{\mathrm{rat}}(\pi^\vee,k)(\pi) = \frac{\tilde{c}(\rho(k),k)}{\tilde{c}(\rho(k+1),k+1)}$$

This result has important consequences. By applying (8.4.21) to the constant function 1 and using Corollary 8.4.4 one can prove the following result [63] which was conjectured by Yano and Sekiguchi [78].

Corollary 8.4.5 *Let $k = s.1 \in \mathcal{K}$, and let $\mathbb{A}$ be the Weyl algebra of polynomial linear partial differential operators on the affine space $W\backslash\mathfrak{t}$. Let $D \in \mathbb{A} \otimes \mathbb{C}[s]$ be defined by $Df = \pi^{-1}T^{\mathrm{rat}}(\pi^\vee, -\frac{1}{2} - k)f$ for all $f \in S\mathfrak{t}^W$. Then*

$$D\pi^{2(s+1)} = |W|b(s)\pi^{2s} \tag{8.4.25}$$

where $b(s)$ is the Bernstein-Sato polynomial of the discriminant $\pi^2 \in S\mathfrak{t}^W$. Moreover $b(s)$ is explicitly given by $b(s) := \prod_{i=1}^n \prod_{j=1}^{d_i-1}(d_i(s+\frac{1}{2})+j)$ where the d_i denote the primitive degrees of W.

Also one may use Corollary 8.4.4 to prove the Macdonald-Mehta conjecture for crystallographic Weyl groups [63]:

Corollary 8.4.6 *Let γ denote the Gaussian measure on the Euclidean vector space $\mathfrak{a}$, i.e. $d\gamma(x) = (2\pi)^{-\frac{n}{2}}e^{-\frac{1}{2}|x|^2}dx$ where dx denotes the Lebesque measure on $\mathfrak{a}$. Let us denote by $\pi(x;k) := \prod_{\alpha\in R_+}(\tilde{\alpha}^2(x))^{k_\alpha}$. where $\tilde{\alpha} = \frac{\sqrt{2}}{|\alpha|}\alpha$. Then for all $k \in \mathcal{K}$ such that $\mathrm{Re}(k_\alpha) \geq 0$ we have:*

$$\int_{\mathfrak{a}} \pi(x;k)d\gamma(x) = \frac{|W|}{\tilde{c}(\rho(k),k)} \tag{8.4.26}$$

8.5 The Calogero–Moser system

In this section we shall view the weight function of the hypergeometric system

$$\delta(k;t) = \prod_{\alpha>0}(t^{\frac{1}{2}\alpha} - t^{-\frac{1}{2}\alpha})^{2k_\alpha} = \prod_{\alpha>0}(t^\alpha + t^{-\alpha} - 2)^{k_\alpha}$$

just formally as a multivalued function of determination order one. Since we shall only conjugate linear differential operators by the square root of δ the multivaluedness is of no concern. Alternatively one could work on the regular part of the real vector subgroup T_v where $(t^\alpha + t^{-\alpha} - 2)$ is positive.

Theorem 8.5.1 *If $\xi_1, \cdots, \xi_n$ is an orthonormal basis of $\mathfrak{t}_v$ then conjugation of the quadratic operator $L(k) = D(\sum_i \xi_i^2, k)$ by the square root of the weight function $\delta(k)$ is given by*

$$\delta(k;t)^{\frac{1}{2}} \circ L(k) \circ \delta(k;t)^{-\frac{1}{2}} = \sum_i \partial(\xi_i)^2 + \sum_{\alpha>0} \frac{k_\alpha(1-k_\alpha-2k_{2\alpha})(\alpha,\alpha)}{(t^{\frac{1}{2}\alpha} - t^{-\frac{1}{2}\alpha})^2}$$

as equality in $\mathbb{D}[T_{\mathrm{reg}}]$.

Proof For $\xi \in \mathfrak{t}$ we have

$$\delta(k)^{-\frac{1}{2}} \circ \partial(\xi) \circ \delta(k)^{\frac{1}{2}} = \partial(\xi) + \tfrac{1}{2}\partial(\xi)(\log\delta(k))$$
$$\delta(k)^{-\frac{1}{2}} \circ \partial(\xi)^2 \circ \delta(k)^{\frac{1}{2}} = \partial(\xi)^2 + \partial(\xi)(\log\delta(k))\partial(\xi) + \delta(k)^{-\frac{1}{2}}\partial(\xi)^2(\delta(k)^{\frac{1}{2}})$$

with

$$\tfrac{1}{2}\partial(\xi)(\log\delta(k)) = \sum_{\alpha>0}\tfrac{1}{2}k_\alpha\alpha(\xi)\frac{t^{\frac{1}{2}\alpha}+t^{-\frac{1}{2}\alpha}}{t^{\frac{1}{2}\alpha}-t^{-\frac{1}{2}\alpha}}$$

the constant term of the first expression. In turn we get

$$\sum_i \partial(\xi_i)(\log\delta(k))\partial(\xi_i) = \sum_{\alpha>0}k_\alpha\frac{t^{\frac{1}{2}\alpha}+t^{-\frac{1}{2}\alpha}}{t^{\frac{1}{2}\alpha}-t^{-\frac{1}{2}\alpha}}\partial(\alpha)$$

which is precisely the first order term of the differential operator $D(\sum\xi_i^2, k)$ in Theorem 8.3.10. If we write $\Box = \sum\partial(\xi_i)^2$ then

$$\tfrac{1}{2}\Box(\log\delta(k)) = -\sum_{\alpha>0}\frac{k_\alpha(\alpha,\alpha)}{(t^{\frac{1}{2}\alpha}-t^{-\frac{1}{2}\alpha})^2}$$

and so

$$\delta(k)^{-\frac{1}{2}}\Box(\delta(k)^{\frac{1}{2}}) = -\sum_{\alpha>0}\frac{k_\alpha(\alpha,\alpha)}{(t^{\frac{1}{2}\alpha}-t^{-\frac{1}{2}\alpha})^2} + \sum_{\alpha,\beta>0}\tfrac{1}{4}k_\alpha k_\beta(\alpha,\beta)\frac{(t^{\frac{1}{2}\alpha}+t^{-\frac{1}{2}\alpha})(t^{\frac{1}{2}\beta}+t^{-\frac{1}{2}\beta})}{(t^{\frac{1}{2}\alpha}-t^{-\frac{1}{2}\alpha})(t^{\frac{1}{2}\beta}-t^{-\frac{1}{2}\beta})}\ .$$

We rewrite the second term on the right hand side as

$$(\rho(k),\rho(k)) + \sum_{\alpha,\beta>0}\tfrac{1}{4}k_\alpha k_\beta(\alpha,\beta)\frac{2(t^{\frac{1}{2}(\alpha-\beta)}+t^{-\frac{1}{2}(\alpha-\beta)})}{(t^{\frac{1}{2}\alpha}-t^{-\frac{1}{2}\alpha})(t^{\frac{1}{2}\beta}-t^{-\frac{1}{2}\beta})}$$

or equivalently as

$$(\rho(k),\rho(k)) + \sum_{\alpha>0}\frac{k_\alpha(k_\alpha+2k_{2\alpha})(\alpha,\alpha)}{(t^{\frac{1}{2}\alpha}-t^{-\frac{1}{2}\alpha})^2} + \sum_{\alpha,\beta>0,\alpha\nsim\beta}\tfrac{1}{4}k_\alpha k_\beta(\alpha,\beta)\frac{2(t^{\frac{1}{2}(\alpha-\beta)}+t^{-\frac{1}{2}(\alpha-\beta)})}{(t^{\frac{1}{2}\alpha}-t^{-\frac{1}{2}\alpha})(t^{\frac{1}{2}\beta}-t^{-\frac{1}{2}\beta})}$$

with $\alpha\nsim\beta$ meaning that α and β are not proportional.

We claim that the third term of this last expression vanishes identically. Indeed it is invariant under W and has simple poles along the zero locus $\cup T_\alpha$ (union over $\alpha\in R_+^0$) of the Weyl denominator Δ. Hence its product with Δ becomes skew invariant under W and is a regular function on T. Since this product if of the form $\sum c_\mu t^\mu$ with $c_\mu = 0$ unless $\mu < \delta$ we conclude that the third term is zero. Here $\delta = \frac{1}{2}\sum\alpha$ is the Weyl vector of R_+^0. The theorem follows by collection of the various terms. □

If we switch from the coupling constant $k\in\mathcal{K}$ to $g\in\mathcal{K}$ by the substitution

$$g_\alpha^2 = \tfrac{1}{2}k_\alpha(k_\alpha+2k_{2\alpha}-1)(\alpha,\alpha)$$

then the differential operator

$$S(g) = \tfrac{1}{2}\sum_i \partial(\xi_i)^2 - \sum_{\alpha>0} \frac{g_\alpha^2}{(t^{\frac{1}{2}\alpha} - t^{-\frac{1}{2}\alpha})^2}$$

is called the Schrödinger operator for the periodic Calogero–Moser system. If we denote by $\mathbb{D}[T_{\mathrm{reg}}]^{W,L(k)}$ the commutant of $L(k)$ in $\mathbb{D}[T_{\mathrm{reg}}]^W$ and likewise $\mathbb{D}[T_{\mathrm{reg}}]^{W,S(g)}$ for the commutant of $S(g)$ then the conjugation map

$$\mathbb{D}[T_{\mathrm{reg}}]^{W,L(k)} \to \mathbb{D}[T_{\mathrm{reg}}]^{W,S(g)}\ ,\ D \mapsto \delta(k)^{\frac{1}{2}} \circ D \circ \delta(k)^{-\frac{1}{2}}$$

is an isomorphism of algebras, which are both isomorphic to $S\mathfrak{t}^W$. The conclusion is that the Calogero–Moser system is a completely integrable quantum system, and the results of the previous section are just an exact solution of this integrable quantum system.

An element $D(p,k) \in \mathbb{D}[T_{\mathrm{reg}}]^{W,L(k)}$ for $p \in \mathbb{D}[T_{\mathrm{reg}}]^{W,L(k)}$ has an asymptotic expansion of the form

$$D(p,k) = \sum_{\mu\le 0} t^\mu \partial(p_\mu)$$

for $p_\mu \in S\mathfrak{t}$. In the previous section we have shown that the constant term $p_0 \in S\mathfrak{t} \cong P\mathfrak{t}^*$ is given by $p_0(\lambda) = p(\lambda + \rho(k))$ for all $\lambda \in \mathfrak{t}^*$. Likewise after conjugation by $\delta^{\frac{1}{2}}$ the operator $\delta^{\frac{1}{2}} \circ D(p,k) \circ \delta^{-\frac{1}{2}}$ in $\mathbb{D}[T_{\mathrm{reg}}]^{W,S(g)}$ has an asymptotic expansion

$$\delta^{\frac{1}{2}} \circ D(p,k) \circ \delta^{-\frac{1}{2}} = \sum_{\mu\le 0} t^\mu \partial(q_\mu)$$

with $q_\mu \in S\mathfrak{t}$ and constant term given by

$$q_0(\lambda) = p_0(\lambda - \rho(k)) = p(\lambda)$$

for $p \in P(\mathfrak{t}^*)^W$. These convergent asymptotic expansion are valid in the interior of the positive chamber in the vector group T_v. Because the constant term corresponds to the case $g = 0$ of a free particle the Calogero–Moser system is called asymptotically free.

The commutation relation $[D, S(g)] = 0$ for $D \in \mathbb{D}[T_{\mathrm{reg}}]^{W,S(g)}$ of the form $\sum_{\mu\le 0} t^\mu \partial(q_\mu)$ amounts to a system of recurrence relations

$$(2\lambda + \mu, \mu) q_\mu(\lambda) = -2 \sum_{\alpha>0} g_\alpha^2 \sum_{j\ge 1} j\{q_{\mu+j\alpha}(\lambda - j\alpha) - q_{\mu+j\alpha}(\lambda)\}$$

by a direct verification. Apparently one can pick the initial polynomial $q_0 \in P(\mathfrak{t}^*)^W$ freely for the constant term, and then solve the $q_\mu \in P(\mathfrak{t}^*)$ recurrently. Evidently these recurrence relations can be solved uniquely for chosen q_0 inside the algebra of rational functions on $\mathfrak{t}^*$ with poles on certain hyperplanes. The amazing fact of the integrability of the Calogero–Moser system is that all divisions can be carried out in the algebra $P(\mathfrak{t}^*)$, which is not at all clear from the recurrence relations. However there is one nontrivial conclusion we can draw from these recurrence relations, namely that all differential operators in the commutant $\mathbb{D}[T_{\mathrm{reg}}]^{W,S(g)}$ of $S(g)$ must have polynomial dependence on the coupling constants $g^2 \in \mathcal{K}$. This was not clear before since the substition $\mathcal{K} \ni k \mapsto g \in \mathcal{K}$ is algebraic.

Example 8.5.2 *For the root system of type A_n given by*

$$\mathfrak{t}_v = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1}; \sum_j x_j = 0\}\ ,\ R = \{\alpha \in \mathfrak{t}_v \cap \mathbb{Z}^{n+1}; (\alpha, \alpha) = 2\}$$

all the roots are conjugated under the symmetric group $W = S_{n+1}$ and we have just one coupling parameter $g^2 = k(k-1)$. On the compact torus T_u the Schrödinger operator becomes

$$S(g) = -\tfrac{1}{2}\sum_j \partial(y_j)^2 + \sum_{j<k} \frac{g^2}{4\sin^2(\frac{1}{2}(y_j - y_k))}$$

using complex coordinates $z_j = x_j + iy_j$. Since

$$|e^{iy_j} - e^{iy_k}|^2 = 4\sin^2(\tfrac{1}{2}(y_j - y_k))$$

the periodic Calogero–Moser system describes a system of $n+1$ identical particles on the unit circle $\mathbb{R}/2\pi\mathbb{Z}$ in $\mathbb{C}$ with an inverse square potential. This was the original example studied by Calogero [6] and Moser [55].

Since T is a complex torus the cotangent bundle T^*T_{reg} is canonically isomorphic to the direct product $T_{\mathrm{reg}} \times \mathfrak{t}^*$. The Hamiltonian of the periodic Calogero–Moser system is defined by

$$H(g, t, \lambda) = -\tfrac{1}{2}(\lambda, \lambda) - \sum_{\alpha>0} \frac{g_\alpha^2}{(t^{\frac{1}{2}\alpha} - t^{-\frac{1}{2}\alpha})^2}$$

viewed as a function of $(g, t, \lambda) \in \mathcal{K} \times T_{\mathrm{reg}} \times \mathfrak{t}^*$. The commutative algebra

$$\mathbb{C}[\mathcal{K}] \otimes \mathbb{C}[T_{\mathrm{reg}}] \otimes S\mathfrak{t}$$

of functions on $\mathcal{K} \times T_{\mathrm{reg}} \times \mathfrak{t}^* \simeq \mathcal{K} \times T^*T_{\mathrm{reg}}$ has a natural Poisson bracket (with K taken as space of constant parameters). This Poisson bracket is derived from the filtration on the differential operator algebra

$$\mathbb{C}[\mathcal{K}] \otimes \mathbb{C}[T_{\mathrm{reg}}] \otimes U\mathfrak{t}$$

by taking the sum of the polynomial degrees in $\mathbb{C}[\mathcal{K}]$ and $U\mathfrak{t} \simeq S\mathfrak{t}$ as the total degree. The associated graded of the commutative algebra of hypergeometric differential operators (twisted by conjugation with $\delta(k)^{\frac{1}{2}}$) yields a Poisson commutative algebra containing the Calogero–Moser Hamiltonian $H(g, t, \lambda)$ as homogeneous function of degree 2. In other words, the quantum integrability of the Calogero–Moser system gives the classical integrability of the Calogero–Moser system by taking the classical limit.

The original proof by Moser [55] established the classical integrability for the root system of type A_n using a Lax pair representation. This proof was extended (under a linear parameter constraint) for the other classical root systems, by Olshanetsky–Perelomov [60]. However, for the exceptional root systems the only known proof of the classical integrability of the Calogero–Moser system is the above approach through quantum integrability.

8.6 The hypergeometric function

Recall the system of hypergeometric differential equation on T_{reg} of Definition 8.3.9. If $\mathrm{Re}(k_\alpha) \geq 0$ for all $\alpha \in R$ and $\lambda = \mu + \rho(k) \in \mathfrak{t}^*$, then the Jacobi polynomial $P(\mu, k) \in \mathbb{C}[T]^W$ is a solution of (8.3.15). For other values of the spectral parameter $\lambda \in W\backslash\mathfrak{t}^*$ the solutions of (8.3.15) do not extend holomorphically to all of T. Remarkably, there always exists a unique solution $F(\lambda, k)$ of (8.3.15) which extends to a W-invariant holomorphic function on a tubular neighborhood of $T_v \subset T$ and which is normalized by $F(\lambda, k; e) = 1$. This solution of (8.3.15) is called the *hypergeometric function for root systems*. Before we look at the general theory establishing the existence and uniqueness of this function, let us consider its meaning in the context of Riemannian symmetric spaces. Just like the Jacobi polynomials $P(\lambda, k)$ for a root system R with multiplicity parameters $k \in \mathcal{K}$ could be viewed as a generalization of the elementary zonal spherical functions on a compact Riemannian symmetric space $X_u = U/K$, these hypergeometric functions can be thought of as a natural generalization of Harish-Chandra's spherical functions on the noncompact dual $X = G/K$ of X_u.

8.6.1 Hypergeometric functions and spherical functions

Recall the setup of paragraph 8.4.1. In this subsection we shall discuss the relation between the hypergeometric functions for special multiplicity parameters and the theory of spherical functions on Riemannian symmetric spaces. A standard reference for the latter theory is Helgason' s text book [34]. Let G have Iwasawa decomposition $G = NAK$, and let $G \ni g \to a(g)$ denote the associated Iwasawa projection onto the split maximal Abelian subgroup $A \subset G$. Let $\mathfrak{a}$ denote the Lie algebra of A, and let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. In the harmonic analysis on X one defines the Harish-Chandra spherical function ϕ_λ on X by the integral formula

$$\phi_\lambda(gK) := \int_K a(kg)^{\lambda+\rho} dk \tag{8.6.1}$$

where dk is the normalized Haar measure of K. Here $\rho := \frac{1}{2} \sum_{\alpha\in\Sigma_+} m_\alpha \alpha \in \mathfrak{a}^*$ is the Weyl vector for the minimal parabolic subgroup normalizing NA, where $\Sigma \subset \mathfrak{a}^*$ denotes the restricted root system of G and m_α the multiplicity of the restricted root α. This function is obviously bi-K-invariant as a function of $g \in G$ and satisfies $\phi_\lambda(eK) = 1$. It is well known that it is a joint eigenfunction of the algebra $\mathcal{D}(X)$ of G-invariant differential operators on X. More specifically, if $\gamma : \mathcal{D}(X) \to S\mathfrak{t}^W$ denotes the Harish-Chandra isomorphism then we have for all $\Delta \in \mathcal{D}(X)$:

$$\Delta\phi_\lambda = \gamma(\Delta)(\lambda)\phi_\lambda \tag{8.6.2}$$

Using the K-invariance of ϕ_λ and of the operators we can separate the variables in this system of differential equations and reduce to $A \cong AK \subset G/K$. Taking the radial component of elements of $\mathcal{D}(X)$ yields an embedding of $\mathcal{D}(X)$ into the algebra of W-invariant partial linear differential operators on A_{reg}. Let us denote its image by $\mathcal{R}(X)$. We factor the Harish-Chandra homomorphism γ via the radial component isomorphism $\mathcal{D}(X) \to \mathcal{R}(X)$ and (by abuse of notation) denote the resulting algebra isomorphism also as $\gamma : \mathcal{R}(X) \to S\mathfrak{t}^W$. Hence for all $D \in \mathcal{R}(X)$ we have:

$$D(\phi_\lambda|_{A_{\mathrm{reg}}}) = \gamma(D)(\lambda)\phi_\lambda|_{A_{\mathrm{reg}}} \tag{8.6.3}$$

This is a system of eigenfunction equations for the commutative algebra $\mathcal{R}(X)$ of W-invariant linear partial differential operators on $T_{v,\mathrm{reg}} = A_{\mathrm{reg}}$. It follows from the material in paragraph 8.4.1 that the Jacobi polynomials $P(\mu, k)$ diagonalize the algebra $\mathcal{R}(X)$. It can be easily seen that $\mathcal{R}(X)$ is nothing but the algebra of the W-invariant differential operators $D(p, k)$ with $p \in S\mathfrak{t}^W$, where we make the same identifications as in paragraph 8.4.1, i.e. we take $R = 2\Sigma$ (with $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ the restricted root system of $\mathfrak{g}$) and $m_\alpha = 2k_{2\alpha}$ for all $\alpha \in \Sigma$. The system of eigenfunction equations (8.6.3) identifies with the hypergeometric system (8.3.15) in this way, and we have therefore established the existence of a W-invariant solution $\phi_\lambda|_{T_v}$ on T_v of (8.3.15) in this special situation. It is easy to see that ϕ_λ actually extends holomorphically to the W-invariant tubular neighborhood $T_v\exp(\pi i\Omega) \subset T$, where $\Omega \subset \mathfrak{a}$ is defined by $\Omega := \{X \in \mathfrak{a}; |\alpha(X)| < 1 \text{ for all } \alpha \in R\}$.

When we consider the hypergeometric system (8.3.15) for more general multiplicity parameters k_α we loose the group theoretical techniques considered above to construct solutions. Yet it turns out that the essential features of the space of solutions of (8.3.15) remain intact.

8.6.2 Asymptotic freedom and monodromy

The first important observation is the generic asymptotic freedom of solutions of the system (8.3.15) as a function of $t \in T_{\mathrm{reg}}$ when $|t| \in T_v^+$ is deep in the positive chamber. Indeed, when we plug in (following Harish-Chandra in the group case) an asymptotic series of the form

$$\Phi(\lambda - \rho(k), k; t) = \sum_{\kappa \in Q_-} \Gamma_\kappa(\lambda, k) t^{\lambda - \rho(k) + \kappa} \tag{8.6.4}$$

(with $\lambda \in \mathfrak{t}^*$, $|t| \in T_v^+$, and $\Gamma_0(\lambda, k) = 1$) into the single eigenfunction equation

$$D(\sum_{i=1}^n \xi_i^2, k)\Phi(\lambda - \rho(k), k; t) = (\lambda, \lambda)\Phi(\lambda - \rho(k), k; t) \tag{8.6.5}$$

we obtain the recurrence relations

$$-(2\lambda + \kappa, \kappa)\Gamma_\kappa(\lambda, k) = 2\sum_{\alpha > 0} k_\alpha \sum_{j \geq 1} (\lambda - \rho(k) + \kappa + j\alpha, \alpha)\Gamma_{\kappa + j\alpha}(\lambda, k) \tag{8.6.6}$$

for the $\Gamma_\kappa(\lambda, k)$ which have a unique solution in the field of rational functions in λ and k. In view of (8.6.6) the coefficients $\Gamma_\kappa(\lambda, k)$ may have poles along the hyperplanes of the form $(2\lambda + \kappa', \kappa') = 0$ for certain $\kappa' \in Q_- \backslash \{0\}$. By [62] these poles are removable except possibly when $\kappa' \in -\mathbb{N}R_+$. This is a locally finite collection of affine hyperplanes in the spectral parameter space $\mathfrak{t}^*$. If λ is in the complement of the collection of hyperplanes then we can evaluate the coefficients of $\Phi(\lambda - \rho(k), k; t)$ at λ to obtain a formal solution of (8.6.5). It is not hard to show that such formal solutions are always convergent for all t with $|t| \in T_v^+$.

The fundamental group Π of the the regular orbit space $W\backslash T_{\text{reg}}$ has the following description. Consider the sequence of unramified covering maps

$$\mathfrak{t}_{\text{reg}} \to T_{\text{reg}} \to W\backslash T_{\text{reg}}$$

with $\mathfrak{t}_{\text{reg}} = \{x \in \mathfrak{t}; \alpha(x) \notin 2\pi i\mathbb{Z}\ \forall\alpha \in R^0\}$ the regular points in $\mathfrak{t}$ for the action of the affine Weyl group with translation lattice $2\pi i Q^\vee$. Choose a base point $*$ inside a fundamental alcove (with the origin in its closure) in $\mathfrak{t}_{u,\text{reg}}$. The line segment $[0,1] \ni t \mapsto (1-t)*+tw*$ hits the singular locus in a finite number of points and, going around them through the complex upper half plane (with coordinate t), we obtain elements $T_w \in \Pi$ with $T_{w_1}T_{w_2} = T_{w_3}$ if $w_1, w_2, w_3 = w_1w_2 \in 2\pi i Q^\vee \rtimes W$ and their lengths add up. In fact this gives a presentation of Π as the affine braid or affine Artin group. For translations over $2\pi i Q^{\vee,+}$ in the direction of the positive chamber containing the alcove the lengths add up, and so $2\pi i Q^{\vee,+}$ reproduces itself as an Abelian monoid inside Π. Another presentation of Π generated by this Abelian group, which can be thought of as the fundamental group of T_{reg}, and the Artin group for the finite Weyl group W was obtained by van der Lek and Looijenga [48]. Using the theory of torus compactifications (more specifically the fact that a mirror intersects a one dimensional boundary stratum normally) it is in fact easy to show that

$$T_{s_i}T_x = T_xT_{s_i} \tag{8.6.7}$$

for all $x \in 2\pi i Q^{\vee,+}$ which are fixed by s_i, and the presentation of van der Lek and Looijenga is a further refinement of these relations.

Proposition 8.6.1 *For $\lambda \in \mathfrak{t}^*$ generic and $k \in \mathcal{K}$ such that* $\text{Re}(k) \in \mathcal{K}_+$*, the solution*

$$\tilde{c}(\lambda,k)\Phi(\lambda-\rho(k),k;t) + \tilde{c}(s_i\lambda,k)\Phi(s_i\lambda-\rho(k),k;t) \tag{8.6.8}$$

of the hypergeometric system (8.3.15), which is a priori defined as a holomorphic function for $|t| \in T_v^+$, has a holomorphic continuation over the wall of T_v^+ corresponding to the simple reflection $s_i \in W$, which is invariant under the transformation s_i of T.

Proof For $\lambda \in \mathfrak{t}^*$ generic the series $\Phi(w\lambda-\rho(k),k;t)$ for $|t| \in T_v^+$ are a basis of the solution space of the hypergeometric system (8.3.15) as w runs over the Weyl group W. The commutation relations (8.6.7) imply that the span of the two basis vectors with indices w and s_iw are invariant under the monodromy operator of T_{s_i}. By asymptotics along this wall the monodromy calculation of T_{s_i} in these two basis vectors can be reduced to the monodromy calculation for the rank one Gauss hypergeometric system. This ultimately follows from the Kummer continuation formula for the Gauss hypergeometric function

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-z)^{-\alpha}F(\alpha,\alpha-\gamma+1,\alpha-\beta+1;1/z)$$

$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta}F(\beta,\beta-\gamma+1,\beta-\alpha+1;1/z)$$

by holomorphic continuation in z along the negative real axis $(-\infty,0)$. In turn, this shows that the given linear combination in (8.6.8) extends over the wall to a meromorphic function invariant under s_i. A computation of the local exponents of the hypergeometric system along the wall gives 0 and $1 - \gamma_i$ (with $\gamma_i = \frac{1}{2} + k_{\alpha_i/2} + k_{\alpha_i}$) both with multiplicity $|W|/2$. Therefore the expresssion (8.6.8) extends in fact holomorphically over the wall. □

Using Hartog's theorem the holomorphic extension over the walls in codimension one implies in fact a holomorphic continuation to a suitable tubular neighborhood of T_v in T.

Corollary 8.6.2 *The function $\tilde{F}(\lambda, k; t) = \sum_w \tilde{c}(w\lambda, k)\Phi(w\lambda - \rho(k); t)$ extends from $T_v^+ T_u$ to a holomorphic W-invariant function on the tubular neighborhood*

$$T_v \exp(\pi i \Omega)\ ,\ \Omega = \{x \in \mathfrak{t}_v; |\alpha(X)| < 1\ \forall \alpha \in R\}$$

of T_v in $T = T_v T_u$.

So far $\lambda \in \mathfrak{t}^*$ has been a generic parameter, but the various functions have in fact meromorphic behaviour in $\lambda \in \mathfrak{t}^*$ and k. A careful analysis of the loci of poles in λ of $\tilde{c}(\lambda, k)$ and of $\Phi(w\lambda - \rho(k), k)$ (via the recurrence relations (8.6.6)) shows [62] the following result:

Theorem 8.6.3 *The function*

$$(\lambda, k, t) \mapsto \tilde{F}(\lambda, k; t)$$

has a holomorphic extension to $\mathfrak{t}^* \times \mathcal{K} \times T_v \exp(\pi i \Omega)$ as a W-invariant function both in the spectral variable λ and the space variable t.

For reasons which will become clear in Section 8.6.4 we renormalize this solution by the multiplying factor $\tilde{c}(\rho(k), k)^{-1}$ (see (8.4.9)) and denote this renormalized function by $F(\lambda, k; t)$. By (8.4.9) and Corollary 8.6.2 the asymptotic expansion formula for $F(\lambda, k)$ in terms of the asymptotically free solutions $\Phi(w\lambda - \rho(k), k; t)$ becomes

$$F(\lambda, k; t) = \sum_w c(w\lambda, k)\Phi(w\lambda - \rho(k), k; t) \tag{8.6.9}$$

Definition 8.6.4 *The function $F(\lambda, k)$ is called the hypergeometric function for the root system R with multiplicity parameter $k \in \mathcal{K}$ and spectral parameter $\lambda \in \mathfrak{t}^*$.*

We claim that the meromorphic function $\mathcal{K} \ni k \to \tilde{c}(\rho(k), k)$ is entire. Indeed, from (8.4.7) it is clear that for any $k_0 \in \mathcal{K}$, there exists an $n \in \mathbb{N}$ such that $\tilde{c}(\rho(k), k)$ is holomorphic in a neighborhood of $k = k_0 + n$. Using the fact that the left hand side of the equation of Corollary 8.4.4 is a polynomial (namely the polynomial $T^{\mathrm{rat}}(\pi^\vee, k)(\pi)$ in the variables $k_\alpha + k_{\alpha/2}$ (where α runs over R_+^0)), the claim follows. By Corollary 8.4.6 it also follows that $\tilde{c}(\rho(k), k)$ is nonzero if $k \in \mathcal{K}^+$, where

$$\mathcal{K}^+ = \{k \in \mathcal{K} \mid \mathrm{Re}(k_\alpha + k_{\alpha/2}) \geq 0,\ \forall \alpha \in R^0\} \tag{8.6.10}$$

Let $\mathcal{S} \subset \mathcal{K}$ denote the set of zeroes of the entire function $\tilde{c}(\rho(k), k)$, and let $\mathcal{S}_0 \subset \mathcal{K}$ be the set of zeroes of the poynomial $T^{\mathrm{rat}}(\pi^\vee, k)(\pi)$ as in Corollary 8.4.4. By the above and Corollary 8.4.4 we have:

$$\mathcal{S} = \cup_{n \geq 0}(S_0 - n.1) \tag{8.6.11}$$

(where $1 \in \mathcal{K}$ denotes the characteristic function of R^0). The set $\mathcal{S}_0$ is a finite union of hyperplanes in $\mathcal{K}$ which has been described explicitly in all cases [18].

Using Theorem 8.6.3, one has:

Proposition 8.6.5 *The function F extends to a meromorphic function on $\mathfrak{t}^* \times \mathcal{K} \times T_v \exp(\pi i \Omega)$ which is holomorphic on $\mathfrak{t}^* \times (\mathcal{K} \backslash \mathcal{S}) \times T_v \exp(\pi i \Omega)$. The set $\mathcal{K} \backslash \mathcal{S}$ is the complement of locally finite union of hyperplanes, and contains the closed set $\mathcal{K}^+$.*

8.6.3 Knizhnik-Zamolodchikov and Matsuo's isomorphism

The commuting Dunkl operators $T(\xi, k)$ of Definition 8.3.2 are deformations of the constant vector fields $\partial(\xi)$ on T_{reg}. This deformation satisfies the Leibniz rule

$$T(\xi, k)(fg) = (\partial(\xi)(f))g + f(T(\xi, k)(g)) \tag{8.6.12}$$

if f is a W-invariant function on T_{reg}. This shows that we can think of the operators $T(\xi, k)$ as the covariant differentiations of a W-equivariant integrable connection on $W \backslash T_{\text{reg}}$ on the free $\mathbb{C}[T_{\text{reg}}]^W$-module $\mathbb{C}[T_{\text{reg}}]$. If $\lambda \in \mathfrak{t}^*$ then exactly similar considerations apply to the operators $T(\xi, k) - \lambda(\xi)$.

To compute the connection form of such integrable connections explicitly, let us choose a point $Wt \in W \backslash T_{\text{reg}}$. We define a tangent vector at Wt by fixing an element $t \in Wt$ and choosing a tangent vector $\xi \in \mathfrak{t}$. Here we view $\mathfrak{t}$ as the tangent space at $t \in T_{\text{reg}}$ in the usual way, by identifying elements of $\mathfrak{t}$ as constant vector fields on T_{reg}. Let us denote this tangent vector at Wt by (t, ξ), then obviously $(wt, w\xi) = (t, \xi)$ for all $w \in W$. Let us denote by $\hat{O}_t$ the completed local ring of $t \in T_{\text{reg}}$, and by O^W_{Wt} the completed local ring of $Wt \in W \backslash T_{\text{reg}}$. Then $\hat{O}_{Wt} := \hat{O}^W_{Wt} \otimes_{\mathbb{C}[T_{\text{reg}}]^W} \mathbb{C}[T_{\text{reg}}]$ is isomorphic to $\oplus_{w \in W} \hat{O}_{wt}$ as $\hat{O}^W_{Wt}[W]$-algebra, by the Chinese remainder theorem. After fixing $t \in Wt$ (as we did above) we can thus identify a germ $\phi \in \hat{O}_{Wt}$ with a collection of germs $(\phi_w)_{w \in W}$ with $\phi_w \in \hat{O}_{wt}$ for all $w \in W$ via this isomorphism. With this notation we define an isomorphism $L_t : \hat{O}_{Wt} \to \hat{O}_t \otimes \mathbb{C}[W]$ of $\hat{O}^W_{Wt}[W]$-modules by

$$L_t : \phi = (\phi_w)_{w \in W} \to \sum_{w \in W} {}^w\phi_{w^{-1}} \otimes w \tag{8.6.13}$$

Here ${}^w\phi_{w^{-1}} := \phi_{w^{-1}} \circ w^{-1} \in \hat{O}_t$. The W-action on $\hat{O}_{Wt}$ corresponds via L_t with the right regular action of W on the right tensor leg $\mathbb{C}[W]$ of $\hat{O}_t \otimes \mathbb{C}[W]$, and multiplication by an element $(f_w)_{w \in W} \in \hat{O}^W_{Wt}$ in $\hat{O}_{Wt}$ corresponds with the multiplication on the left by $f_e \in \hat{O}_t$. The inverse of this isomorphism L_t maps $\psi \otimes w \in \hat{O}_t \otimes \mathbb{C}[W]$ to ${}^{w^{-1}}\psi \in \hat{O}_{w^{-1}t} \subset \hat{O}_{Wt}$. Hence the integrable connection $\nabla(\lambda, k)$ on $\hat{O}_t \otimes \mathbb{C}[W]$ whose covariant derivative with respect to the tangent vector (t, ξ) at Wt is given by the operator $\oplus_{w \in W}(T(w\xi, k) - \lambda(w\xi))$ on $\hat{O}_{Wt} = \oplus_{w \in W} \hat{O}_{wt}$ satisfies

$$\nabla_{(t,\xi)}(\lambda, k)(\psi \otimes w) = L_t((T(w^{-1}\xi, k) - w\lambda(\xi))({}^{w^{-1}}\psi)) \tag{8.6.14}$$

A straightforward computation shows that

$$\nabla_{(t,\xi)}(\lambda,k) = \partial_\xi\psi\otimes\mathrm{id} - \mathrm{id}\otimes D(\lambda,\xi) + \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha\alpha(\xi)\left(\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\otimes(1-r_\alpha) + \mathrm{id}\otimes r_\alpha\epsilon_\alpha\right)$$

where $\epsilon_\alpha(w) = -\mathrm{sgn}(w^{-1}\alpha)w$ and $D(\lambda,\xi)(w) = w\lambda(\xi)w$.

It is easy to see that the connection $\nabla(\lambda,k)$ does not depend on the choice of $t\in Wt$, and that $\nabla(\lambda,k)$ is equivariant with respect to the diagonal action of W on $O(T_{\mathrm{reg}})\otimes\mathbb{C}[W]$ (where we act via the left regular action of W on the right tensor leg $\mathbb{C}[W]$).

Definition 8.6.6 *The integrable, W-equivariant connection $\nabla(\lambda,k)$ on $O(T_{\mathrm{reg}})\otimes\mathbb{C}[W]$ we just defined is called the (trigonometric)* Knizhnik-Zamolodchikov connection *(KZ-connection in the sequel).*

For a further discussion of KZ-equations see Chapter 11 by Tarasov and Varchenko.

Corollary 8.6.7 (Matsuo's isomorphism [54], [7], [65]) *Assume that for all $\alpha\in R^0_+$ we have $\lambda(\alpha^\vee)\neq k_\alpha+\frac{1}{2}k_{\alpha/2}$. The $\mathbb{D}$-module on $W\backslash T_{\mathrm{reg}}$ defined by the W-equivariant integrable connection $\nabla(\lambda,k)$ on the trivial vector bundle $O(T_{\mathrm{reg}})\otimes\mathbb{C}[W]$ over T_{reg} is equivalent to the cyclic D-module on $W\backslash T_{\mathrm{reg}}$ defined by the hypergeometric system (8.3.15) via the map $\hat{O}_t\otimes\mathbb{C}[W]\ni\psi\otimes w\to\psi\in\hat{O}_t$.*

Proof It is enough to show that this map restricts to an isomorphism between the sheaf of flat sections of $\nabla(\lambda,k)$ and the sheaf of solutions of (8.3.15). By (8.6.14) we see that a flat section of $\nabla(\lambda,k)$ is of the form $L_t(\psi)$ where $\psi\in\hat{O}_{Wt}$ is a joint eigenfunction of the $T(\xi,k)$ satisfying $T(\xi,k)\psi=\lambda(\xi)\psi$. When we extend the corresponding image under the Matsuo map of this flat section in a W-invariant way we simply obtain $\overline{\psi}:=\sum_{w\in W}{}^w\psi\in\hat{O}^W_{Wt}$. Observe that $\hat{O}_{Wt}$ is an $\mathbb{H}$-module via the Dunkl representation, and that ψ (and hence also $\overline{\psi}$) as above belong to the submodule $S(\lambda,k)$ of $\hat{O}_{Wt}$ on which $Z(\mathbb{H})\simeq S\mathfrak{t}^W$ acts by the central character $W\lambda$. Recall the minimal principal series module $M(\lambda)=\mathbb{H}\otimes_{S\mathfrak{t}}\mathbb{C}_\lambda$ of the degenerate affine Hecke algebra $\mathbb{H}$. This module has central character $W\lambda$ and always contains a one dimensional subspace $M(\lambda)^W$ of W-invariant vectors. If $\lambda(\xi)\neq\pm(k_\alpha+\frac{1}{2}k_{\alpha/2})$ for all $\alpha\in R^0$ then this W-dimensional module of $\mathbb{H}$ is known to be irreducible by a well known theorem of Shinichi Kato [40]. If $M(\lambda)$ is irreducible then it is easy to see that the joint $\mathfrak{t}$-eigenspace $M(\lambda)_\lambda$ is one dimensional, and that the symmetrization map $\sum_w w$ with respect to W defines an isomorphism from $M(\lambda)_\lambda$ to $M(\lambda)^W$. By a dimension count it follows, using Definition 8.3.6 and Proposition 8.3.7, that the quotient algebra $\mathbb{H}_\lambda$ by the maximal $Z(\mathbb{H})$-ideal of the point $W\lambda\in W\backslash\mathfrak{t}^*$ has dimension $|W|^2$. Hence by Kato's theorem, if $\lambda(\xi)\neq\pm(k_\alpha+\frac{1}{2}k_{\alpha/2})$ for all $\alpha\in R^0$ then $\mathbb{H}_\lambda$ is isomorphic to the finite dimensional simple $\mathbb{C}$-algebra $\mathrm{End}(M(\lambda))$. Hence $S(\lambda,k)$ is semisimple in this case, and isomorphic to $M(\lambda)^d$ where $d=\dim(S(\lambda,k)^W$. In particular, the symmetrization map $\sum_w w$ defines a linear isomorphism from the space $S(\lambda,k)_\lambda$ of joint eigenfunctions ψ of the $T(\xi,k)$ with joint eigenvalue λ onto $S(\lambda,k)^W$. Recall that $S(\lambda,k)^W$ is equal to the local solution space at Wt of the hypergeometric system (8.3.15), and via the map L_t defined above the space $S(\lambda,k)_\lambda$ is isomorphic to the space of flat sections of $\nabla(\lambda,k)$ locally at t. Via L_t the symmetrization map $\sum_w w$ corresponds to Matsuo's map on this space of flat sections. Therefore Matsuo's map defines an isomorphims onto $S(\lambda,k)^W$ if $\lambda(\xi) \neq \pm(k_\alpha + \frac{1}{2}k_{\alpha/2})$ for all $\alpha \in R^0$. Its inverse can be written down explicitly as differential operator using Dunkl's operators; this even shows that this Matsuo's map is an isomorphism whenever $\lambda(\xi) \neq k_\alpha + \frac{1}{2}k_{\alpha/2}$ (see [65]). □

Corollary 8.6.8 *The hypergeometric system (8.3.15) is holonomic of rank* $|W|$ *and is regular singular on* $W\backslash T_{\mathrm{reg}}$.

Proof It is not difficult to prove the claim that the system (8.3.15) is always holonomic of rank $|W|$ using the fact that the commuting differential operators $D(p,k)$ with $p \in S\mathfrak{t}^W$ are of the form $\partial(p)$ + l.o.t.. When $\lambda(\alpha^\vee) \neq k_\alpha + \frac{1}{2}k_{\alpha/2}$ then Matsuo's isomorphism proves that the system is equivalent to an algebraic integrable connection on the trivial vector bundle with fibre $\mathbb{C}[W]$ on the smooth quasi-projective variety $W\backslash T_{\mathrm{reg}}$, exhibiting simple poles only. Hence the system is clearly regular singular in that case. Since the holonomic rank of the system is constant equal to $|W|$ in the parameters, the regularity is detected by the regularity of the restrictions of a rank $|W|$ connection with holomorphic dependence on the parameters (λ,k) on punctured disks such that generically in (λ,k) the connection is regular singular at the puncture. By rewriting the first order system of ordinary differential equations for the flat sections of this connection on the punctured disk as a higher order ordinary differential equation with holomorphic coefficients and holomorphic dependence on (λ,k) it is clear that the generic regularity in (λ,k) of the singularity implies the regularity for all values of (λ,k). □

8.6.4 Normalization at e and summation formulae

We have introduced the hypergeometric function $F(\lambda,k)$ of Definition 8.6.4 via its asymptotically free expansion deep in the Weyl chamber. However its normalization is motivated by the evaluation at e, as we will see in this section. The main theorem of this subsection is:

Theorem 8.6.9 *We have* $F(\lambda,k;e) = 1$ *for all* $\lambda \in \mathfrak{t}^*$ *and* $k \in \mathcal{K}$.

As a first step we look at the special case $\lambda = \mu+\rho(k)$ where $\mu \in P^+$. We have $F(\lambda,k) = c(\mu+\rho(k),k)P(\mu,k)$, since both expressions are meromorphic in k (for fixed μ) and represent a W-invariant holomorphic solution of (8.3.15) in a tubular neighborhood of T_v. For generic k, and hence for all k, they must be proportional therefore, and we conclude that the asymptotically free expansion (8.6.9) of the left hand side coincides with the right hand side if $\lambda = \mu + \rho(k)$ with μ dominant and integral. By Theorem 8.4.2 we conclude that $F(\lambda,k;e) = 1$ if $\lambda = \mu+\rho(k)$ with μ dominant and integral.

Using the theory of hypergeometric shift operators we see more generally that the meromorphic function $(\lambda,k) \to F(\lambda,k;e)$ is periodic for translations of k in the integral lattice $\mathcal{K}^{\mathbb{Z}} \subset \mathcal{K}$ (for translation which are integral multiples of the constant multiplicity functions 1 this follows from (8.4.24) and Corollary 8.4.4, and this argument can be generalized to general integral translations in k). This statement is indeed more general than the former, since for the polynomial case where $\mu = \lambda - \rho(k)$ is integral and dominant the periodicity in k clearly implies that $F(\lambda, k; e)$ is constant in k, and hence constant equal to 1. We would now like to extend this result to arbitrary $\lambda \in \mathfrak{t}^*$ using some kind of interpolation result like Carlson's Theorem, although this is not literally possible. We begin with a bound on the growth order:

Lemma 8.6.10 *Let $k_0 \in \mathcal{K}^{\mathbb{Z}}$ and $\lambda \in \mathfrak{t}^*$. The entire function $\mathbb{C} \ni x \to \epsilon^{k_0}(x) := F(\lambda, xk_0; e)$ is periodic with period* 1 *and has growth order at most* 1.

Proof The periodicity follows immediately from the above text. It is enough to prove that the growth order of the entire function $\tilde{\epsilon}^{k_0}(x) := \tilde{F}(\lambda, xk_0; e)$ is at most 1, using the (nontrivial) fact that the growth order of an entire function that can be written as a quotient f/g of two entire functions of finite growth order has a finite growth order bounded by the maximum of the growth orders of f and g. By a well-known result of Gruman and Lelong [49], the growth order $\rho(z)$ of a holomorphic family of entire functions $w \to f(z, w)$ has the property that its smallest upper half continuous majorant $z \to \rho^*(z)$ is plurisubharmonic, hence in particular satisfies the maximum principle. Applying this to $\mathfrak{t}^* \times T_v \exp(\pi i \Omega) \ni (\lambda, z) \to \tilde{F}(\lambda, k; z)$ it suffices therefore, in view of Corollary 8.6.2, to prove that the order as an entire function of k of the holomorphic family $(\lambda, z) \to \Phi(\lambda - \rho(k), k; z)$ is bounded by 1, for λ outside the locally finite set of hyperplanes $(\lambda, \kappa^\vee) + 1 = 0$ and $z \in T_v^+ \exp(\pi i \Omega)$ regular. This is a consequence of the recurrence relations (8.6.6), see [64]. □

Proof of Theorem 8.6.9. The next two lemmas yield the non-vanishing of $F(\lambda, k; e)$ if $k \in \mathcal{K}^+$, which is an important intermediate result.

Lemma 8.6.11 *There exists an open neighborhood $\mathcal{U}$ of $\mathcal{K}^+$ with the following property: If F is a nonzero holomorphic W-invariant solution of (8.3.15) defined in a neighborhood of e for some $(\lambda, k) \in \mathfrak{t}^* \times \mathcal{U}$, then $F(e) \neq 0$.*

Proof Let G be a joint eigenfunction of the $T(\xi, k)$ with eigenvalue $\lambda \in \mathfrak{t}^*$, for some $k \in \mathcal{K}$. Suppose that G is nonzero and that $G(e) = 0$. Then the lowest homogeneous term g of the expansion of G at e is a nonzero homogeneous polynomial of positive degree d on $\mathfrak{t}$ which is killed by all operators $T^{\text{rat}}(\xi, k)$, and hence by the degree preserving operator $E(k) := \sum_i x_i T^{\text{rat}}(\xi_i, k)$. We have:

$$E(k) = \sum_i x_i \partial(\xi_i) + \sum_{\alpha \in R_+^0} (k_\alpha + k_{\alpha/2})(1 - s_\alpha). \tag{8.6.15}$$

Observe that $\sum_i x_i \partial(\xi_i)(g) = dg$ with $d > 0$, and that $\sum_{\alpha \in R_+^0} (k_\alpha + k_{\alpha/2})(1 - s_\alpha)$ is a scalar operator on any irreducible W-module by Schur's lemma. Let $\text{Harm}_+ \subset \text{Harm}$ denote the space of W-harmonic polynomials on $\mathfrak{t}$ which vanish at 0, and define

$$\mathcal{U} = \{k \in \mathcal{K} \mid \text{Re}(\epsilon(k)) > 0, \ \forall \text{ eigenvalues } \epsilon(k) \text{ of } E(k) \text{ on } \text{Harm}_+\} \tag{8.6.16}$$

Then $\mathcal{U} \subset \mathcal{K}$ is open, and nonempty since $0 \in \mathcal{U}$. Since the space of all polynomials on $\mathfrak{t}$ is a free $(S\mathfrak{t}^*)^W$-module with basis Harm it follows that for all $k \in \mathcal{U}$ and any degree $d > 0$, every eigenvalue of $E(k)$ on the space of all homogeneous polynomials of degree d has a strictly positive real part. In particular, if $k \in \mathcal{U}$ then $E(k)$ is a degree preserving linear isomorphism on the space of polynomials on $\mathfrak{t}$ of positive degree. We conclude that if $k \in \mathcal{U}$ then $G \neq 0$ implies that $G(e) \neq 0$.

Now consider the $\mathbb{H}$-module $M = \mathbb{H}F$. It is obvious that $M = \mathbb{H}F$ has dimension $\leq |W|$, central character $W\lambda$, and that the trivial representation of W occurs in M with multiplicity 1. Let $G \in M$ be any nonzero simultaneous eigenvector of the $T(\xi, k)$. By the above we see that $G(e) \neq 0$. Hence we may and will assume that $G(e) = 1$. We conclude that $0 \neq \sum_{w \in W} {}^w G \in M^W$, and that $F = F(e)|W|^{-1} \sum_{w \in W} {}^w G$. Hence $F(e) \neq 0$.

Finally note that class functions of W of the form $\sum_\alpha (1 - s_\alpha)$, where α runs over the set of all positive roots of a fixed length, act as a nonnegative constant in any irreducible W-module. Hence $\mathcal{K}^+ \subset \mathcal{U}$, and combined with the above this finishes the proof. □

Lemma 8.6.12 *If* $(\lambda, k) \in \mathfrak{t}^* \times \mathcal{K}^+$ *then* $F(\lambda, k, e) \neq 0$.

Proof Let $\mathcal{U}, \mathcal{S} \subset \mathcal{K}$ be the subsets defined by (8.6.16) and (8.6.11) respectively, and put $\mathcal{V} = \mathcal{U} \backslash (\mathcal{S} \cap \mathcal{U})$. Notice that $\mathcal{V}$ is open, connected and $\mathcal{K}^+ \subset \mathcal{V}$.

By Proposition 8.6.5 $F(\lambda, k)$ is a holomorphic family for $(\lambda, k) \in \mathfrak{t}^* \times \mathcal{V}$. Let $Z_{F,e}$ denote the zero locus of the holomorphic function $\mathfrak{t}^* \times \mathcal{V} \ni (\lambda, k) \to F(\lambda, k; e)$. Thus $Z_{F,e} \subset \mathfrak{t}^* \times \mathcal{V}$ is an analytic hypersurface, and in particular a complex space. By Lemma 8.6.11 we see that $Z_{F,e} \times T_v \exp(\pi i \Omega) \subset Z_F$, where Z_F denotes the zero locus of F viewed as holomorphic function on $\mathfrak{t}^* \times \mathcal{V} \times T_v \exp(\pi i \Omega)$.

On the other hand, observe that if all summands of the asymptotic expansion (8.6.9) are well defined, nonvanishing, and the points $w\lambda$ are distinct modulo Q when w varies in W, then $F(\lambda, k) \neq 0$ (i.e. $F(\lambda, k; t)$ does not vanish identically for all $t \in T_v \exp(\pi i \Omega)$).

From (8.4.7), (8.6.9) and (8.6.6) and the above we conclude that the analytic hypersurface $Z_{F,e}$ is contained in the locally finite union $U \subset \mathfrak{t}^* \times \mathcal{V}$ of hyperplanes $H_{\kappa,1}$ defined by the equation $(\lambda, \kappa^\vee) + 1 = 0$ for some $\kappa \in Q \backslash \{0\}$, and root hyperplanes $H_{\alpha,0}$ defined by the equation $(\lambda, \alpha^\vee) = 0$ for some $\alpha \in R$.

By the irreducible decomposition of $Z_{F,e}$ [25, 9.2] and the inclusion $Z_{F,e} \subset U$ it follows that $Z_{F,e}$ is a union of a subset of the set of hyperplanes H contained in U. Suppose that $Z_{F,e}$ is nonempty, and let $(\lambda_0, k_0) \in Z_{F,e}$. Then there exists a hyperplane of the form $H_{\kappa,c}$ (given by an equation of the form $(\lambda, \kappa^\vee) + c = 0$) such that $(\lambda_0, k_0) \in H_{\kappa,c} \subset Z_{F,e}$. In particular, $(\lambda_0, 0) \in Z_{F,e}$ since $(\lambda_0, 0) \in H_{\kappa,c}$. But this is not possible since (as one easily checks) $F(\lambda, 0; e) = 1$ for all $\lambda \in \mathfrak{t}^*$. Hence $Z_{F,e}$ must be the empty set, finishing the proof. □

We now continue the proof of the main Theorem 8.6.9. The previous lemma and the 1-periodicity of the entire function $\mathbb{C} \ni x \to \epsilon^{k_0}(x)$ imply that this function is non-vanishing if $k_0 \in \mathcal{K}^{\mathbb{Z},+} := \mathcal{K}^{\mathbb{Z}} \cap \mathcal{K}^+$, and by Lemma 8.6.10 it has growth order at most 1. This implies that $\epsilon^{k_0}(x) = \exp(p(x))$ where p is a polynomial of degree at most 1. Since this is a 1-periodic function we see that $p(x)$ has to be of the form $p_{n,c}(x) = 2\pi i n x + c$ for some $n \in \mathbb{Z}$ and $c \in \mathbb{C}$. It is also clear that for real λ we have $\epsilon^{k_0}(x) \in \mathbb{R}$ if $x \in \mathbb{R}$. Hence we have $n = 0$, proving that ϵ^{k_0} is a constant function. Hence for λ real and $k \in \mathcal{K}^+$ rational we see that $F(\lambda, k; e) = \epsilon^{k_0}(q) = \epsilon^{k_0}(0) = 1$, where we wrote $k = qk_0$ for some $k_0 \in \mathcal{K}^{\mathbb{Z},+}$ and $q \in \mathbb{Q}_+$. Since $F(\lambda, k; e)$ is meromorphic in λ and k the desired result follows. □

The following uniqueness result is based on the argument in the proof of Lemma 8.6.11:

Theorem 8.6.13 *The meromorphic family $F(\lambda, k)$ is holomorphic on $\mathfrak{t}^* \times \mathcal{U}$. If $(\lambda, k) \in \mathfrak{t}^* \times \mathcal{U}$ then $F(\lambda, k)$ is the* unique *holomorphic W-invariant solution of (8.3.15) defined in a neighborhood of e, up to scalar multiplication.*

Proof If the space holomorphic W-invariant solutions of (8.3.15) defined in a neighborhood of e for a parameter $(\lambda, k) \in \mathfrak{t}^* \times \mathcal{U}$ has dimension higher than 1, then there also exists such a nonzero holomorphic W-invariant solution F_0 of (8.3.15) defined in a neighborhood of e such that $F_0(e) = 0$. This is impossible by Lemma 8.6.11, proving the uniqueness claim.

Hence by Theorem 8.6.9 we have $F(\lambda, k) = \tilde{F}(\lambda, k; e)^{-1}\tilde{F}(\lambda, k)$, a meromorphic family of solutions of (8.3.15), holomorphic and W-invariant on $T_v \exp(\pi i\Omega)$. If this meromorphic family would have poles at some $(\lambda, k) \in \mathfrak{t}^* \times \mathcal{U}$, then removing these poles would again yield a nonzero solution F_0 of (8.3.15) at the parameter (λ, k) such that $F_0(e) = 0$, contradicting Lemma 8.6.11. This finishes the proof. □

A more refined analysis then was outlined in the above proof of Theorem 8.6.9 makes it possible to evaluate the asymptotically free solutions $\Phi(\lambda, k; e)$ at e whenever this is possible. It is an analog of the Gauss summation formula for the classical hypergeometric function:

Theorem 8.6.14 ([64]) *Let $k \in -\mathcal{U}$ with $\mathcal{U}$ as in (8.6.16). Then*

$$\lim_{A_+ \ni z \to e} \Phi(\lambda - \rho(k), k; z) = \frac{c^*(\rho(k), k)}{c^*(\lambda, k)} \tag{8.6.17}$$

Such formulae also enable one to evaluate explicitly $F(\lambda, k; p)$ at special points $p \in \overline{\exp(\pi i\Omega)}$.

Theorem 8.6.9 and the above summation formulae are multivariate examples of explicit (partial) solutions to a "connection problem" for a regular singular system of linear partial differential equations. Connection problems play a central role in the theory ordinary linear differential equations with regular singularities on the projective line. There has been quite some progress in the theory for these one-dimensional connection problems, in particular in the case of rigid local systems, by the work of Katz [41], Crawley-Boevey [13], Oshima and others (see [68] for an account of these developments). Oshima and Shimeno [69] observed that the solution to the connection problem for rigid local systems in one dimension is relevant in the multivariate situation of Theorem 8.6.9 as well, by "restricting" the hypergeometric system to various one-dimensional strata of the singular locus. This yields a different approach than the one presented here. Though natural and possibly more elementary than the above proof, it seems inevitable that a case-by-case analysis will be a part of such an approach. Perhaps the key step to a both uniform and satisfactory proof of theorem 8.6.9 is the "global hypergeometric function" for root systems introduced by Cherednik. This functions is a q-hypergeometric functions version of the $F(\lambda, k; x)$. Techniques of Stokman [75] handle the normalization of this global hypergeometric function for root systems quite naturally, and Theorem 8.6.9 should follow by taking the limit when q tends to 1.

8.6.5 Noncompact harmonic analysis

In Subsection 8.6 (also see Subsection 8.4.1) we have seen that the restriction of $F(\lambda, k)$ to T_v can be viewed as a generalization of the elementary zonal spherical function on a Riemannian symmetric space $X = G/K$ of noncompact type restricted to a maximally flat subspace $T_v = AK/K \subset G/K$. In the previous subsection we have seen that $F(\lambda, k; e) = 1$, generalizing a basic property of the family of elementary zonal spherical functions. It turns out that the spherical Plancherel formula for G/K generalizes as well, as long as $k \in \mathcal{K}_+$. This is the "noncompact version" of the theory of the orthogonal basis of Jacobi polynomials for $\mathbb{C}[T]^W$ on $W\backslash T_u$ as discussed in Subsection 8.4.1, and can be viewed as a common generalization of Harish-Chandra's spherical Plancherel formula [26], [27] for noncompact Riemannian symmetric spaces and the Jacobi transform for even functions on the real line [45].

We identify T_v with $\mathfrak{t}_v$ via the exponential mapping exp normalized such that for all $\alpha \in R$ we have $(\exp(\xi))^\alpha = \exp \alpha(\xi)$. Let dx denote a Haar measure on $\mathfrak{t}_v$ normalized such that the co-volume of $2\pi Q^\vee$ equals 1, and let $d\lambda$ denote the Haar measure on $\mathfrak{t}^*$ defined by duality. Let da denote the Haar measure on T_v corresponding to dx via the identification. We equip $C_c^\infty(T_v)$ with a pre-Hilbertian structure by the Hermitian form

$$(f, g) := \int_{T_v} \overline{f(a)} g(a) d\mu(a) \tag{8.6.18}$$

where

$$d\mu(a) := |W|^{-1} \prod_{\alpha \in R_+} |a^{\frac{1}{2}\alpha} - a^{-\frac{1}{2}\alpha}|^{2k_\alpha} da \tag{8.6.19}$$

Let us define an absolutely continuous measure ν on $i\mathfrak{t}_v^*$ by the formula

$$d\nu(\lambda) := \frac{(2\pi)^{-n}}{\tilde{c}(\lambda, k)\tilde{c}_{w_0}(w_0\lambda, k)} d\lambda \tag{8.6.20}$$

For $f \in C_c^\infty(T_v)^W$ we define its hypergeometric Fourier transform (with respect to the root system R and parameter function k) as the W-invariant function $\mathcal{H}(f)$ of $\lambda \in \mathfrak{t}^*$ defined by:

$$\mathcal{H}(f)(\lambda) := \int_{a \in T_v} f(a) F(-\lambda, k; a) d\mu(a) \tag{8.6.21}$$

By Proposition 8.6.5 this is well defined, obviously W-invariant, and holomorphic in λ. In the opposite direction we define a wave packet operator $\mathcal{J}$. If h is a nice W-invariant function on $i\mathfrak{t}_v^*$ (say an integrable function with respect to the measure ν) then we define

$$\mathcal{J}(h)(a) := \int_{\lambda \in i\mathfrak{t}_v^*} h(\lambda) F(\lambda, k; a) d\nu(\lambda) \tag{8.6.22}$$

The transforms $\mathcal{H}$ and $\mathcal{J}$ can be extended to various more general types of functions, and are in a formal sense adjoint to each other if we give these respective function spaces the Hermitian inner product structures associated to the measures μ and ν respectively. By abuse of language we will not make any notational distinction between all these extensions of the transforms $\mathcal{H}$ and $\mathcal{J}$. The main results on these transforms state that $\mathcal{H}$ and $\mathcal{J}$ are inverse to each other, with important refinements describing the behavior of various important spaces of functions under these transforms. Proving such results is based on various types of estimates for the kernel $F(\lambda, k; a)$.

The following uniform estimate (both in $\lambda \in \mathfrak{t}^*$ and in $a \in T_v$) plays an important role.

Theorem 8.6.15 ([65]) *We have*

$$|F(\lambda, k; a)| \leq |W|^{1/2} H_a(\mathrm{Re}(\lambda)) \tag{8.6.23}$$

where for $\lambda \in \mathfrak{t}_v^*$ and $a \in T_v$ one defines $H_a(\lambda) := \max\{a^{w\lambda} \mid w \in W\}$.

Using this estimate one proves the Paley-Wiener Theorem. Define for $a \in T_v$ the (W-invariant) Paley-Wiener space $PW(a)^W$ consisting of all W-invariant entire complex functions h on $\mathfrak{t}^*$ such that for all $N \in \mathbb{N}$ there exists a constant C_N such that

$$|h(\lambda)| \leq C_N (1 + \|\lambda\|)^{-N} H_a(-\mathrm{Re}(\lambda)) \tag{8.6.24}$$

Let C_a denote the convex hull of the orbit Wa in T_v. Using Theorem 8.6.15 one shows easily that $\mathcal{H}(C_c^\infty(C_a)^W) \subset PW(a)^W$, where $C_c^\infty(C_a)$ denotes the space of compactly supported smooth functions on T_v whose support is contained inside C_a. The converse statement can be proved by an argument which goes back to Rosenberg [70] using the asymptotic expansion of Corollary 8.6.2 of the kernel $F(\lambda, k; a)$ in the positive chamber T_v^+. This yields the result $\mathcal{J}(PW(a)^W) \subset C_c^\infty(C_a)^W$. By an argument due to Schlichtkrull and Van den Ban [2] one can now prove the Paley-Wiener Theorem:

Theorem 8.6.16 ([65]) *The transforms $\mathcal{H} : C_c^\infty(C_a)^W \to PW(a)^W$ and $\mathcal{J} : PW(a)^W \to C_c^\infty(C_a)^W$ are inverse isomorphisms.*

In combination with the formal adjointness of $\mathcal{H}$ and $\mathcal{J}$ we immediately obtain the L^2 version of this result:

Theorem 8.6.17 ([65]) *The transforms $\mathcal{H}$ and $\mathcal{J}$ admit a unique extension to inverse unitary isomorphisms between $L^2(T_v, \mu)$ and $L^2(i\mathfrak{t}_v^*, \nu)$.*

This result was further generalized by [66] to include also the case of not necessarily positive root parameters k_α subject to the condition that μ is a locally integrable function on T_v. It is interesting that the spectrum is no longer continuous in this generality, but consists of series of various dimensions. This corresponds to the possible occurrence of spherical discrete series of graded affine Hecke algebras if the parameters are not necessarily positive. The result was also refined by [65] by extending the transforms beyond W-invariant functions. In this version the transforms $\mathcal{H}$ and $\mathcal{J}$ extend to intertwining isomorphism between $\mathbb{H}$-modules. A further refinement was provided by Delorme [15], who defined the natural Schwartz spaces and proved that $\mathcal{H}$ and $\mathcal{J}$ (in the non W-invariant version, and in the generality where we only require μ to be a locally integrable function) restrict to inverse topological isomorphisms between these Schwartz spaces. In the "repulsive" case where k_α is positive for all $\alpha > 0$ the argument of Delorme was simplified by Schapira [73] by means of a beautiful sharpening of the uniform estimate Theorem 8.6.15. He proved the following striking results

Theorem 8.6.18 ([73]) *Let $\lambda = \sigma + i\tau$ with $\sigma, \tau \in \mathfrak{t}_v^*$. Let $a = \exp x \in \overline{T_v^+}$, and let $k \in \mathcal{K}^+$. Then*

(i). $|F(\lambda, k; a)| \leq F(\sigma, k; a)$

(ii). $F(\sigma, k; a) \leq F(0, k; a) H_a(\sigma)$

(iii). $F(0, k; a) \asymp \prod_{\alpha \in R_{0,+}} (1 + \alpha(x)) a^{-\rho(k)}$

Following Harish-Chandra, Delorme defined the Schwartz space for $\mathcal{H}$ on T_v as the space $\mathcal{C}(T_v)$ consisting of smooth functions f on T_v such that for all constant coefficient differential operators D on T_v, and all $N \in \mathbb{N}$ one has

$$\sup_{a \in T_v} (1 + \| \log(a) \|)^N F(0, k; a)^{-1} |Df(a)| < \infty \tag{8.6.25}$$

The space $\mathcal{C}(T_v)$, equipped with its natural family of seminorms arising from (8.6.25), is a nuclear Fréchet space.

The results of Delorme and Schapira, restricted to the case at hand of W-invariant functions and positive root parameters k_α, can now be stated as follows:

Theorem 8.6.19 ([15],[73]) *The transform $\mathcal{H}$ maps $\mathcal{C}(T_v)^W$ onto the space of W-invariant elements of the classical Schwartz space $\mathcal{S}(i\mathfrak{t}_v^*)$. This yields an isomorphism $\mathcal{H} : \mathcal{C}(T_v)^W \to \mathcal{S}(i\mathfrak{t}_v^*)^W$ of topological vector spaces, whose inverse is $\mathcal{J}$ (considered on the classical Fréchet space $\mathcal{S}(i\mathfrak{t}_v^*)^W$).*

8.6.6 Further comments

Schapira's estimates have been improved by Theorem 3.3 in [72]. The sharp asymptotics in Theorem 8.6.18(iii) have an analog at every λ in the closed positive Weyl chamber. They are stated in [77], Remark 3.1. Their proof requires some work: see Theorem 3.4 in [57].

In this section we only gave an account of the L^2-harmonic analysis. For the L^p-harmonic analysis with $p \geq 1$ much less is known: The characterization of the $F(\lambda, k)$ which are bounded is given in [57, Theorem 4.2], but the product formulas and the L^1-convolution structure are longstanding open problems; see [19] and some partial progress in [71] and [77].

8.7 Special cases

8.7.1 Jack polynomials

Let $\mathcal{P}$ denote the set of integer partitions and $\mathcal{P}_n \subset \mathcal{P}$ the subset of partitions in at most n parts. The Jack polynomials $J_\lambda(x; \alpha)$ in $x = (x_1, \ldots, x_n)$ (with $\lambda \in \mathcal{P}_n$ and $\alpha \in \mathbb{C}$) can be naturally considered as the GL_n-type extension of the Jacobi polynomials for the root system of type

A_{n-1}. They form a $\mathbb{C}$-basis of the ring of symmetric polynomials in $x_1, \ldots, x_n$. If $\alpha = 1$ they reduce to the well known Schur polynomials, up to normalization. The Jack polynomials [38] form a very important class of symmetric polynomials, with remarkably deep applications, interpretations and special properties. There are several elegant and meaningful definitions accordingly (see e.g. [53]). We will presently define the Jack polynomial via its relation with the type A_{n-1} Jacobi polynomial, and comment on the more conventional definitions afterwards.

Let λ be an integer partition and $\alpha \in \mathbb{C}$. We define the lower hook length product $h_*(\lambda, \alpha)$ by

$$h_*(\lambda, \alpha) = \prod_{(i,j)\in\lambda} (\lambda'_j - i + 1 + \alpha(\lambda_i - j))$$

where (i, j) runs over the set of coordinates of the boxes of λ when represented as a Young diagram in the usual way, and λ' denotes the conjugate partition (so that $\lambda'_j - i + 1 + \lambda_i - j$ equals the length of the "hook" inside λ with upper leftmost corner the box with coordinates (i, j)).

Recall that an integer partition $\lambda \in \mathcal{P}_n$ determines canonically a dominant weight $\pi(\lambda)$ of the root system of type A_{n-1}.

Definition 8.7.1 *Let λ be an integer partition with at most n parts. The n-variable Jack polynomial $J_\lambda(x; \alpha)$ is the unique symmetric polynomial of homogeneous degree $|\lambda|$ in $x_1, \ldots, x_n$ characterized by the property that its restriction to the complex algebraic torus T_A defined by $\{(x_1, \ldots, x_n) \mid x_1 \ldots x_n = 1\}$ equals $h_*(\lambda, \alpha)P_A(\pi(\lambda), \alpha^{-1}; (x_1, \ldots, x_n))$ where $P_A(\mu, k; x)$ denotes the Jacobi polynomial of type A_{n-1} with highest weight μ.*

We now recall some of the striking properties of the $J_\lambda(x; \alpha)$ which can not be easily understood directly in terms of the Jacobi polynomials.

First of all, they are stable with respect to the number of variables. That is, we have, if $m \geq n \geq l(\lambda)$ (where $l(\lambda) = \lambda'_1$ denotes the number of parts of λ), then

$$J_\lambda(x_1, \ldots, x_m; \alpha)|_{x_{n+1}=\cdots=x_m=0} = J_\lambda(x_1, \ldots, x_n; \alpha) \tag{8.7.1}$$

For this reason it is possible to view the $J_\lambda(x; \alpha)$ as restrictions to a finite set of variables of symmetric functions $J_\lambda(\alpha)$ in an infinite set of variables.

Definition 8.7.2 *The $J_\lambda(\alpha)$ are called the Jack functions.*

The $\{J_\lambda(\alpha) \mid \lambda \in \mathcal{P}\}$ form a basis of the ring symmetric functions. The expansion of $J_\lambda(\alpha)$ in terms of the basis of monomial symmetric functions m_μ only involves partitions μ which are smaller than or equal to λ in the dominance ordering of integer partitions. In particular it makes sense to speak about the coefficient of the monomial symmetric function $m_{(1^l)}$ where $l = |\lambda|$. This reveals a much more natural definition of the normalization of the family $J_\lambda(\alpha)$: the coefficient of $m_{(1^l)}$ equals $l!$. In fact this normalization is part of the original definition of the Jack polynomials by Jack [38], [53], and the equivalence with our normalization in a fixed number of variables can be derived from a nontrivial result due to Stanley [74] (see [3]).

It follows in a straightforward way from our definition that when α is a positive real number then the n-variable polynomials $J_\lambda(x;\alpha)$ are orthogonal with respect to the measure

$$\prod_{i<j} |x_i - x_j|^{2/\alpha} x_1^{-1} \dots x_n^{-1} dx_1 \wedge \cdots \wedge dx_n$$

on the torus $T_B = \{(x_1, \dots, x_n) \mid \forall i = 1, \dots, n : |x_i| = 1\}$. There exists a quite different inner product with respect to which the Jack *functions* $J_\lambda(\alpha)$ are orthogonal. Define

$$z_\lambda := \prod_{i\geq 1} (i^{m_i} m_i!) \tag{8.7.2}$$

where $m_i = m_i(\lambda)$ denotes the number of parts of λ that are equal to i. Let $(\cdot,\cdot)_\alpha$ denote the scalar product on the ring of symmetric functions such that

$$(p_\lambda, p_\mu)_\alpha = \delta_{\lambda,\mu} z_\lambda \alpha^{l(\lambda)} \tag{8.7.3}$$

Theorem 8.7.3 *The Jack functions $J_\lambda(\alpha)$ are orthogonal with respect to the inner product $(\cdot,\cdot)_\alpha$.*

Another remarkable property of the $J_\lambda(\alpha)$ is the positivity and integrality of its coefficients when expressed with respect to the basis of normalized monomial symmetric functions $\tilde{m}_\lambda = n_\lambda m_\lambda$, where $n_\lambda := \prod_{i\geq 1} m_i!$.

Theorem 8.7.4 ([42]) *The coefficients of $J_\lambda(\alpha)$ with respect to the basis $\{\tilde{m}_\mu \mid \mu \in \mathcal{P}\}$ of the ring of symmetric functions are polynomials in α with nonnegative integral coefficients.*

The Jack polynomials are also discussed in Chapter 7 by Dunkl.

8.7.2 The hypergeometric function of matrix argument

Hypergeometric functions of matrix argument arose as certain zonal spherical functions on the cone of positive definite real symmetric $n \times n$ matrices in the work of Bochner [4] and Herz [35]. This theory was generalized and further developed by Constantine [10], James [39], Muirhead [56], and Takemura [76]. The hypergeometric functions of matrix argument find applications in random matrix models, number theory, and quantum theory. The most general special functions of this type were introduced independently by Macdonald [52] and Korányi [47]. Macdonald [52] defined his functions as formal series in terms of the Jack functions $J_\lambda(\alpha)$, hence in infinitely many variables. When we restrict to n variables by setting $x_i = 0$ for all $i > n$ then one obtains a symmetric formal power series in $x_1, \dots, x_n$. These formal power series are convergent if $|x_i| < 1$ for all $i = 1, \dots, n$. By interpreting the x_i as the eigenvalues of an $n \times n$ matrix then we can think of these functions as functions of "matrix argument".

First define the dual Jack polynomials $J_\lambda^*(x;\alpha)$ by

$$J_\lambda^*(\alpha) := J_\lambda(\alpha)/(J_\lambda(\alpha), J_\lambda(\alpha))_\alpha$$

We note in passing that it was shown by Stanley [74] that

$$(J_\lambda(\alpha), J_\lambda(\alpha))_\alpha = h^*(\lambda,\alpha) h_*(\lambda,\alpha)$$

where

$$h^*(\lambda,\alpha) = \prod_{(i,j)\in\lambda} (\lambda'_j - i + \alpha(\lambda_i - j + 1))$$

is the product over the boxes of the diagram of λ of the so-called "upper hook lengths". Hence when we restrict to n-variables $x_1,\ldots,x_n$ we have

$$J^*_\lambda(x;\alpha) = P_A(\lambda,\alpha^{-1};x)/h^*(\lambda,\alpha)$$

Finally we define the "C-normalization" of the Jack polynomials by

$$C_\lambda(\alpha) := \alpha^{|\lambda|}|\lambda|!J^*_\lambda(\alpha) \tag{8.7.4}$$

Recall the Pochhammer symbol $(a)_s$: If $a\in\mathbb{C}$ and $s\in\mathbb{Z}_{\geq 0}$ it is defined by

$$(a)_s = a(a+1)\ldots(a+s-1)$$

By convention $(a)_0 = 1$ for all $a\in\mathbb{C}$. Given $\lambda\in\mathcal{P}$ and a parameter $k\in\mathbb{C}$ we define a generalized Pochhammer symbol by

$$(a)^{(\alpha)}_\lambda := \prod_{i\geq 1}(a-\alpha^{-1}(i-1))_{\lambda_i}$$

We are now ready to formulate Macdonald's definition [52] of the generalized hypergeometric function ${}_pF_q$ of matrix argument.

Definition 8.7.5 *Let $a_1,\ldots,a_p$; $b_1,\ldots,b_q$ and α be complex parameters. The hypergeometric function ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\alpha)$ is the formal series given by:*

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\alpha) := \sum_{\lambda\in\mathcal{P}} \frac{(a_1)^{(\alpha)}_\lambda\ldots(a_p)^{(\alpha)}_\lambda}{(b_1)^{(\alpha)}_\lambda\ldots(b_q)^{(\alpha)}_\lambda|\lambda|!}C_\lambda(\alpha)$$

We denote the restriction of ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\alpha)$ to n variables by setting $x_i = 0$ for all $i > n$ by ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x;\alpha)$ with $x = (x_1,\ldots,x_n)$. This symmetric power series in $x_1,\ldots,x_n$ is convergent on the polydisk defined by $|x_i| < 1$ for all i.

When $\alpha = \frac{1}{2}, 1$ or 2 then the Jack polynomials in this power series can be interpreted as zonal polynomials on the cone of quaternionic, complex or real positive definite matrices respectively, and in this way one can establish the link for these special parameter values between the functions ${}_pF_q$ defined by Macdonald and Korányi and the original functions of matrix argument studied by Constantine, James and Muirhead.

Let us now restrict our attention to the special case $p = 2, q = 1$ of the generalized hypergeometric functions ${}_pF_q$ of matrix argument.

The power series ${}_2F_1$ is characterized uniquely by a system of n linear partial differential equations of order 2. Explicitly these equations are given as follows. Here and below we will always write $k = \alpha^{-1}$. Define

$$\Delta_i(a,b,c;k) = x_i(1-x_i)\partial_i^2 + (c - k(n-1) - (a+b+1-k(n-1))x_i)\,\partial_i + k\sum_{j=1;j\neq i}^{n} \frac{x_i(1-x_i)}{(x_i-x_j)}\partial_i - k\sum_{j=1;j\neq i}^{n} \frac{x_j(1-x_j)}{(x_i-x_j)}\partial_j$$

Theorem 8.7.6 ([47]) *The hypergeometric function ${}_2F_1(a,b;c;x;\alpha)$ is the unique symmetric function in the n variables $x_1,\ldots,x_n$ that satisfies*

$$\Delta_i(a,b,c;\alpha^{-1})F = abF,\ \forall i = 1,2,\ldots,n \tag{8.7.5}$$

which is analytic at $x_1 = \cdots = x_n = 0$ and normalized by $F(0) = 1$.

Observe that $\Delta_i(a,b,c;\alpha^{-1})$ depends only on $a+b$ and that (8.7.5) is symmetric in a and b. Note that if a symmetric function F is an eigenfunction of the operators $\Delta_i(a',b',c;\alpha^{-1})$ then the eigenvalues are independent of i and we can choose a and b such that $a+b = a'+b'$ and F satisfies (8.7.5).

Let us now turn to the relation with the hypergeometric function for root systems. We have seen in section 8.6 that the hypergeometric system (8.3.9) is a holonomic system of linear partial differential equations of rank $|W|$. For generic values of (λ,k) the system is irreducible. This can be seen for instance from the generic irreducibility of the monodromy representation of the hypergeometric system. For special values of the parameters $\lambda \in \mathfrak{t}^*$ and $k \in \mathcal{K}$ the hypergeometric system (8.3.9) may no longer be irreducible. This happens for example if the system of equations (8.3.9) factorizes via a holonomic system of smaller rank, i.e. if there exists a holonomic system of linear partial differential equations of smaller rank whose solutions are also solutions of (8.3.9). The holonomic systems which appear as factors of the hypergeometric system (8.3.9) are often interesting in their own right.

The hypergeometric function ${}_2F_1$ of matrix argument is a case in point. It was shown by Beerends and Opdam [3] that the system of differential equations of Theorem 8.7.6 is a factor (in the above sense) of the hypergeometric system (8.3.9) for the root system of type BC_n and a special choice of its parameters, a result that we will explain below. For a good account of this result and of the holonomic system defined by the system of differential equations of Theorem 8.7.6, we refer to [37].

It follows in particular that ${}_2F_1$ is an explicit power series expansion at 0 of the BC_n-type hypergeometric function $F(\lambda,k;t)$ for these parameters. We remark that in general such power series expansions are not known. We refer the reader to [3] and the references therein for a more extensive historical background on this type of hypergeometric series of matrix argument.

Let $(e_1,\ldots,e_n)$ be the standard basis in $\mathbb{R}^n$. We equip $\mathbb{R}^n$ with the Euclidean inner product $\langle\cdot,\cdot\rangle$ with respect to which the standard basis is an orthonormal basis. The set

$$R_B := \{\pm e_i, \pm 2e_i, \pm(e_k \pm e_l) \mid i = 1,\ldots,n;\ 1 \le k < l \le n\} \tag{8.7.6}$$

forms a root system of type BC_n. The set

$$S_B := \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n\} \tag{8.7.7}$$

is a set of simple roots. The torus is of the form $T = \{(t_1, \ldots, t_n) \mid t_i \in C^\times\}$, where t_i is identified with the character of T corresponding to the root e_i of R_B.

The Weyl group W_B acts naturally on the complex algebraic torus T. It is the hyperoctahedral group $W_B = W_A \ltimes N$, where W_A is the symmetric group of permutations of the coordinates t_i, and $N \approx C_2^n$ is the group of sequences of signs of length n, acting on T by raising t_i to the power of the $i - th$ sign in the sequence. The space $N\backslash T$ of N-orbits in T is isomorphic to the n-dimensional complex affine space. We equip this orbit space with coordinates $x_i := \frac{1}{2} - \frac{1}{4}(t_i + t_i^{-1})$, giving $N\backslash T$ the structure of an n-dimensional complex vector space $V = \mathbb{C}^n$, with linear action by the permutation group W_A. Obviously we have a canonical identification $W_A\backslash V = W_B\backslash T$, and we have

$$\mathbb{C}[V]^{W_A} = \mathbb{C}[x_1, \ldots, x_n]^{W_A} \approx \mathbb{C}[T]^{W_B} \tag{8.7.8}$$

Given (a, b, c) and α we define mutiplicity parameters $k_1 = k_{e_i}$, $k_2 = k_{2e_i}$ and $k_3 = k_{e_i \pm e_j}$ for the root R_B of type BC_n as follows:

$$\begin{aligned} k_1 &= 2c - a - b - 1 - \alpha^{-1}(n-1) \\ k_2 &= a + b + \frac{1}{2} - c \\ k_3 &= \alpha^{-1} \end{aligned} \tag{8.7.9}$$

We record that in terms of these parameters we have

$$a + b = k_1 + 2k_2 + (n-1)k_3 \tag{8.7.10}$$

and

$$\rho(k) = \frac{1}{2}(a+b)\omega_n + \alpha^{-1}\rho_A \tag{8.7.11}$$

where $\omega_n = e_1 + \cdots + e_n$ is the n-th fundamental weight with respect to the basis S_B of simple roots, and where $\rho_A = \frac{1}{2}\sum_{i=1}^n (n - 2i + 1)e_i$ is the half the sum of the positive type A-roots $e_i - e_j$ (with $i < j$) of R_B. Observe that ω_n and ρ_A are orthogonal vectors. We define a spectral parameter λ for R_B by

$$\lambda = -a\omega_n + \rho(k) \tag{8.7.12}$$

The main result of [3] states that:

Theorem 8.7.7 *Via the identification (8.7.8) we consider ${}_2F_1(a, b; c; x_1, \ldots, x_n; \alpha)$ as a W_B-invariant holomorphic function in an open neighborhood of $e \in T$. Then*

$${}_2F_1(a, b; c; x_1, \ldots, x_n; \alpha) = F_B(\lambda, k; t_1, \ldots, t_n) \tag{8.7.13}$$

where F_B denotes the hypergeometric function for the root system R_B, and the parameters (λ, k) are defined in terms of $(a, b, c; \alpha)$ by (8.7.9), (8.7.12).

In view of Definition 8.7.5, this result yields a series expansion of the special type of BC$_n$-Jacobi polynomials which appear on the right hand side of (8.7.13) in terms of Jack polynomials, with explicit hypergeometric coefficients. More generally, Macdonald [52] considered expansions for arbitrary type BC_n-Jacobi polynomials in terms of Jack polynomials. He derived combinatorial expressions for the coefficients as certain tableau sums. Results of this kind can also be derived from the binomial formulae due to Okounkov [58], [59] for Koornwinder and Macdonald polynomials in terms of so-called interpolation polynomials, cf. [46].

One may also express Korányi's second order operators $\Delta_i(a, b, c; \alpha^{-1})$ directly in terms of the Dunkl-Heckman operators $S_\xi(k)$ [29] for R_B. These operators are defined by:

$$\begin{aligned} S_\xi(k) :=& \partial_\xi + \frac{1}{2} \sum_{\alpha \in R_{B,+}} k_\alpha \alpha(\xi) \frac{1 + t^{-\alpha}}{1 - t^{-\alpha}} (1 - s_\alpha) \\ =& \frac{1}{2} (T_\xi(k) + w_0 \circ T_{-\xi}(k) \circ w_0) \end{aligned}$$

where $w_0 : T \to T$ is given by $w_0(t) = t^{-1}$, the action of the longest Weyl group element of W_B on T. These operators are W_B-equivariant (i.e. for any $w \in W_B$ we have $w \circ S_\xi(k) \circ w^{-1} = S_{w\xi}(k)$) but they do not commute. The W_B-equivariance of the operators $S_\xi(k)$ implies that $S^2_{e_i}(k)$ defines a differential-reflection operator on $\mathbb{C}[T]^N = \mathbb{C}[x_1, \dots, x_n]$ for every $i = 1, \dots, n$.

Proposition 8.7.8 *Let $D_i(k)$ denote the unique linear partial differential operator on $\mathbb{C}[x_1, \dots, x_n]$ which coincides with $S^2_{e_i}(k)$ on the subring $\mathbb{C}[x_1, \dots, x_n]^{W_A}$. For all $i = 1, \dots, n$ we have:*

$$D_i(k) = (\rho(k), e_i)^2 - \Delta_i(a, b, c; \alpha^{-1}) \tag{8.7.14}$$

Proof This is a straightforward but tedious direct computation. □

8.7.3 The missing Euler integral

The multivariable hypergeometric function associated with a root system generalizes the classical Euler–Gauss $F(\alpha, \beta, \gamma; z)$ in all its properties, except for one crucial missing insight: the Euler integral representation

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1} (1 - t)^{\gamma-\alpha-1} (1 - tz)^{-\beta} dt$$

for $|z| < 1$ or by analytic continuation for $z \in \mathbb{C}$ minus a cut with the interval $[1, \infty]$. For rational parameters α, β, γ the integrand is an algebraic function of t, which becomes single valued on a suitable finite cover of the complex plane ramified over the four points $t = 0, 1, \infty, 1/z$. As such it can be viewed as a period of a meromorphic differential on a one parameter (namely z) family of Riemann surfaces. This is the modern algebraic geometric view on the hypergeometric equation, and has been generalized to the concept of the Gauss-Manin connection (the attribution to Gauss is of course wrong, and should be to Euler, but wrong attributions in mathematics happen quite often).

For multivariable hypergeometric functions of Appell and Lauricella type integral representations are classical. For integral representations for the KZ-equation we refer to Chapter 11 by Tarasov and Varchenko.

Hence the search for an Euler type integral representation in the multivariable root system context is urgent, but unfortunately progress has been small. It can be shown that for a "special" spectral parameter $\lambda \in \mathfrak{h}^*$, depending linearly on the coupling parameter $k \in \mathcal{K}$, the hypergeometric system becomes highly reducible and has a subsystem with dimension of the solution space equal to $\mathrm{rk}(R) + 1$. The corresponding monodromy is the reflection representation of the affine Hecke algebra. Let us call this the "special" hypergeometric system associated with R [12].

The natural generalization of the Schwartz map defines a projective structure on the T°/W with T° the complement of the mirrors. For $k \in \mathcal{K}$ positive and sufficiently small it even defines a hyperbolic structure on T°/W with conic singularities along the mirrors. The problem for which of these $k \in \mathcal{K}$ the space

$$T^\circ/W$$

becomes a Heegner divisor complement in a ball quotient can be answered. In the analogous Bessel equation this has been completely answered in [11] and the list is quite substantial. This work generalizes the results of Deligne and Mostow on the Lauricella F_D hypergeometric function [14] to the root system context. In the toric root system setting an announcement of similar results was discussed in [12] but complete details have not been published yet. The toric setting in interesting because it provides a uniform framework for the period maps of the moduli space of del Pezzo surfaces of degree $d = 3, 2, 1$ to ball quotients of dimension $4, 6, 8$ respectively. These period maps were found by Allcock–Carlson–Toledo for $d = 3$ (cubic surfaces) [1], by Kondo for $d = 2$ (quartic curves) [43] and by Heckman and Looijenga for $d = 1$ (rational elliptic surfaces) [31].

But despite all this progress on the special hypergeometric system for particular values of $k \in \mathcal{K}$ (satisfying the Schwarz conditions) we do not even have an integral representation for the "special" hypergeometric function for arbitrary $k \in \mathcal{K}$.

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