COXETER GROUPS
(Unfinished and comments are welcome)

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Contents

Preface

Finite reflection groups are a central subject in mathematics with a long and rich history. The group of symmetries of a regular m-gon in the plane, that is the convex hull in the complex plane of the m^{th} roots of unity, is the dihedral group $\mathcal{D}_m$ of order $2m$, which is the simplest example of a reflection group. Similarly the groups of symmetries of a tetrahedron, octahedron and icosahedron in Euclidean three space are isomorphic to $\mathcal{S}_4$, $\mathcal{S}_4 \times \mathcal{C}_2$ and $\mathcal{A}_5 \times \mathcal{C}_2$ respectively, and are again finite reflection groups. More generally, the symmetry groups of a regular polytope in Euclidean space of dimension n are examples of finite reflection groups. Such polytopes were classified by Ludwig Schläfli in 1852 [50]. In dimension $n \geq 5$ there are just three of them, the simplex of dimension n and the dual pair of hyperoctahedron and hypercube. Their symmetry groups are the symmetric group $\mathcal{S}_{n+1}$ and the hyperoctahedral group $\mathcal{H}_n$ respectively. In dimension $n = 4$ there are three additional regular polytopes, and all their symmetry groups are finite reflection groups [23], [4].

Reflection groups were studied in a systematic way by Donald Coxeter (1907-2003) in the nineteen thirties. Not just the finite reflection groups in a Euclidean vector space, but also infinite reflection groups in a Euclidean affine space or a Lobachevsky space. If you like to see a glimpse of Coxeter then you can look up "Coxeter discusses the math behind Escher's circle limit" on YouTube. Their theory was further developped by Jacques Tits, leading to the concept of Coxeter groups [9] and giving a basis for Tits geometry. Hyperbolic reflection groups were studied extensively by Ernest Vinberg and his Russian colleagues [61]. The delightful biography by Shiobhan Roberts about the life and work of Donald Coxeter is highly recommended [49].

The true reason of their importance in mathematics is the fundamental role played by reflection groups in the theory of simple algebraic groups and simple Lie algebras, as established in the late 19^{th} and early 20^{th} century by Wilhelm Killing, Élie Cartan and Hermann Weyl [33], [13], [66]. Both in their classification and for the formulation of the the basic results reflection groups play a pivitol role. But there are many more subjects, for example the study of surface singularities, the theory of hypergeometric functions, the theory of integrable models or in various parts of algebraic geometry, where reflection groups again play a crucial role. Reflection groups are a true ubiquity in modern mathematics in a remarkable and almost unreasonable variety of different ways.

There are several good text books, in which the material of Chapters 2, 3, 4 and 5 is covered. The Bourbaki text Chapitres 4, 5 et 6 of Groupes et Algèbres de Lie from 1968 is a jewel (and by far the most successful book in the whole Bourbaki series, as can for example be seen from a citation count of N. Bourbaki in MathSciNet). Besides the text of this Bourbaki volume the given Tables are also quite useful. Another well written book on the subject is the text by James Humphreys on Reflection and Coxeter groups from 1990, but his exposition on hyperbolic reflection groups is somewhat limited.

1 Regular Polytopes

1.1 Convex Sets

Let V be a Euclidean vector space of dimension n with scalar product $(\cdot,\cdot)$. We shall denote scalars by Roman letters $x, y, \cdots$ and vectors in V by Greek letters $\xi, \eta, \cdots$.

Definition 1.1. *A subset $A \subset V$ is called affine if for all $\xi, \eta \in A$ and all $x \in \mathbb{R}$ we have $(1-x)\xi + x\eta \in A$.*

Obviously the intersection of a collection of affine sets in V is again affine. For $X \subset V$ a subset of V we denote by $\mathrm{aff}(X)$ the intersection of all affine subsets in V containing X. Clearly the affine hull $\mathrm{aff}(X)$ is the smallest affine subset of V containing X.

Lemma 1.2. *If $A \subset V$ is a nonempty affine set and $\xi_0, \xi_1, \cdots, \xi_m \in A$ and $x_0, x_1, \cdots, x_m \in \mathbb{R}$ with $\sum x_i = 1$ then $\sum x_i\xi_i \in A$ (with sum over $0 \leq i \leq m$).*

Proof. This follows easily by induction of the natural number m. The case $m = 1$ is clear from the definition of affine set. Now suppose $m \geq 2$ and say $x_m \neq 1$. Then we can write

$$\sum_0^m x_i\xi_i = (1-x_m)(\sum_0^{m-1} y_j\xi_j) + x_m\xi_m$$

with $y_j = x_j/(1-x_m)$ and $\sum y_j = (\sum x_j)/(1-x_m) = 1$ (with sum over $0 \leq j \leq m-1$). By the induction hypothesis $\sum y_j\xi_j \in A$ and hence also $\sum x_i\xi_i \in A$. $\square$

Corollary 1.3. *If A is a nonempty affine set in V and $\xi_0 \in A$ then A is just the translate $\xi_0 + L$ over ξ_0 of a linear subspace $L \subset V$.*

Proof. We take $A = \xi_0 + L$ as definition of the subset L of V. If $x, y \in \mathbb{R}$ and $\xi, \eta \in L$ then $\xi_0 + x\xi + y\eta = (1-x-y)\xi_0 + x(\xi_0+\xi) + y(\xi_0+\eta) \in A$. Hence L is a linear subspace of V. $\square$

Hence we can speak of the dimension of an affine subset A of V: $\dim A = \dim L$. If $\dim A = 0, 1, 2, m-1$ then A is called a point, a line, a plane or a hyperplane respectively.

Definition 1.4. *A subset $C \subset V$ is called convex if for all $\xi, \eta \in C$ and all $0 \leq x \leq 1$ we have $(1-x)\xi + x\eta \in C$.*

The intersection of a collection of convex sets in V is again convex. For $X \subset V$ the convex hull $\mathrm{conv}(X)$ is defined as the intersection of all convex sets in V containing X. Clearly $\mathrm{conv}(X)$ is the smallest convex set in V containing X.

Lemma 1.5. *If $C \subset V$ is a nonempty convex set and $\xi_0, \xi_1, \cdots, \xi_m \in C$ and $x_0, x_1, \cdots, x_m \geq 0$ with $\sum x_i = 1$ then $\sum x_i \xi_i \in C$ (with sum over $0 \leq i \leq m$).*

Proof. The proof is the same as that for Lemma 1.2. □

Lemma 1.6. *If $C \subset V$ is a nonempty convex set then the relative interior* $\mathrm{relint}(C)$ *of C in the affine hull* $\mathrm{aff}(C)$ *is nonempty.*

Proof. If $\mathrm{aff}(C)$ has dimension m then we can choose $(m+1)$ distinct points $\xi_0, \xi_1, \cdots, \xi_m \in C$ with $\mathrm{aff}(C) = \mathrm{aff}(\{\xi_0, \xi_1, \cdots, \xi_m\})$. Now the $m-$simplex $\mathrm{conv}(\{\xi_0, \xi_1, \cdots, \xi_m\})$ is contained in C and has nonempty relative interior in $\mathrm{aff}(C)$. □

Hence we can define the dimension of a nonempty convex set $C \subset V$ by $\dim C = \dim \mathrm{aff}(C)$. *Throughout the remaining part of this section let C be a nonempty closed convex set in V.* Suppose $\xi \in V$ is a point. Because C is closed there is a point $\zeta \in C$ at minimal distance from ξ: $|\xi - \zeta| \leq |\xi - \eta|$ for all $\eta \in C$. Because C is convex this point $\zeta \in C$ is unique and we write $\zeta = p(\xi) = p_C(\xi)$.

Definition 1.7. *The map $p = p_C : V \to C$ is called the metric projection of V on C.*

Definition 1.8. *For $\zeta \in C$ the set*

$$N(\zeta) = N_C(\zeta) = \{\nu \in V; (\nu, \zeta - \eta) \geq 0 \ \forall \eta \in C\}$$

is called the outer normal cone of C at ζ.

Lemma 1.9. *For $\zeta \in C$ the set $N(\zeta)$ is a closed convex cone.*

Proof. Clearly $N(\zeta)$ is closed as intersection of closed sets. For $\nu_1, \nu_2 \in N(\zeta)$ and $x_1, x_2 \geq 0$ we have

$$(x_1\nu_1 + x_2\nu_2, \zeta - \eta) = x_1(\nu_1, \zeta - \eta) + x_2(\nu_2, \zeta - \eta) \geq 0$$

for all $\eta \in C$, and therefore $x_1\nu_1 + x_2\nu_2 \in N(\zeta)$. □

Proposition 1.10. *For $\xi \in V$ and $\zeta \in C$ we have $p(\xi) = \zeta$ if and only if $(\xi - \zeta) \in N(\zeta)$.*

Proof. For $\xi \in V$ and $\zeta \in C$ we have $p(\xi) = \zeta$ if and only if

$$(\xi - \zeta, \xi - \zeta) - (\xi - (1-x)\zeta - x\eta, \xi - (1-x)\zeta - x\eta) \leq 0$$

for all $\eta \in C$ and all $0 \leq x \leq 1$. In turn this is equivalent to

$$-(2(\xi - \zeta) + x(\zeta - \eta), x(\zeta - \eta)) \leq 0$$

for all $\eta \in C$ and all $0 \leq x \leq 1$, or equivalently

$$(2(\xi - \zeta) + x(\zeta - \eta), \zeta - \eta) \geq 0$$

for all $\eta \in C$ and all $0 \leq x \leq 1$, or equivalently

$$(2(\xi - \zeta), \zeta - \eta) \geq 0$$

for all $\eta \in C$. This amounts to $(\xi - \zeta) \in N(\zeta)$. $\square$

Theorem 1.11. *The metric projection $p : V \to C$ is a contraction, meaning*

$$|p(\xi) - p(\eta)| \leq |\xi - \eta|$$

for all $\xi, \eta \in V$.

Proof. By the previous proposition we have

$$(\xi - p(\xi), p(\xi) - p(\eta)) \geq 0 \ , \ (\eta - p(\eta), p(\eta) - p(\xi)) \geq 0$$

for all $\xi, \eta \in V$. Hence we get

$$\begin{aligned}(\xi - \eta, p(\xi) - p(\eta)) &= (\xi - p(\xi) + p(\xi) - p(\eta) + p(\eta) - \eta, p(\xi) - p(\eta)) \\ &= (\xi - p(\xi), p(\xi) - p(\eta)) + |p(\xi) - p(\eta)|^2 + (\eta - p(\eta), p(\eta) - p(\xi)) \\ &\geq (p(\xi) - p(\eta), p(\xi) - p(\eta))\end{aligned}$$

for all $\xi, \eta \in V$, and therefore

$$(\xi - \eta, p(\xi) - p(\eta))^2 \geq (p(\xi) - p(\eta), p(\xi) - p(\eta))^2$$

for all $\xi, \eta \in V$. On the other hand we have

$$(\xi - \eta, \xi - \eta)(p(\xi) - p(\eta), p(\xi) - p(\eta)) \geq (\xi - \eta, p(\xi) - p(\eta))^2$$

by the Cauchy inequality. Combining these two inequalities gives

$$(\xi - \eta, \xi - \eta) \geq (p(\xi) - p(\eta), p(\xi) - p(\eta))$$

for all $\xi, \eta \in V$, which proves the theorem. $\square$

Corollary 1.12. *The metric projection $p : V \to C$ is continuous.*

Proposition 1.13. *A point $\zeta \in C$ is interior point of C if and only if $N(\zeta) = \{0\}$.*

Proof. It is obvious from Definition 1.8 that for $\zeta \in C$ an interior point of C we have $N(\zeta) = \{0\}$. Now suppose $\zeta \in C$ is a boundary point of C. Then there exists a sequence $\xi_j \in V - C$ with $|\xi_j - \zeta| < 1$ and $\lim \xi_j = \zeta$. Hence also $|p(\xi_j) - \zeta| < 1$ and $\lim p(\xi_j) = \zeta$. Let $\eta_j = p(\xi_j) + x_j(\xi_j - p(\xi_j))$ with $x_j > 1$ such that $|\eta_j - \zeta| = 1$. By Lemma 1.9 and Proposition 1.10 we have $p(\eta_j) = p(\xi_j)$. Choosing a convergent subsequence we can assume that $\lim \eta_j = \eta$ exists with $|\eta - \zeta| = 1$ and $p(\eta) = \lim p(\eta_j) = \lim p(\xi_j) = \zeta$. Hence we get $0 \neq \eta - \zeta \in N(\zeta)$. $\square$

Definition 1.14. *A subset $F \subset C$ is called a face of C if F is a closed convex subset of C and if for all $\xi, \eta \in C$ with $(1-x)\xi + x\eta \in F$ for some $0 < x < 1$ we have $\xi, \eta \in F$. A point $\zeta \in C$ is called an extremal point of C if $\{\zeta\}$ is a face of C.*

Clearly C itself is a face of C, called the trivial face of C. All other faces of C are called proper faces. It is clear that the intersection of faces of C is again a face of C. It is also obvious from the definition that if F is a face of C and G is a face of F then G is a face of C.

Proposition 1.15. *For $\zeta \in C$ the set*

$$F(\zeta) = \{\eta \in C; (\nu, \zeta - \eta) = 0 \ \forall \nu \in N(\zeta)\}$$

is a face of C, and such face is called an exposed face of C.

Proof. It is obvious that $F(\zeta)$ is nonempty, closed and convex. Now suppose $\xi, \eta \in C$ with $(1-x)\xi + x\eta \in F(\zeta)$ for some $0 < x < 1$. Then we get

$$0 = (\nu, \zeta - (1-x)\xi - x\eta) = (1-x)(\nu, \zeta - \xi) + x(\nu, \zeta - \eta)$$

for all $\nu \in N(\zeta)$. Since $(1-x), x > 0$ and $(\nu, \zeta - \xi), (\nu, \zeta - \eta) \geq 0$ we get $(\nu, \zeta - \xi) = (\nu, \zeta - \eta) = 0$ for all $\nu \in N(\zeta)$. Hence $\xi, \eta \in F(\zeta)$ and so $F(\zeta)$ is a face of C. $\square$

Exercise 1.16. *Show that not necessarily each face of C is an exposed face.*

The following fundamental theorem was obtained in 1911 by Hermann Minkowski [41].

Theorem 1.17. *A nonempty compact convex set C in V is the convex hull of its extremal points.*

Proof. By induction on the dimension n of C. We may assume that the interior of C is nonempty, otherwise replace V by the affine hull of C. Using Proposition 1.13 and Proposition 1.15 it follows that the boundary of C is a union of proper compact faces, all having codimension at least one. Clearly each point of C is a convex combination of two boundary points, one of which may be assumed to be an extremal point. The other boundary point lies in a face F of codimension at least one, and by induction is a convex combination of the extremal points of F (and in fact a convex combination of at most n extremal points of F). Hence the original point of C is a convex combination of at most $(n+1)$ extremal points of C. $\square$

Clearly each proper face of C is contained in an exposed face. Note that codimension one faces are always exposed faces.

Definition 1.18. *A convex polytope $P \subset V$ is a nonempty compact convex subset of V whose set of extremal points is a finite set.*

Let P be a convex polytope in V. All faces of P are convex polytopes as well. Note that the number of faces of P is finite, as it is bounded by 2 to the power the number of extremal points of P. The extremal points of P are also called the vertices of P, while the faces of dimension one are called the edges of P. The faces of P of codimension one are called the facets of P. A convex polytope of dimension two or three is called a convex polygon or a convex polyhedron respectively.

Definition 1.19. *If P is a convex polytope in V of dimension n then a sequence*

$$F_0 \subset F_1 \subset \cdots F_{n-1} \subset F_n = P$$

with F_i a face of P of dimension i is called a flag of faces of P.

Definition 1.20. *A convex polytope P in V is called regular if the group $G(P)$ of isometries of V leaving P invariant acts transitively on the set of flags of faces of P.*

Exercise 1.21. *An* isometry *(or* motion*) of $\mathbb{R}^n$ is a distance preserving transformation of $\mathbb{R}^n$. Show that the set of all motions of $\mathbb{R}^n$ is a group with respect to composition of maps, the so called* motion group $\mathrm{M}(\mathbb{R}^n)$, *isomorphic to the semidirect product $\mathbb{R}^n \rtimes \mathrm{O}(\mathbb{R}^n)$, acting by orthogonal linear transformations and translations on $\mathbb{R}^n$.*

If P is a regular convex polytope in V of dimension n then each face of P is contained in a flag of faces of P, and therefore the symmetry group $G(P)$ acts transitively on all faces of a fixed dimension. In particular, $G(P)$ acts transitively on the set of all vertices of P. After translation we may assume that the sum of all vertices of P is equal to the origin of V. Since isometries of V leaving the origin fixed are orthogonal linear transformations the group $G(P)$ becomes a subgroup of the orthogonal group $\mathrm{O}(V)$ of V. In turn this implies (still under the assumption that P has nonempty interior in V) that $G(P)$ acts simply transitively on the set of all flags of faces of P.

The regular convex polyhedra are called the Platonic solids. Up to scale and symmetry there are just five Platonic solids: the tetrahedron, the hexahedron (or cube), the octahedron, the dodecahedron and the icosahedron. They were described in the (last) Book XIII by Euclid.

Regular convex polytopes in arbitray dimension were introduced and classified by Ludwig Schläfli in 1852. Schläfli was one of the first mathematicians who conceived the possibility of geometry in more than three dimensions. The other famous example was Bernhard Riemann in 1854, who initiated in his Habilitationsvortrag the inner differential geometry of spaces of arbitrary dimension n as a generalization of the work by Gauss on surfaces (of dimension two) in $\mathbb{R}^3$.

This work by Schläfli was little appreciated at the time, and his long paper [50] was rejected. Between 1881 and 1900 his results were rediscovered independently by several mathematicians, as evidence that at last the time was ripe for his ideas. We refer to Coxeter's classic book Regular Polytopes, in particular the historical remarks at the end of Chapter VII are worth reading [23].

1.2 Examples of Regular Polytopes

There are four series of regular convex polytopes, which are very well known. We shall describe their standard forms, but any motion or dilation of the standard form, will also be a regular polytope of that type. The first series are the regular polygons of dimension 2, indexed by the number $m \geq 3$ of vertices.

Example 1.22. *For $m \geq 3$ the* standard regular m-gon *is the convex polygon in the Cartesian plane $\mathbb{R}^2$ with vertices $(\cos 2\pi j/m, \sin 2\pi j/m)$ for $j = 1, \cdots, m$. Its symmetry group is the dihedral group $\mathcal{D}_m$ of order $2m$, generated by the rotation r around the origin over an angle $2\pi/m$ and the reflection t in the x-axis. The symmetry group has m rotations and m reflections, with mirrors the lines through a vertex and its opposite edge in case*

m is odd, and the $m/2$ lines through opposite vertices and the $m/2$ lines through midpoints of opposite edges in case m is even.

The other three series are indexed by the dimension n of the regular convex polytope.

Example 1.23. *The* standard $n-$simplex *is the convex polytope in $\mathbb{R}^{n+1}$ with vertices the standard basis vectors $\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_n$, and so is given as $\{\sum_0^n x_i\varepsilon_i ; \sum_i x_i = 1, x_i \geq 0 \ \forall \ i\}$. Its symmetry group is the symmetric group $\mathcal{S}_{n+1}$ acting as permutation group of the $n+1$ vertices. Clearly it acts transitively on flags of faces, making the simplex a regular convex polytope.*

Example 1.24. *The* standard hyperoctahedron *in $\mathbb{R}^n$ is the convex polytope in $\mathbb{R}^n$ with the $2n$ vertices $\pm\varepsilon_1, \pm\varepsilon_2, \cdots, \pm\varepsilon_n$. The* standard hypercube *in $\mathbb{R}^n$ is the convex polytope in $\mathbb{R}^n$ with the 2^n vertices $\pm\varepsilon_1 \pm \varepsilon_2 \pm \cdots \pm \varepsilon_n$. Both convex polytopes have the hyperoctahedral group $\mathcal{C}_2^n \rtimes \mathcal{S}_n$ as symmetry group, acting on $\mathbb{R}^n$ by permutations and sign changes of the coordinates. The facets of the hyperoctahedron in $\mathbb{R}^n$ are simplices of dimension $n-1$, and there are 2^n of them. The facets of the hypercube in $\mathbb{R}^n$ are hypercubes of dimension one less, and there are $2n$ of them. It is easy to check that both the hyperoctahedron and the hypercube are regular polytopes.*

Note that the vertices of the hyperocatahedron are the centers of the facets of the hypercube, and conversely the vertices of the hypercube are the centers of the facets of the with factor n rescaled hyperocathedron. We say that the hyperocathedron and the hypercube are dual regular polytopes. In particular they have the same symmetry groups. The simplex is selfdual.

For dimension $n = 3$ the simplex is usually called the tetrahedron, and we simply speak of the octahedron and cube leaving the word hyper for the case of dimension $n \geq 4$. The cube is also called the hexahedron.

Example 1.25. *The octahedron in $\mathbb{R}^3$ has 12 edges, and on each edge we choose a point dividing the edge in a fixed ratio $l : s$ (with $x = l/s > 1$), such that for each triangular facet of the octahedron the corresponding three points form an equilateral triangle (with edges of length $\sqrt{l^2 + s^2 - ls}$ by the cosine rule). This can be done for all facets in a compatible way, because at each vertex of the octahedron 4 edges come together. Choosing $x > 1$ appropriately, such that*

$$\sqrt{2} = \sqrt{x^2 + 1 - x} \Leftrightarrow x^2 = x + 1 \Leftrightarrow x = \tau = \frac{1 + \sqrt{5}}{2} = 1.611803 \cdots$$

is the golden ratio, the convex polyhedron with these 12 points as vertices is bounded by $8 + 2 \times 6 = 20$ *(one for each facet, and two for each vertex) equilateral triangles. Taking the rescaled octahedron with vertices*

$$(\pm(\tau + 1), 0, 0), (0, \pm(\tau + 1), 0), (0, 0, \pm(\tau + 1))$$

the 12 points

$$(\pm\tau, 0, \pm1), (\pm1, \pm\tau, 0), (0, \pm1, \pm\tau)$$

form the vertices of the standard icosahedron. The dual of the icosahedron is called the dodecahedron. The vertices of the standard dodecahedron are taken as

$$(\pm1, \pm1, \pm1), (\pm\tau, \pm\tau', 0), (0, \pm\tau, \pm\tau'), (\pm\tau', 0, \pm\tau)$$

with $\tau' = (1 - \sqrt{5})/2$*, altogether* $8 + 3 \times 4 = 20$ *in total. The dodecahedron has 12 pentagonal facets. The symmetry group of the icosahedron and dodecahedron has order* 120*, and turns out to be isomorphic to* $\mathcal{C}_2 \times \mathcal{A}_5$*. The icosahedron has* 30 *edges, and the* 15 *lines through the centers of opposite edges form five groups of orthogonal triples. The simple alternating group* $\mathcal{A}_5$ *of rotations of the icosahedron acts on these five groups by even permutations. It can be checked that the icosahedron and the dodecahedron are regular polyhedra.*

The words tetra, hexa, octa, dodeca and icosa mean 4, 6, 8, 12 and 20 in Greek, and refer to the number of facets of the corresponding polyhedron. These five regular polyhedra are called the Platonic solids. But already in ancient cultures before the Hellenistic period copies of dodecahedra have been found, as I noticed in the Gallo-Roman Museum of Tongeren in Belgium.

For dimension $n = 4$ the simplex, hypercube and hyperocatahedron are called the 5−cell, 8−cell and 16−cell respectively. The number m of the $m-$cell refers to the number of facets: 5 tetrahedra, 8 cubes and 16 tetrahedra respectively. Besides these we shall describe three more regular polytopes in dimension $n = 4$, the $m-$cells for $m = 24, 120, 600$.

Example 1.26. *The standard* $24-$*cell is the convex polytope in* $\mathbb{R}^4$ *with* 24 *vertices the* 8 *vertices* $(\pm2, 0, 0, 0), (0, \pm2, 0, 0), (0, 0, \pm2, 0), (0, 0, 0, \pm2)$ *of the hyperoctahedron together with the* 16 *vertices* $(\pm1, \pm1, \pm1 \pm 1)$ *of the hypercube. The vertex* $(2, 0, 0, 0)$ *is connected by edges to the* 8 *vertices* $(1, \pm1, \pm1, \pm1)$*, which are the* 8 *vertices of a cube. Hence the faces of the* $24-$*cell of dimension* j *containing* $(2, 0, 0, 0)$ *are in bijection (by taking the* *intersection with the hyperplane $x_1 = 1$) with the faces of dimension $j-1$ of this cube. In particular, the symmetry group $G(8-cell) = G(16-cell)$ acts transitively on flags of faces containing the vertex $(2,0,0,0)$. It remains to show that the symmetry group $G(24-cell)$ acts transitively on the 24 vertices, and in fact it suffices to find an isometry mapping $(2,0,0,0)$ to $(1,1,1,1)$. For this we can take the orhogonal reflection in the hyperplane $x_1 - x_2 - x_3 - x_4 = 0$. Hence the 24−cell is regular. The order of the group $G(24-cell)$ is equal to $24 \times 48 = 1152$. The 24−cell has 24 octahedra as facets. Just like the $n-$simplex the 24−cell is selfdual.*

Example 1.27. *The standard 600−cell is the convex polytope in $\mathbb{R}^4$ with vertices the 24 vertices of the 24−cell together with the 96 points obtained by even permutations of the coordinates of the 8 points $(\pm\tau, \pm1, \pm\tau', 0)$. The vertex $(2,0,0,0)$ with maximal first coordinate is connected by edges to the 12 vertices $(\tau, \pm1, \pm\tau', 0), (\tau, 0, \pm1, \pm\tau'), (\tau, \pm\tau', 0, \pm1)$ with first coordinate equal to τ. These 12 points are the vertices of an icosahedron. Intersection of the 600−cell with hyperplanes $x_1 = c$ yields the following pattern: $c = \pm2$ gives just one vertex, $c = \pm\tau$ or $\pm\tau'$ gives the 12 vertices of an icosahedron, $c = \pm1$ gives 20 vertices of a dodecahedron, and finally $c = 0$ gives the 30 centers of the edges of an icosahedron, making in total $2+48+40+30 = 120$ vertices as should. Hence we obtain an injection of $G(icosahedron)$ into $G(600-cell)$, and all vertices with a given value $x_1 = c$ are conjugated.*

With the same argument as in the previous example it suffices to show that $G(600-cell)$ acts transitively on the 120 vertices. Via symmetries of the group $\mathcal{A}_4$ of even permutations of the coordinates the vertices with $x_1 = c$ for any c are all conjugated: $c = 2 \mapsto 0 \mapsto \pm\tau \mapsto \pm1 \mapsto \pm\tau'$ and $0 \mapsto -2$, which can be achieved using the Klein Vierergruppe in $\mathcal{A}_4$. Hence the 600−cell is a regular polytope. The 600−cell has 600 tetrahedra as facets. The order of the group $G(600-cell)$ equals $120 \times 120 = 14400$. The dual of the 600−cell is the 120−cell, which has 120 dodecahedra as facets.

This ends the explicit construction of the regular polytopes. In the next section we will prove that the above enumeration is complete. In all dimensions there are three series: the simplex, the hyperoctahedron and the hypercube. In dimensions $n = 2, 3, 4$ there are a handful of exceptional regular polytopes. For $n = 2$ there are besides the equilateral triangle and square the regular $m-$gon for $m \geq 5$. For $n = 3$ there are besides the tetrahedron, octahedron and cube the icosahedron and dodecahedron. For $n = 4$ there are besides the classical 5−cell, 8−cell and 16−cell the 24−cell and the dual pair of the 600−cell and the 120−cell.

This illustrates a philosophical principle formulated by the René Thom about *rich structures* and *poor structures* in mathematics. The classification of rich structures has the property that with increasing dimension or size the number of possibilities goes down: there are series and some exceptions for small dimension or small size. However, the classification of poor structures explodes with increasing dimension or size.

For example, the topological type of convex polytopes in $\mathbb{R}^n$ is a poor structure and their classifcation explodes with increasing n. However, the regular polytopes form a rich structure, and their classification exhibits the pattern described by Thom. Finite groups are a poor structure, but finite simple groups form a rich structure. The classification of finite groups up to isomorphism by their order N is totally impossible for large N. However, the classification of the finite simple groups consists of various classical series, and a few exceptional series, and a handful of true exceptions: the so called 26 sporadic groups. Similar patterns hold if one replaces the word finite group by compact Lie group or complex algebraic group. Conjugation classes of finite subgroups of the orthogonal groups $\mathrm{O}(\mathbb{R}^n)$ are a poor structure, but the subclass of groups generated by reflections form a rich structure, as we shall see in the next chapters.

Exercise 1.28. *Show that* $\tau = 2\cos(\pi/5)$.

1.3 Classification of Regular Polytopes

Let P be regular convex polytope in a Euclidean vector space V with center at the origin. Without loss of generality we may assume that P and V have the same dimension n.

Definition 1.29. *Let ξ be a vertex of P. The* vertex figure *of P at ξ is the convex polytope, whose vertices are those vertices of P, which are connected to ξ by an edge of P. We shall denote it $V(P,\xi)$.*

Since P is a regular polytope the vertex figure $V(P,\xi)$ has dimension $n-1$, and can be obtained as the intersection of P with the hyperplane U, which is the affine hull of $V(P,\xi)$. We have a bijection between faces of P of dimension $j \geq 1$ containing ξ and faces of $V(P,\xi)$ of dimension $j-1$, by intersection with U. Hence $V(P,\xi)$ is again a regular convex polytope with symmetry group $G(V(P,\xi)) = \{g \in G(P); g\xi = \xi\}$, the stabilizer in $G(P)$ of ξ. Vertex figures at different vertices of P are all isometric, and we denote by $V(P)$ the isometry class of the vertex figures of P. Clearly $|G(P)|$ is equal to $|G(V(P))|$ times the number of vertices of P.

Definition 1.30. *The Schläfli symbol of a regular convex polytope P of dimension n is a sequence $\{m_1(P), m_2(P), \cdots, m_{n-1}(P)\}$ of $n-1$ natural numbers ≥ 3 defined inductively by:*

- *$m_1(P)$ is the number of vertices of a two dimensional face of P,*
- *$\{m_2(P), \cdots, m_{n-1}(P)\}$ is the Schläfli symbol of $V(P)$.*

The Schläfli symbol of the regular $m-$gon is equal to $\{m\}$. The Schläfli symbols of tetrahedron, octahedron and icosahedron are $\{3, m\}$ for $m = 3, 4, 5$ respectively, and those of the cube and dodecahedron are $\{m, 3\}$ for $m = 4, 5$ respectively.

Proposition 1.31. *The Schläfli symbol of the isometry class $F(P)$ of the facets of P is equal to $\{m_1(P), m_2(P), \cdots, m_{n-2}(P)\}$.*

Proof. This is clear by induction on the dimension n, since the vertex figure $V(F(P))$ of the facet $F(P)$ is equal to the facet $F(V(P))$ of the vertex figure $V(P)$. $\square$

The next result was obtained in 1852 by the Swiss mathematician Ludwig Schläfli.

Theorem 1.32. *The complete list of Schläfli symbols of the regular convex polytopes in a Euclidean space of dimension $n \geq 2$ is given by*

- *$n = 2$: $\{m\}$ with $m \geq 3$,*
- *$n = 3$: $\{3, 3\}, \{3, 4\}, \{4, 3\}, \{3, 5\}, \{5, 3\}$,*
- *$n = 4$: $\{3, 3, 3\}, \{3, 3, 4\}, \{4, 3, 3\}, \{3, 4, 3\}, \{3, 3, 5\}, \{5, 3, 3\}$,*
- *$n \geq 5$: $\{3, 3, \cdots, 3, 3\}, \{3, 3, \cdots, 3, 4\}, \{4, 3, \cdots, 3, 3\}$.*

Proof. Let P be a regular convex polytope in V and $\{m_1, m_2, \cdots, m_{n-1}\}$ its Schläfli symbol. Let

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = P$$

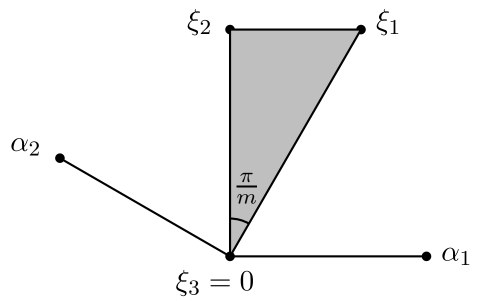
be a flag of faces of P, and let ξ_{i+1} be the center of F_i with the center ξ_{n+1} of P taken at the origin 0 of V. Note that the vectors ξ_i for $i = 1, \cdots, n$ form a basis of V. The convex hull of the $n+1$ points $\xi_1, \xi_2, \cdots, \xi_n, 0$ is called an orthoscheme of P. Any two orthoschemes are isometric, since $G(P)$ acts transitively on the flags of faces of P and hence also on the set of all orthoschemes.

Choose a rescaled dual basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ of length 2 vectors of the basis $\xi_1, \cdots, \xi_n$ of V, that is we require $(\xi_i, \alpha_j) = 0$ for all $i \neq j$, and $(\xi_i, \alpha_i) > 0$ and $(\alpha_i, \alpha_i) = 2$ for all i. We claim that

$$(\alpha_i, \alpha_{i+1}) = -2\cos(\pi/m_i) \ , \ (\alpha_i, \alpha_j) = 0$$

for all i, j with $|i - j| \geq 2$. The proof goes by induction on n.

If $n = 2$ and P is the regular $m-$gon then the orthoscheme is a rectangular triangle. The angle at ξ_1 is equal to $(m-2)\pi/(2m)$ and with a right angle at ξ_2 the angle at $\xi_3 = 0$ becomes π/m. Hence $(\alpha_1, \alpha_2) = -2\cos(\pi/m)$, which proves the statement for $n = 2$.



Let us now assume that $n \geq 3$. By construction the vectors $\alpha_1, \cdots, \alpha_{n-1}$ are perpendicular to ξ_n and in fact equal to the rescaled dual basis (in U) of the orthogonal projection of the vectors $\xi_1, \cdots, \xi_{n-1}$ on the linear hyperplane U perpendicular to ξ_n. The intersection of the affine hyperplane $\xi_n + U$ with P is a facet F of P with Schläfli symbol $\{m_1, m_2, \cdots, m_{n-2}\}$ by the previous proposition. By induction we get

$$(\alpha_i, \alpha_{i+1}) = -2\cos(\pi/m_i) \ , \ (\alpha_i, \alpha_j) = 0$$

for all subindices from $\{1, \cdots, n-1\}$ with $|i - j| \geq 2$. It remains to show that

$$(\alpha_i, \alpha_n) = 0 \ , \ (\alpha_{n-1}, \alpha_n) = -2\cos(\pi/m_{n-1})$$

for all $i = 1, \cdots, n-2$. The first equality follows because the vectors α_i for $i = 1, \cdots, n-2$ lie in the span of $\xi_1 - \xi_{n-1}, \cdots, \xi_{n-2} - \xi_{n-1}$ and so are clearly orthogonal to α_n. Finally the second equality follows by duality, since the Schläfli symbol of the dual regular polytope is obtained from the original Schläfli symbol by order reversion.

Let us denote by $D_n = D_n(m_1, \cdots, m_{n-1})$ the determinant of the Gram matrix

$$\begin{pmatrix} 2 & -2\cos(\pi/m_1) & 0 & \cdots & 0 \\ -2\cos(\pi/m_1) & 2 & -2\cos(\pi/m_2) & \cdots & 0 \\ 0 & -2\cos(\pi/m_2) & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

of the basis $\alpha_1, \cdots, \alpha_n$ of V. Since this matrix is tridiagonal its determinant can be evaluated by the recursive formula

$$D_{k+1} = 2D_k - 4\cos^2(\pi/m_k)D_{k-1}$$

for all $k = 1, \cdots, n-1$ with initial values $D_0 = 1, D_1 = 2$.

The determinant of the Gram matrix of an independent set in V is always > 0, and working out the conditions $D_k > 0$ for $k = 0, 1, \cdots, n$ inductively gives the list in a straight forward way. The details of this are indicated in the next exercise, and will also be carried out in greater generality in a later chapter. ☐

Exercise 1.33. *Check that $D_2(m) = 4\sin^2(\pi/m) > 0$ for all $m \geq 3$. Check that $D_3(p,q) > 0$ for $3 \leq p \leq q$ if and only if $p = 3, q = 3, 4, 5$. Check that $D_n(3, \cdots, 3) = n+1, D_n(3, \cdots, 3, 4) = 2$ and $D_n(4, 3, \cdots, 3, 4) = 0$. Check that $D_4(3, 4, 3) = 1$ and $D_5(3, 3, 4, 3) = 0$. Check that $D_n(3, \cdots, 3, 5) = (1-\tau)n + (\tau + 1) > 0$ if and only if $n \leq 4$. Check that $D_4(3, 5, 3) < 0$ and $D_5(5, 3, 3, 5) < D_5(5, 3, 3, 4) = 2 - 2\tau < 0$.*

Derive the list of Schläfli symbols of the regular convex polytopes in a Euclidean space of dimension $n \geq 2$ as given in the previous theorem.

Exercise 1.34. *Show that the Schläfli symbol of the dual regular polytope is obtained from the Schläfli symbol of the original polytope by order reversion.*

Exercise 1.35. *Match the Schläfli symbols listed in the Schläfli theorem to the regular polytopes described in the previous section.*

Remark 1.36. *An Archimedan solid P is a convex polyhedron (so of dimension three), such that its symmetry group $G(P)$ acts transitively on the vertices of P and moreover all facets of P are regular polygons of at least two different types (thereby excluding the Platonic solids). There are the truncated Platonic solids with Coxeter–Schläfli symbols*

$$t\{3,3\}, t\{3,4\}, t\{4,3\}, t\{3,5\}, t\{5,3\}$$

and also the $r\{4,3\}, r\{5,3\}$*, called the cuboctahedron and the icosidodecahedron. There are also the* $rr\{4,3\}, rr\{5,3\}$*, called the rhombicuboctahedron and the rhombicosidodecahedron, and the* $tr\{4,3\}, tr\{5,3\}$*, called the truncated cuboctahedron and the truncated icosidodecahedron. These* 11 *Archimedean solids have the same reflectional and rotational symmetry as the corresponding Platonic solid. Finally, there are still two more Archimedean solids* $sr\{4,3\}, sr\{5,3\}$*, called the snub cube and the snub dodecahedron, with symmetry group the index two subgroup of rotations in the symmetry group of the corresponding Platonic solid. Together these form the* 13 *Archimedean solids, which were discussed by Archimedes of Syracuse in the third century BC. For those, who enjoy these semiregular solids, the details can be found on the internet, but their structure is maybe easiest grasped after having understood the next chapter on finite reflection groups.*

2 Finite Reflection Groups

2.1 Normalized Root Systems

Suppose V is a finite dimensional Euclidean vector space with scalar product $(\cdot,\cdot)$. Elements in V are denoted by small Greek letters $\alpha, \beta, \cdots, \lambda, \mu, \cdots$. For $\alpha \in V, \alpha \neq 0$ the orthogonal reflection s_α with mirror the hyperplane perpendicular to α is given by

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$$

for $\lambda \in V$. Indeed $s_\alpha(\alpha) = -\alpha$ while $s_\alpha(\lambda) = \lambda$ for $(\lambda,\alpha) = 0$. If we denote $\mathrm{O}(V) = \{g \in \mathrm{GL}(V); (g\lambda, g\mu) = (\lambda,\mu)\ \forall \lambda, \mu \in V\}$ for the orthogonal group of V then $s_\alpha \in \mathrm{O}(V)$ and satisfies $s_\alpha^2 = 1$. It is easy to check from the above formula that

$$s_{g\alpha} = g s_\alpha g^{-1}$$

for all $g \in \mathrm{O}(V)$.

Definition 2.1. *A normalized root system R in V is a finite subset of V normalized by $(\alpha,\alpha) = 2$ for all $\alpha \in R$ such that $s_\alpha(\beta) \in R$ for all $\alpha, \beta \in R$.*

Let R be a normalized root system in V. The elements of R are called the roots, and the normalization is chosen such that the above formula simplifies to $s_\alpha(\lambda) = \lambda - (\lambda,\alpha)\alpha$. Since $s_\alpha(\alpha) = -\alpha$ we get $R = -R$.

Definition 2.2. *The subgroup $W = W(R)$ of $\mathrm{O}(V)$ generated by the reflections s_α for $\alpha \in R$ is called the finite reflection group or the Weyl group associated with the normalized root system R.*

Clearly W is finite as permutation group of the finite set R because W acts trivially on the orthogonal complement of R.

Example 2.3. *Let V be $\mathbb{R}^n$ with standard basis $\varepsilon_1, \cdots, \varepsilon_n$ and standard inner product $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. The set*

$$R = \{\pm(\varepsilon_i - \varepsilon_j); i < j\}$$

is a normalized root system in V, and the Weyl group associated to R is the symmetric group $\mathcal{S}_n$ in its standard representation. Indeed, the reflection in the root $\varepsilon_i - \varepsilon_j$ interchanges the two basis vectors $\varepsilon_i, \varepsilon_j$ and leaves the remaining ones fixed.

Example 2.4. *Let V be as in the previous example. The set*

$$R = \{\pm\sqrt{2}\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j; i < j\}$$

is a normalized root system in V with Weyl group the hyperoctahedral group $\mathcal{H}_n = (\mathcal{C}_2)^n \rtimes \mathcal{S}_n$ of sign changes and permutations of the standard basis.

Example 2.5. *Let V be as in the previous example. The set*

$$R = \{\pm\varepsilon_i \pm \varepsilon_j; i < j\}$$

is a normalized root system in V. The reflection in the root $\varepsilon_i - \varepsilon_j$ is the permutation matrix interchanging ε_i and ε_j and leaving the remaining basis vectors ε_k with $k \neq i, j$ fixed. The Weyl group element $s_{\varepsilon_i+\varepsilon_j} s_{\varepsilon_i-\varepsilon_j}$ is minus the identity on the plane $\mathbb{R}\varepsilon_i + \mathbb{R}\varepsilon_j$ and leaves the remaining basis vectors ε_k with $k \neq i, j$ fixed. The conclusion is that the Weyl group is the semidirect of the group of an even number of sign changes with the permutation group of the standard basis. It is an index two subgroup of the hyperoctahedral group.

Example 2.6. *If we identify the complex plane $\mathbb{C}$ with the Euclidean plane $\mathbb{R}^2$ via $z = x + iy \mapsto (x, y)$ then the set*

$$R = \{\sqrt{2}\exp(\pi ij/m); j = 0, 1, \cdots, 2m - 1\}$$

of renormalized $2m^{th}$ roots of unity has Weyl group equal to the dihedral group $\mathcal{D}_m = \mathcal{C}_m \rtimes \mathcal{S}_2$ of order $2m$, containing m reflections and m rotations of order a divisor of m. Indeed, the composition of two reflections is a rotation over twice the angle between their mirrors.

Definition 2.7. *A vector $\lambda \in V$ is called regular if $(\lambda, \alpha) \neq 0$ for all $\alpha \in R$. The set of all regular vectors in V is just the complement of all mirrors and is denoted V°. A connected component of V° is an open convex polyhedral cone, called a Weyl chamber.*

Fix, once and for all, a Weyl chamber and denote it by V_+. We shall call V_+ the positive Weyl chamber. This gives a corresponding partition

$$R = R_+ \sqcup R_-$$

of R into positive and negative roots. By definition positive roots have positive inner products with all vectors in V_+ while negative roots are minus positive roots. We shall write $\alpha > 0$ if $\alpha \in R_+$. It is clear that

$$R_+ = \{\alpha \in R; (\lambda, \alpha) > 0 \; \forall \lambda \in V_+\}$$
$$V_+ = \{\lambda \in V; (\lambda, \alpha) > 0 \; \forall \alpha \in R_+\}$$

and so V_+ and R_+ mutually determine each other.

Proposition 2.8. *The relation $\leq$ on V defined by*

$$\lambda \leq \mu \Leftrightarrow \mu - \lambda = \sum_{\alpha > 0} x_\alpha \alpha \ , \ x_\alpha \geq 0$$

is a partial ordering.

Proof. For a partial ordering we have to verify that

$$\lambda \leq \mu, \mu \leq \nu \Rightarrow \lambda \leq \nu \quad , \quad \lambda \leq \mu, \mu \leq \lambda \Rightarrow \lambda = \mu \ .$$

The first condition is trivially verified. Now suppose that $\lambda \leq \mu, \mu \leq \lambda$. Then $(\mu - \lambda, \nu) \geq 0$ and $(\lambda - \mu, \nu) \geq 0$ for all $\nu \in V_+$. Hence $(\lambda - \mu, \nu) = 0$ for all $\nu \in V_+$ and since V_+ is a non empty open subset of V we deduce $\lambda = \mu$. □

Definition 2.9. *A positive root $\alpha \in R_+$ is called simple in R_+ if α is not of the form $\alpha = x_1\alpha_1 + x_2\alpha_2$ with $x_1, x_2 \geq 1$ and $\alpha_1, \alpha_2 \in R_+$.*

Proposition 2.10. *Let $\{\alpha_1, \cdots, \alpha_n\}$ be the set of simple roots in R_+. Then every $\alpha \in R_+$ is of the form $\alpha = \sum x_j\alpha_j$ with $x_j \geq 0$ and $x_j \geq 1$ if $x_j \neq 0$.*

Proof. We prove the statement by induction on the partial ordering on V. If $\alpha \in R_+$ is not simple then $\alpha = y_1\beta_1 + y_2\beta_2$ with $y_1, y_2 \geq 1$ and $\beta_1, \beta_2 \in R_+$. Clearly $\beta_1, \beta_2 < \alpha$. Hence the minimal elements in R_+ are simple, and for these the statement is trivial. By induction $\beta_1 = \sum x'_j\alpha_j$ and $\beta_2 = \sum x''_j\alpha_j$ with $x'_j, x''_j \geq 0$ and ≥ 1 if $\neq 0$. Hence $\alpha = \sum x_j\alpha_j$ with $x_j = y_1x'_j + y_2x''_j \geq 0$ and ≥ 1 if $\neq 0$. □

Corollary 2.11. *If $\{\alpha_1, \cdots, \alpha_n\}$ is the set of simple roots in R_+ then*

$$V_+ = \{\lambda \in V; (\lambda, \alpha_j) > 0 \,\forall j\}$$
$$\mathrm{Clos}(V_+) = \{\lambda \in V; (\lambda, \alpha_j) \geq 0 \,\forall j\}$$

with $\mathrm{Clos}(V_+)$ *the topological closure of V_+.*

Theorem 2.12. *Let R be a normalized root system in V, W the Weyl group of R, V_+ a positive Weyl chamber, R_+ the corresponding set of positive roots, $\{\alpha_1, \cdots, \alpha_n\}$ the set of simple roots in R_+, and $s_1, \cdots, s_n \in W$ the corresponding simple reflections. Then we have*

1. *For every vector $\lambda \in V$ there exists $w \in W$ with $w\lambda \in \mathrm{Clos}(V_+)$.*
2. *For every Weyl chamber C there exists $w \in W$ with $wC = V_+$.*

3. For every root $\alpha \in R$ there exists $w \in W$ with $w\alpha$ simple in R_+.

4. The Weyl group W is generated by the simple reflections $s_1, \cdots, s_n$.

Proof. Let W' be the subgroup of W generated by $s_1, \cdots, s_n$. We shall prove the first three items for W' in stead of W and conclude in the last item that $W' = W$.

1. Fix $\mu \in V_+$. Let $w \in W'$ such that $(w\lambda, \mu) \geq (v\lambda, \mu)$ for all $v \in W'$. In particular $(w\lambda, \mu) \geq (s_j w\lambda, \mu)$ for all j, and so $(w\lambda, \alpha_j)(\alpha_j, \mu) \geq 0$ for all j. Hence $(w\lambda, \alpha_j) \geq 0$ for all j, and therefore $w\lambda \in \mathrm{Clos}(V_+)$ by Corollary 2.11.

2. Choose $\lambda \in C$ and let $w \in W'$ with $w\lambda \in \mathrm{Clos}(V_+)$. Since λ is regular also $w\lambda$ is regular, and so $w\lambda \in V_+$. Hence $wC \cap V_+$ is not empty and by definition $wC = V_+$.

3. Let $\alpha \in R$. Choose $\lambda \in V$ with $(\lambda, \alpha) = 0$ but $(\lambda, \beta) \neq 0$ for all $\beta \in R$ with $\beta \neq \pm\alpha$. Choose $w \in W'$ with $w\lambda \in \mathrm{Clos}(V_+)$. By Corollary 2.11 we get $(w\lambda, \alpha_j) = 0$ for some j. Hence $w\alpha = \pm\alpha_j$ and we get either $\alpha_j = w\alpha$ or $\alpha_j = s_j w\alpha$.

4. By definition W is the group generated by s_α for $\alpha \in R$. Given $\alpha \in R$ there exists $w \in W'$ with $w\alpha = \alpha_j$ simple in R_+. Hence $s_\alpha = w^{-1} s_j w \in W'$ and we conclude that $W = W'$.

$\square$

2.2 The Dihedral Normalized Root System

Identify the Cartesian plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by $(x, y) \mapsto z = x + iy$. Let $R \subset \mathbb{R}^2$ be a normalized root system with $|R| = 2m$ and $m \geq 2$, so that the rank of R is equal to 2. Let

$$R = \{\beta_1, \cdots, \beta_m, \beta_{m+1} = -\beta_1, \cdots, \beta_{2m} = -\beta_m\}$$

be the roots in R numbered according to positive orientation, and such that $(\beta_1, \beta_2) \geq (\beta_j, \beta_{j+1})$ for all j. If $(\beta_1, \beta_2) = 2\cos\theta$ then $0 < \theta \leq \pi/m$ and $w = s_{\beta_2} s_{\beta_1} \in W$ is a rotation over an angle 2θ. Since w leaves R invariant we get $(\beta_j, \beta_{j+1}) = 2\cos\theta$ for all j and

$$R = \{\beta_1 z; z^{2m} = 1\}$$

and by choosing a new orthonormal basis we may assume that $\beta_1 = \sqrt{2}$. This brings us back to Example 2.6.

Corollary 2.13. *The set* $R = \{\sqrt{2}\exp(\pi ij/m); j = 0, \cdots, 2m-1\}$ *is a normalized root system in* $\mathbb{R}^2 \cong \mathbb{C}$*, and up to a rotation every normalized root system in* $\mathbb{R}^2$ *is of this form.*

If we choose the positive Weyl chamber V_+ that contains the point $i+\epsilon$ with $\epsilon > 0$ small, then we get

$$R_+ = \{\sqrt{2}\exp(\pi ij/m); j = 0, \cdots, m-1\} .$$

We claim that $\alpha_1 = \sqrt{2}, \alpha_2 = \sqrt{2}\exp(\pi i(m-1)/m)$ are the simple roots in R_+. Indeed for $j = 1, \cdots, m-2$ we have

$$\sqrt{2}\exp(\pi ij/m) = x_1\sqrt{2} + x_2\sqrt{2}\exp(\pi i(m-1)/m)$$

with

$$x_2 = \frac{\sin(\pi j/m)}{\sin(\pi/m)} \geq 1$$

by taking the imaginary part, and

$$x_1 = \frac{\cos(\pi j/m)\sin(\pi/m) + \sin(\pi j/m)\cos(\pi/m)}{\sin(\pi/m)} = \frac{\sin(\pi(j+1)/m)}{\sin(\pi/m)} \geq 1$$

by taking the real part. Hence the roots $\sqrt{2}\exp(\pi ij/m)$ for $j = 1, \cdots, m-2$ can not be simple, and by Proposition 2.10 the remaning two positive roots $\{\alpha_1, \alpha_2\}$ are simple.

Corollary 2.14. *The set of simple roots in*

$$R_+ = \{\sqrt{2}\exp(\pi ij/m); j = 0, \cdots, m-1\}$$

is equal to $\{\alpha_1 = \sqrt{2}, \alpha_2 = \sqrt{2}\exp(\pi(m-1)/m)\}$*. Hence we get* $(\alpha_1, \alpha_2) = -2\cos(\pi/m)$ *for the dihedral root system with* $2m$ *roots.*

The fact that for any normalized root system R of rank two with $2m$ roots and for any subset R_+ of positive roots there are just two simple roots α_1, α_2 in R_+ with $(\alpha_1, \alpha_2) = -2\cos(\pi/m)$ will play an important role in the general theory of normalized root systems in the next section.

2.3 The Basis of Simple Roots

Let R be a normalized root system in V. Fix a positive Weyl chamber V_+ and let R_+ be the corresponding set of positive roots.

Proposition 2.15. *If α_i, α_j are distinct simple roots in R_+ then we have $(\alpha_i, \alpha_j) = -2\cos(\pi/m_{ij})$ for some $m_{ij} \in \mathbb{Z}, m_{ij} \geq 2$.*

Proof. Let R' be the intersection of R with the real span of α_i and α_j. Then R' is a rank two normalized root system in V. Let V'_+ be the Weyl chamber for R' that contains V_+, and let R'_+ be the corresponding set of positive roots in R'. Then $R'_+ = R' \cap R_+$. If $\alpha \in R'_+$ and α is simple in R_+ then α is also simple in R'_+. Hence the proposition is a direct consequence of Corollary 2.14. □

Theorem 2.16. *The simple roots in R_+ are linearly independent.*

Proof. Let $\{\alpha_1, \cdots, \alpha_n\}$ be the set of simple roots in R_+. By the previous proposition this set is obtuse: $(\alpha_i, \alpha_j) \leq 0$ for all $i \neq j$. Together with the fact that the simple roots all lie on one side of a hyperplane this will imply the statement.

Suppose $\sum x_j \alpha_j = 0$ is a linear relation between the simple roots. Separating indices for which $x_j \geq 0$ and $x_j \leq 0$ we get $\lambda = \sum y_j \alpha_j = \sum z_j \alpha_j$ with $y_j = \max\{x_j, 0\} \geq 0, z_j = \max\{-x_j, 0\} \geq 0$ and $y_j z_j = 0$ for all j. Hence

$$0 \leq (\lambda, \lambda) = \sum y_i z_j (\alpha_i, \alpha_j) \leq 0$$

and therefore $\lambda = 0$. For $\mu \in V_+$ we have

$$0 = (\lambda, \mu) = \sum y_j (\alpha_j, \mu) = \sum z_j (\alpha_j, \mu)$$

with $y_j, z_j \geq 0$ and $(\alpha_j, \mu) > 0$ for all j. Hence $y_j = z_j = 0$ and so $x_j = 0$ for all j. □

Corollary 2.17. *If R spans V then a Weyl chamber is the interior of a simplicial cone.*

Proof. By assumption R spans V and so the simple roots $\alpha_1, \cdots, \alpha_n$ form a basis of V by the previous theorem. Let $\lambda_1, \cdots, \lambda_n$ be the dual basis in V characterized by $(\alpha_i, \lambda_j) = \delta_{ij}$. Then we have

$$V_+ = \{\lambda \in V; (\lambda, \alpha) > 0 \, \forall \alpha > 0\} = \{\lambda \in V; (\lambda, \alpha_j) > 0 \, \forall j\}$$

which is the simplicial cone spanned by the basis $\lambda_1, \cdots, \lambda_n$ over the positive real numbers. □

Theorem 2.18. *Let V, V' be Euclidean vector spaces of dimension n and $R \subset V, R' \subset V'$ two normalized root systems of rank n. Let V_+, V'_+ be Weyl chambers, R_+ and R'_+ the corresponding set of positive roots, and $\alpha_1, \cdots, \alpha_n$ and $\alpha'_1, \cdots, \alpha'_n$ the set of simple roots in R_+ and R'_+ respectivly. If $(\alpha_i, \alpha_j) = (\alpha'_i, \alpha'_j)$ for all i, j then the orthogonal isomorphism $g : V \to V'$ defined by $g(\alpha_j) = \alpha'_j$ gives a bijection $g : R \to R'$.*

Proof. Let $s_1, \cdots, s_n \in W$ and $s'_1, \cdots, s'_n \in W'$ denote the corresponding simple reflections. Then it is clear that $s'_j = g s_j g^{-1}$. Since the Weyl group is generated by the simple reflections conjugation by g gives an isomorphism between $W = \langle s_1, \cdots, s_n \rangle$ and $W' = \langle s'_1, \cdots, s'_n \rangle$. Since roots are transforms under the Weyl group of simple roots we conclude that $g : R \to R'$ is a bijection. □

Definition 2.19. *If $\alpha_1, \cdots, \alpha_n$ is a basis of simple roots for $R_+ \subset R$ then the matrix $M = (m_{ij})$ defined by $(\alpha_i, \alpha_j) = -2\cos(\pi/m_{ij})$ is called the Coxeter matrix of R.*

Clearly $m_{ii} = 1$ for all i and we have seen that $m_{ij} \in \mathbb{Z}, m_{ij} \geq 2$ for all $i \neq j$. Apart from a numbering of the simple roots the Coxeter matrix determines and is determined by the normalized root system R up to orthogonal isomorphism.

Definition 2.20. *The Coxeter diagram associated with the Coxeter matrix $M = (m_{ij})$ is a marked graph with n nodes labelled $1, \cdots, n$. The i^{th} and j^{th} node are connected by a bond if and only if $m_{ij} \geq 3$, and in case the number $m_{ij} \geq 4$ the number m_{ij} is marked to the corresponding bond.*

Up to orthogonal isomorphism a normalized root system is determined by its Coxeter diagram.

2.4 The Classification of Elliptic Coxeter Diagrams

Definition 2.21. *A Coxeter matrix is a symmetric matrix*

$$M = (m_{ij})_{1 \leq i,j \leq n}$$

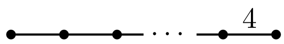
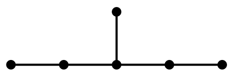
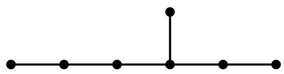
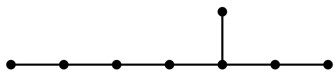
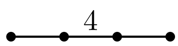
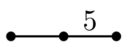
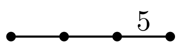
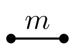
with $m_{ii} = 1$ and $m_{ij} \in \mathbb{Z}, m_{ij} \geq 2$ fo all $i \neq j$. The Coxeter diagram of M is defined in the same way as in Definition 2.20. The Gram matrix $G(M)$ of M is defined by

$$G(M) = (g_{ij} = -2\cos(\pi/m_{ij}))_{1 \leq i,j \leq n}$$

and the Coxeter matrix M or its Coxeter diagram is called elliptic if its Gram matrix $G(M)$ is positive definite.

A Coxeter diagram is elliptic if and only if all of its connected components are elliptic Coxeter diagrams, and so the classification of elliptic Coxeter diagrams reduces to the classification of connected elliptic Coxeter diagrams. In this section we will prove the following classification theorem.

Theorem 2.22. *The connected elliptic Coxeter diagrams are given in the following table. The first column is their name (Cartan symbol), the second column is the Coxeter diagram. The two remaining colums of the Coxeter number h and the sequence of exponents will appear in later sections.*

name	Coxeter diagram	h	exponents
A_n		$n+1$	$1, 2, \cdots, n$
B_n		$2n$	$1, 3, 5, \cdots, 2n-1$
D_n		$2n-2$	$1, 3, \cdots, 2n-3, n-1$
E_6		12	$1, 4, 5, 7, 8, 11$
E_7		18	$1, 5, 7, 9, 11, 13, 17$
E_8		30	$1, 7, 11, 13, 17, 19, 23, 29$
F_4		12	$1, 5, 7, 11$
H_3		10	$1, 5, 9$
H_4		30	$1, 11, 19, 29$
$\mathrm{I}_2(m)$		m	$1, m-1$

We have the restrictions $n \geq 1$ *for* A_n, $n \geq 2$ *for* B_n, $n \geq 4$ *for* D_n *and* $m \geq 5$ *for* $\mathrm{I}_2(m)$ *to eliminate coincidences* $\mathrm{B}_1 = \mathrm{A}_1$, $\mathrm{D}_3 = \mathrm{A}_3$, $\mathrm{I}_2(3) = \mathrm{A}_2$ *and* $\mathrm{I}_2(4) = \mathrm{B}_2$.

If we take for the positive Weyl chamber

$$\begin{aligned}
V_+ &= \{\sum x_i\varepsilon_i; x_1 > x_2 > \cdots > x_{n-1} > x_n\} \\
V_+ &= \{\sum x_i\varepsilon_i; x_1 > x_2 > \cdots > x_{n-1} > x_n > 0\} \\
V_+ &= \{\sum x_i\varepsilon_i; x_1 > x_2 > \cdots > x_{n-1} > |x_n|\}
\end{aligned}$$

in the root systems given in Example 2.3, Example 2.4 and Example 2.5 respectively then the bases of simple roots become

$$\begin{aligned}
\{\alpha_i\} &= \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{n-1} - \varepsilon_n\} \\
\{\alpha_i\} &= \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{n-1} - \varepsilon_n, \sqrt{2}\varepsilon_n\} \\
\{\alpha_i\} &= \{\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}
\end{aligned}$$

respectively. Hence the corresponding Coxeter diagrams are of type A_{n-1}, B_n and D_n respectively. It is clear from Corollary 2.14 that the dihedral root system with $2m$ roots is of type $\mathrm{I}_2(m)$. Apparently there are just six more exceptional connected elliptic Coxeter diagrams, three of type E, one of type F and two of type H.

If we have given an elliptic Coxeter matrix $M = (m_{ij})_{1\le i,j\le n}$ then we let V be the Euclidean vector space with basis $\alpha_1, \cdots, \alpha_n$ and with inner product $(\alpha_i, \alpha_j) = -2\cos(\pi/m_{ij})$. Likewise $V_+ = \{\lambda \in V; (\lambda, \alpha_i) > 0\ \forall i\}$ is called the positive Weyl chamber. Of course, if M is the Coxeter matrix of a normalized root system, then this terminolgy coincides with previous definitions. However, if we start with an arbitrary elliptic Coxeter matrix, then a priori we do not know whether this always comes from a normalized root system. This is true though, and can be checked in a case by case manner. In the next chapter we will give a uniform proof, independent of classification.

Remark 2.23. *The classification of elliptic Coxeter diagrams in rank three is easy. Taking the intersection of the positive Weyl chamber with the unit sphere gives a spherical triangle with dihedral angles $\pi/k, \pi/l, \pi/m$ for some integers $k, l, m \ge 2$. Since the angle sum of a spherical triangle is strictly greater than π we have the restriction*

$$1/k + 1/l + 1/m > 1$$

and assuming $2 \le k \le l \le m$ gives the solutions (k, l, m) equal to $(2, 2, m)$ for all $m \ge 2$ and $(2, 3, m)$ with $m = 3, 4, 5$. The first series corresponds to a disconnected elliptic Coxeter diagram of type $\mathrm{A}_1 + \mathrm{I}_2(m)$ and the second *series is of type* $\mathrm{A}_3, \mathrm{B}_3, \mathrm{H}_3$ *respectively, in accordance with the table. The Weyl groups of the second series are the symmetry groups of the Platonic solids tetrahedron, octahedron and icosahedron respectively.*

The proof of the theorem consists of a number of easy lemmas, which occupy the rest of this section.

Lemma 2.24. *Any subdiagram of an elliptic Coxeter diagram obtained by deleting some nodes and all bonds connected to the deleted nodes is again an elliptic Coxeter diagram.*

Proof. If $I \subset \{1, \cdots, n\}$ is the set of remaining nodes then the linear subspace spanned by $\{\alpha_i; i \in I\}$ is a Euclidean vector space, which proves the lemma. $\square$

Lemma 2.25. *An elliptic Coxeter diagram has no loops.*

Proof. Suppose there is an elliptic Coxeter diagram with a loop. Taking a loop with a minimal number of nodes (say n) its Coxeter diagram is elliptic by the previous lemma, and after renumeration

$$m_{12} = m_{21}, m_{23} = m_{32}, \cdots, m_{(n-1)n} = m_{n(n-1)}, m_{n1} = m_{1n} \geq 3$$

and all remaining m_{ij} with $i \neq j$ equal 2. Hence

$$(\sum \alpha_i, \sum \alpha_j) = 2n + 2\sum_{i<j}(\alpha_i, \alpha_j) \leq 2n - 2n \cdot 2\cos(\pi/3) = 0$$

gives a contradiction with V being a Euclidean vector space. $\square$

Lemma 2.26. *The Gram matrix of an elliptic Coxeter diagram satisfies* $\sum_{j \neq i} g_{ij}^2 < 4$ *for all* i.

Proof. Without loss of generality we may assume $i = 1$. If we delete all nodes and bonds of the Coxeter diagram not connected to the first node then the remaining diagram (say with n nodes) satisfies

$$m_{1j} \geq 3\ ,\ m_{ij} = 2$$

for all $i, j \geq 2$ with $i \neq j$. Therefore $\{\alpha_2, \cdots, \alpha_n\}$ is an orthogonal set. Hence

$$\sum_{j \geq 2} g_{1j}^2 = 2(\tfrac{1}{2}\sum_{j \geq 2}(\alpha_1, \alpha_j)\alpha_j, \tfrac{1}{2}\sum_{j \geq 2}(\alpha_1, \alpha_j)\alpha_j) < 2(\alpha_1, \alpha_1) = 4$$

because $\frac{1}{2}\sum_{j \geq 2}(\alpha_1, \alpha_j)\alpha_j$ is just the orthogonal projection of α_1 on the hyperplane spanned by $\alpha_2, \cdots, \alpha_n$. $\square$

Corollary 2.27. *Consider an elliptic Coxeter diagram. One node connects to at most three bonds, and if three bonds connect to one node then all three bonds are unmarked. If one node connects to two bonds then at least one of the bonds is unmarked. If a bond has mark at least* 6 *then no other bonds connect to the two connected nodes.*

Proof. Observe that the map $m_{ij} \mapsto g_{ij}^2 = 4\cos^2(\pi/m_{ij})$ is monotonic and takes the values $1, 2, 3$ for $m_{ij} = 3, 4, 6$ respectively. Hence the corollary is obvious from the previous lemma. $\square$

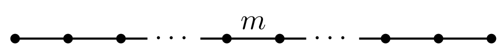
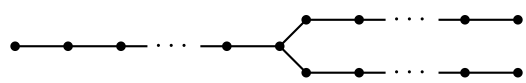
Lemma 2.28. *If an elliptic Coxeter diagram with n nodes has a Coxeter subdiagram of type A_k then the new Coxeter diagram obtained from the old Coxeter diagram by contracting this type A_k subdiagram to a single node is again an elliptic Coxeter diagram.*

Proof. Suppose $(\alpha_i, \alpha_{i+1}) = -1$ for $i = 1, \cdots, k-1$ and $(\alpha_i, \alpha_j) = 0$ for $1 \le i, j \le k$ with $|i - j| \ge 2$. Put $\alpha_0 = \sum_{1 \le i \le k} \alpha_i$. Then we have

$$(\alpha_0, \alpha_0) = 2k + 2 \sum_{1 \le i < j \le k} (\alpha_i, \alpha_j) = 2 \, .$$

Moreover for $j \ge k+1$ we have either $(\alpha_i, \alpha_j) = 0$ for all $i = 1, \cdots, k$ or $(\alpha_i, \alpha_j) \ne 0$ for precisely one i with $1 \le i \le k$. Otherwise there would be a loop in the Coxeter diagram, contradicting Lemma 2.25. Hence we get $(\alpha_i, \alpha_j) = -2\cos(\pi/m_{ij})$ for all $i, j \in \{0, k+1, \cdots, n\}$ with $m_{ii} = 1$ for all i and $m_{ij} = m_{ji} \in \mathbb{Z}$ and $m_{ij} \ge 2$ for all $i \ne j$. Since $\{\alpha_0, \alpha_{k+1}, \cdots, \alpha_n\}$ is linearly independent its Gram matrix is positive definite, and the new Coxeter diagram is elliptic. $\square$

Corollary 2.29. *A connected elliptic Coxeter diagram must be of the form*

name	Coxeter diagram
A_n	
$\mathrm{I}_{pq}(m)$	
T_{pqr}	

with n nodes for A_n, with $p+q$ nodes for $\mathrm{I}_{pq}(m)$ and $m \ge 4$, with $p+q+r-2$ nodes for T_{pqr}, and suitable n, p, q, r. In case $\mathrm{I}_{pq}(m)$ with $m \ge 6$ we have $p = q = 1$ and write $\mathrm{I}_2(m)$.

Proof. A connected elliptic Coxeter diagram can not have more than one bond with mark $m \geq 4$, nor more than one triple node, nor a bond with mark $m \geq 4$ together with a triple node. This is clear from Corollary 2.27 together with Lemma 2.28. $\square$

Indeed, all Coxeter diagrams in Theorem 2.22 are of the form given in the above corollary for suitable m, n, p, q, r.

Lemma 2.30. *Suppose a Coxeter diagram with $n = p+q$ nodes is made out of two Coxeter subdiagrams, one with p nodes and the other with q nodes together with just one bond with mark $m \geq 3$ connecting the two Coxeter subdiagrams. Let $G_n = (g_{ij})_{1 \leq i,j, \leq n}$ be the Gram matrix of the full Coxeter diagram, and let*

$$G_p = (g_{ij})_{1 \leq i,j, \leq p} \ , \ G_q = (g_{ij})_{p+1 \leq i,j, \leq n}$$

be the Gram matrices of the two Coxeter subdiagrams, and say the last node of the first Coxeter subdiagram is connected to the first node of the second Coxeter subdiagram by that bond with mark $m \geq 3$. Let

$$G_{p-1} = (g_{ij})_{1 \leq i,j, \leq p-1} \ , \ G_{q-1} = (g_{ij})_{p+2 \leq i,j, \leq n}$$

be the Gram matrices of the two new Coxeter subdiagrams, one obtained by deleting the last node from the first Coxeter subdiagram, and the other by deleting the first node from the second Coxeter subdiagram, together with all bonds connected to these two nodes. Then we have

$$\det(G_n) = \det(G_p)\det(G_q) - 4\cos^2(\pi/m)\det(G_{p-1})\det(G_{q-1}) \ .$$

Proof. The Gram matrix G_n is almost in two block form. Indeed $g_{ij} = 0$ for all $1 \leq i \leq p, p+1 \leq j \leq n$ and $p+1 \leq i \leq n, 1 \leq j \leq p$, except for

$$g_{p(p+1)} = g_{(p+1)p} = -2\cos(\pi/m) \ .$$

Therefore the lemma follows from the determinant formula

$$\det(G_n) = \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma) \prod_{1 \leq k \leq n} g_{k\sigma(k)}$$

because all terms in the sum over $\sigma \in \mathcal{S}_n$ vanish except for $\sigma \in \mathcal{S}_p \times \mathcal{S}_q$ and for $\sigma \in \mathcal{S}_{p-1} \times (p\, p+1) \times \mathcal{S}_{q-1}$. $\square$

This lemma gives a quick inductive way for computing the determinant of the Gram matrix of a Coxeter diagram.

Lemma 2.31. *The determinant of the Gram matrix of the Coxeter diagrams of Corollary 2.29 are given by*

1. $\det(G(\mathrm{A}_n)) = n + 1$.
2. $\det(G(\mathrm{I}_{pq}(m))) = (p + 1)(q + 1) - 4pq\cos^2(\pi/m)$.
3. $\det(G(\mathrm{T}_{pqr})) = pqr(1/p + 1/q + 1/r - 1)$.

Proof. The proof is straightforward using the pevious lemma, by choosing an appropriate partition of the nodes in two parts.

1. $\det(G(\mathrm{A}_n)) = (p + 1)(q + 1) - pq = p + q + 1 = n + 1$.
2. $\det(G(\mathrm{I}_{pq}(m))) = (p + 1)(q + 1) - 4pq\cos^2(\pi/m)$.
3. $\det(G(\mathrm{T}_{pqr})) = p(q + r) - (p - 1)qr = pqr(1/p + 1/q + 1/r - 1)$.

This proves the lemma. □

Corollary 2.32. *The Coxeter diagrams given in Corollary 2.29 have Gram matrices with positive determinant if (say $p \le q \le r$)*

1. $\det(G(\mathrm{I}_{pq}(4))) > 0 \Leftrightarrow p = 1, q \ge 1$ *or* $p = q = 2$.
2. $\det(G(\mathrm{I}_{pq}(5))) > 0 \Leftrightarrow p = 1, q = 1, 2, 3$.
3. $\det(G(\mathrm{T}_{pqr})) > 0 \Leftrightarrow p = q = 2, r \ge 2$ *or* $p = 2, q = 3, r = 3, 4, 5$.

Proof. Just apply the previous lemma. The first and third item are easy and left to the reader, and we only verify the second item. First observe that $\tau = 2\cos(\pi/5)$ satisfies $\tau^2 = \tau + 1$, and therefore $\tau = (1 + \sqrt{5})/2$ and $\tau^2 = (3 + \sqrt{5})/2$. Hence we get

$$\det(G(\mathrm{I}_{pq}(5))) > 0 \Leftrightarrow 2(1 + \frac{1}{p})(1 + \frac{1}{q}) > 3 + \sqrt{5} \Leftrightarrow -1 + \frac{2}{p} + \frac{2}{q} + \frac{2}{pq} > \sqrt{5}$$

and if $2 \le p \le q$ then the left hand side of this last inequality is at most $3/2 < \sqrt{5}$. Hence $p = 1$ and $1 + 4/q > \sqrt{5}$. Equivalently $q < 1 + \sqrt{5}$ and so $q = 1, 2, 3$. □

The conclusion is that the Coxeter diagrams in the table of Theorem 2.22 are just the Coxeter diagrams in the table of Corollary 2.29 whose Gram matrix has positive determinant. Since the table in Theorem 2.22 is stable under taking connected components of Coxeter subdiagrams the theorem follows from the theorem of Sylvester.

A natural question remains, whether all diagrams in the classification theorem come from a normalized root system. The answer is yes, and follows from the Tits theorem in a later chapter. Here we shall give case by case arguments.

Example 2.33. *A lattice L in a Euclidean space V is the integral span of a basis of V, such that $(\lambda, \mu) \in \mathbb{Z}$ for all $\lambda, \mu \in L$. The determinant of the Gram matrix of a lattice basis is an invariant of the lattice. The lattice L is called even if $(\lambda, \lambda) \in 2\mathbb{Z}$ for all λ in L (or equivalently for all λ from a lattice basis). For an even lattice L the set*

$$R(L) = \{\alpha \in L; (\alpha, \alpha) = 2\}$$

of norm two vectors is a normalized root system with a simply laced Coxeter diagram (so all $m_{ij} \leq 3$). Conversely, each normalized root sytem R in V with $V = \mathbb{R}R$ and with a simply laced Coxeter diagram arises in this way. If $\alpha_1, \cdots, \alpha_n$ are the simple roots in R_+ then the root lattice

$$Q(R) = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n = \mathbb{Z}R$$

determines the root system $R = R(Q(R))$. Indeed, the classification table of connected simply laced Coxeter diagrams

Cartan symbol	A_n	D_n	E_6	E_7	E_8
$\det(G(M))$	$n+1$	4	3	2	1

shows that R is determined by the number pair $(n = \mathrm{rk}(R), \det(G(M)))$. In turn this yields a construction of the normalized root systems of type ADE. *For type* AD *we have*

$$Q(\mathrm{A}_n) = \{x \in \mathbb{Z}^{n+1}; \sum x_i = 0\}\ ,\ Q(D_n) = \{x \in \mathbb{Z}^n; \sum x_i \in 2\mathbb{Z}\}$$

in the notation of Example 2.3 and Example 2.5. The lattice $Q(\mathrm{E}_8)$ is the unique even unimodular lattice of rank 8*. An explicit model for $Q(\mathrm{E}_8)$ is the lattice, which contains both the root lattice $Q(\mathrm{D}_8)$ as an index two sublattice and the vector $(\frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{R}^8$. The root system of type* E_8 *becomes*

$$R(\mathrm{E}_8) = R(\mathrm{D}_8) \sqcup \{(\epsilon_1, \cdots, \epsilon_8)/2; \epsilon_i = \pm 1, \epsilon_1 \cdots \epsilon_8 = 1\}$$

making altogether $112 + 128 = 240$ roots. A basis of simple roots for $R(\mathrm{E}_8)$ can be taken $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq 6$, $\alpha_7 = \varepsilon_6 + \varepsilon_7$, making a basis of simple roots for $R(\mathrm{D}_7)$, and $\alpha_8 = (-\frac{1}{2}, \cdots, -\frac{1}{2})$.

Example 2.34. *The normalized root system $R(\mathrm{F}_4)$ of type F_4 can be taken as the set of all permutations of the coordinates of the three vectors*

$$(\pm 1, \pm 1, 0, 0)\ ,\ (\pm\sqrt{2}, 0, 0, 0)\ ,\ (\pm 1, \pm 1, \pm 1, \pm 1)/\sqrt{2}$$

with basis of simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ given by

$$(0, 1, -1, 0), (0, 0, 1, -1), (0, 0, 0, \sqrt{2}), (1, -1, -1, -1)/\sqrt{2}$$

respectively. Clearly $R(\mathrm{F}_4)$ has $24 + 8 + 16 = 48$ roots.

Example 2.35. *The normalized root system of type H_3 can be taken as the set of all the midpoints of the 30 edges of the icosehedron with 12 vertices all cyclic permutations of the coordinates of $(\pm 1, \pm 1/\tau, 0)/\sqrt{2}$ with the golden section $\tau = (1 + \sqrt{5})/2$ the positive solution of $\tau^2 = \tau + 1$. It is easy to check that $R(\mathrm{H}_3)$ contains all cyclic permutations of the coordinates of the two vectors*

$$(\pm\tau, \pm 1, \pm 1/\tau)/\sqrt{2}\ ,\ (\pm\sqrt{2}, 0, 0)$$

making a total of $24 + 6 = 30$ roots in $R(\mathrm{H}_3)$.

The normalized root system of type H_4 can be taken as the set of all even permutations of the coordinates of the three vectors

$$(\pm\tau, \pm 1, \pm 1/\tau, 0)/\sqrt{2}\ ,\ (\pm\sqrt{2}, 0, 0, 0)\ ,\ (\pm 1, \pm 1, \pm 1, \pm 1)/\sqrt{2}$$

making a total of $96 + 8 + 16 = 120$ roots in $R(\mathrm{H}_4)$. Details are left to the reader.

2.5 The Coxeter Element

Let $R \subset V$ be a normalized root system. Let V_+ be a positive Weyl chamber with corresponding set of positive roots R_+. Let $\{\alpha_1, \cdots, \alpha_n\}$ be the set of simple roots in R_+ and let $s_1, \cdots, s_n$ be the corresponding simple reflections.

Definition 2.36. *An element of the form $c = s_1 \cdots s_n \in W$ is called a Coxeter element for any positive Weyl chamber and any numbering of the set of simple roots.*

Theorem 2.37. *The Coxeter elements form a single conjugacy class in the Weyl group W.*

Proof. Suppose we have given a tree (a graph without loops) with n nodes numbered $1, 2, \cdots, n$ and elements $g_1, g_2, \cdots, g_n$ in some group G such that $g_i g_j = g_j g_i$ if the nodes with number i and j are disconnected. We will show that all elements

$$g_{\sigma(1)} g_{\sigma(2)} \cdots g_{\sigma(n)}$$

with $\sigma \in \mathcal{S}_n$ are conjugated inside G.

The proof is by induction on n. By renumbering we may assume that the node with number n is an extremal node, and possibly only connected to the node with number $n-1$. Hence $g_i g_n = g_n g_i$ for $i = 1, 2, \cdots, n-2$. Put $c = g_1 \cdots g_n$ and $c' = g_{\sigma(1)} \cdots g_{\sigma(n)}$ for some $\sigma \in \mathcal{S}_n$. We have to show that c and c' are conjugated in G, which we denote by $c \sim c'$. If $\sigma(n-1) = n-1$ and $\sigma(n) = n$ then this is clear from the induction hypothesis by deleting the last node from the graph and considering the elements

$$g_1' = g_1, g_2' = g_2, \cdots, g_{n-2}' = g_{n-2}, g_{n-1}' = g_{n-1} g_n$$

in G. If $\sigma(n) = n$ and $\sigma(j) = n-1$ for some $1 \leq j \leq n-2$ then

$$c' = g_{\sigma(1)} \cdots g_{\sigma(j-1)} g_{n-1} g_{\sigma(j+1)} \cdots g_{\sigma(n-1)} g_n$$

is equal to

$$c' = g_{\sigma(1)} \cdots g_{\sigma(j-1)} g_{n-1} g_n g_{\sigma(j+1)} \cdots g_{\sigma(n-1)}$$

and is conjugated to

$$g_{\sigma(j+1)} \cdots g_{\sigma(n-1)} g_{\sigma(1)} \cdots g_{\sigma(j-1)} g_{n-1} g_n \sim g_1 \cdots g_n = c$$

by the previous case. Finally if $\sigma(j) = n$ for some $1 \leq j \leq n-1$ then the element

$$c' = g_{\sigma(1)} \cdots g_{\sigma(j-1)} g_n g_{\sigma(j+1)} \cdots g_{\sigma(n)}$$

is conjugated to

$$g_{\sigma(j+1)} \cdots g_{\sigma(n)} g_{\sigma(1)} \cdots g_{\sigma(j-1)} g_n \sim g_1 \cdots g_n = c$$

by the previous case. The theorem follows because any two Weyl chambers are conjugated under W. $\square$

Example 2.38. *For the symmetric group $W(\mathrm{A}_n) \cong \mathcal{S}_{n+1}$ acting on $\mathbb{R}^{n+1}$ the Coxeter elements are the cycles of maximal length $n+1$ as conjugates of $s_1 \cdots s_n = (1\ 2) \cdots (n\ n+1) = (1\ 2\ \cdots\ n\ n+1)$. For the dihedral group $W(\mathrm{I}_2(m)) \cong \mathcal{D}_m$ of order $2m$ ($m \geq 3$) acting on $\mathbb{R}^2$ there are two Coxeter elements which are the two rotations (clockwise and counterclockwise) over an angle $2\pi/m$.*

For the rest of this section let $c = s_1 \cdots s_n$ be a fixed Coxeter element in W. We wish to compute the eigenvalues of the Coxeter elements. The next lemma is the crucial step.

Lemma 2.39. *If $\beta_j = s_1 \cdots s_{j-1}(\alpha_j)$ then we have*

$$\alpha_j = \beta_j + \sum_{i<j} (\alpha_i, \alpha_j)\beta_i \ , \ c(\alpha_j) = \beta_j - \sum_{i \geq j} (\alpha_i, \alpha_j)\beta_i$$

for $1 \leq j \leq n$.

Proof. For $1 \leq i, j \leq n$ we have

$$s_1 \cdots s_{i-1}(\alpha_j) - s_1 \cdots s_i(\alpha_j) = s_1 \cdots s_{i-1}(\alpha_j - s_i(\alpha_j)) = (\alpha_i, \alpha_j)\beta_i \ .$$

Hence for all j we have

$$\sum_{i<j} (\alpha_i, \alpha_j)\beta_i = \alpha_j - s_1 \cdots s_{j-1}(\alpha_j) = \alpha_j - \beta_j$$

and

$$\sum_{i \geq j} (\alpha_i, \alpha_j)\beta_i = s_1 \cdots s_{j-1}(\alpha_j) - s_1 \cdots s_n(\alpha_j) = \beta_j - c(\alpha_j)$$

which proves the lemma. □

For the rest of this section we assume that $\dim(V) = \mathrm{rk}(R) = n$. The set $\{\beta_1, \cdots, \beta_n\}$ is again a basis of V since it is obtained from the basis $\{\alpha_1, \cdots, \alpha_n\}$ of simple roots by a unipotent lower triangular matrix.

Corollary 2.40. *If $U = (u_{ij})$ and $L = (l_{ij})$ are strict upper triangular and lower triangular matrices respectively of size $n \times n$ defined by*

$$u_{ij} = \begin{cases} (\alpha_i, \alpha_j) & i < j \\ 0 & i \geq j \end{cases} \qquad l_{ij} = \begin{cases} 0 & i < j \\ (\alpha_i, \alpha_j) & i \geq j \end{cases}$$

then the matrix C of the Coxeter element c in the basis $\{\alpha_1, \cdots, \alpha_n\}$ of V is given by $C = (I-L)(I+U)^{-1}$. Moreover $\det(tI-C) = \det((t-1)I+tU+L)$.

Proof. The first statement is obvious from the previous lemma. Hence the characteristic polynomial becomes $\det(tI-C) = \det(tI+tU-I+L)$ because $\det(I+U) = 1$. □

Lemma 2.41. *The characteristic polynomial of the Coxeter element is given by the formula*

$$\det(t-C_n) = \det(t-C_p)\det(t-C_q) - 4t\cos^2(\frac{\pi}{m})\det(t-C_{p-1})\det(t-C_{q-1})$$

in the notation of Lemma 2.30.

Proof. This is obvious from the previous corollary and Lemma 2.30. □

A normalized root system R is called irreducible if every partition of R in two mutually orthogonal normalized root systems is trivial in the sense that one part is empty. Using the Weyl group action on roots it is easy to see that a normalized root system is irreducible if and only if the corresponding Coxeter diagram is connected.

Theorem 2.42. *The characteristic polynomials of the Coxeter elements of the irreducible normalized root systems are given by*

name	$\det(t-c)$
A_n	$t^n + t^{n-1} + \cdots + t + 1$
B_n	$t^n + 1$
D_n	$(t+1)(t^{n-1}+1)$
E_6	$(t^2+t+1)(t^4-t^2+1)$
E_7	$(t+1)(t^6-t^3+1)$
E_8	$t^8+t^7-t^5-t^4-t^3+t+1$
F_4	t^4-t^2+1
H_3	$(t+1)(t^2-\tau t+1)$
H_4	$t^4+(1-\tau)t^3+(1-\tau)t^2+(1-\tau)t+1$
$\mathrm{I}_2(m)$	$t^2-2\cos(2\pi/m)t+1$

with $\tau = (1+\sqrt{5})/2$ *the golden section.*

Proof. This is a straightforward case by case computation using the previous lemma together with induction on n, and is left to the reader. □

Definition 2.43. *Let* $R \subset V$ *be an irreducible normalized root system with* $\dim(V) = \mathrm{rk}(R) = n$*. The order* h *of the Coxeter element is called the Coxeter number of* R*. The eigenvalues of the Coxeter element are of the form* $\exp(2\pi i m_j/h)$ *with*

$$1 \leq m_1 \leq m_2 \leq \cdots \leq m_{n-1} \leq m_n \leq h-1$$

the sequence of exponents of R*.*

Using Theorem 2.42 the Coxeter number and the sequence of exponents can be computed for each of the irreducible normalized root systems in a case by case manner. All one needs to know is a handful of cyclotomic polynomials

$$\Phi_{12}(t) = t^4 - t^2 + 1, \Phi_{18}(t) = t^6 - t^3 + 1, \Phi_{30}(t) = t^8 + t^7 - t^5 - t^4 - t^3 + t + 1$$

and the final answer has already been tabulated in Theorem 2.22. The outcome of these calculations is that 1 is never eigenvalue of the Coxeter element, which we already used in the above definition by writing

$$1 \le m_1 \le \cdots \le m_n \le h - 1$$

but in fact this is also clear from the formula $\det(1 - c) = \det(G(M)) > 0$. In turn we have $m_j + m_{n+1-j} = h$ because $\det(t - c) \in \mathbb{R}[t]$. Note that $m_1 = 1 < m_2$ and likewise $m_{n-1} < m_n = h - 1$. In fact the only case where two exponents are equal is the case $\mathrm{D}_{\mathrm{even}}$. For ADE we have $\det(t - c) \in \mathbb{Z}[t]$ and because $m_1 = 1$ all integers between 1 and $h-1$ that are relatively prime to h are exponents as well. For E_8 there are eight numbers between 1 and 30 that are relatively prime to 30, namely $1, 7, 11, 13, 17, 19, 23, 29$ and so these are all exponents for E_8. It so happens that the numbers greater than 1 are in fact all prime, and it can be shown that 30 is the largest number with this property [47]. For the calculation of the Coxeter number and exponents for H_4 one can verify that $\Phi_{30}(t)$ is in fact the product of $\det(t - c)$ and its Galois conjugate polynomial $t^4 + \tau t^3 + \tau t^2 + \tau t + 1$.

2.6 A Dihedral Subgroup of W

A real symmetric matrix $G = (g_{ij})$ of size $n \times n$ is called indecomposable if there is no nontrivial partion $\{1, \cdots, n\} = I \sqcup J$ with $g_{ij} = 0$ for all $i \in I$ and $j \in J$. Clearly the Gram matrix $G(M) = (-2\cos(\pi/m_{ij}))$ associated with a Coxeter matrix M is indecomposable if and only if the corresponding Coxeter diagram is connected.

The next result was obtained by Oskar Perron (in 1907) and Georg Frobenius (in 1912), and is called the Perron–Frobenius theorem.

Theorem 2.44. *Let $G = (g_{ij})$ be a symmetric matrix of size $n \times n$, which is indecomposable, positive semidefinite and satisfies $g_{ij} \le 0$ for all $i \ne j$. Then the smallest eigenvalue of G has muliplicity one, and the corresponding eigenspace is spanned by a vector whose coordinates are all positive.*

Proof. Suppose $t \geq 0$ is the smallest eigenvalue of G. Replacing G by $G - tI$ we may assume that $t = 0$. Because $g_{ij} \leq 0$ for all $i \neq j$ we have

$$\sum_{i,j} g_{ij}|x_i||x_j| \leq \sum_{i,j} g_{ij}x_ix_j$$

for all $(x_1, \cdots, x_n) \in \mathbb{R}^n$, and therefore $(x_1, \cdots, x_n) \in \ker(G)$ implies that also $(|x_1|, \cdots, |x_n|) \in \ker(G)$.

Now suppose that $0 \neq (x_1, \cdots, x_n) \in \ker(G)$ with $x_i \geq 0$ for all i. Put $I = \{i; x_i = 0\}$ and $J = \{j; x_j > 0\} \neq \emptyset$. For all $i = 1, \cdots, n$ we get

$$\sum_{j \in J} g_{ij}x_j = \sum_{j=1}^{n} g_{ij}x_j = 0$$

and hence $g_{ij} = 0$ for all $i \in I$ and $j \in J$. Since G is indecomposable we get $I = \emptyset$. Hence $\ker(G)$ is one dimensional and spanned by a vector $(x_1, \cdots, x_n)$ all whose coordinates are positive. □

Now let $R \subset V$ be an irreducible normalized root system with $\dim(V) = \mathrm{rk}(R) = n$. Let $\{\alpha_1, \cdots, \alpha_n\}$ be a basis of simple roots in R and let $\lambda_1, \cdots, \lambda_n$ be the dual basis of V defined by $(\lambda_i, \alpha_j) = \delta_{ij}$. Let $g : V \to V$ be the linear map defined by $g(\lambda_i) = \alpha_i$ for all i. Clearly the matrix of g in the basis $\{\lambda_1, \cdots, \lambda_n\}$ is just the Gram matrix $G = (\alpha_i, \alpha_j)$ of the set $\{\alpha_1, \cdots, \alpha_n\}$. We are now in a position to apply the Perron–Frobenius theorem. Let t be the smallest eigenvalue of g and let $\lambda = \sum x_i\lambda_i$ be an eigenvector of g with eigenvalue t and $x_i > 0$ for all i. Hence $\lambda \in V_+$.

Since the Coxeter diagram is a tree it is bipartite, so there is a unique partition $\{1, \cdots, n\} = I \sqcup J$ such that $s_is_j = s_js_i$ for all $i, j \in I$ and all $i, j \in J$. After a possible renumbering of the simple roots we can assume that $I = \{1, \cdots, p\}$ and $J = \{p+1, \cdots, n\}$. The elements

$$r = s_1 \cdots s_p \quad , \quad s = s_{p+1} \cdots s_n$$

are well defined (independent of the numberings of I and J) involutions, and $c = rs$ is a Coxeter element for R. Decompose the eigenvector λ as

$$\lambda = \mu + \nu \ , \ \mu = \sum_{1}^{p} x_i\lambda_i \ , \ \nu = \sum_{p+1}^{n} x_j\lambda_j$$

and let V' be the plane spanned by the two independent vectors μ and ν.

Theorem 2.45. *The action of r, s on the vectors μ, ν is given by*

$$r(\nu) = \nu \ , \ r(2\mu + (t-2)\nu) = -(2\mu + (t-2)\nu)$$
$$s(\mu) = \mu \ , \ s((t-2)\mu + 2\nu) = -((t-2)\mu + 2\nu)$$

with $\mu = \sum x_i\lambda_i$ and $\nu = \sum x_j\lambda_j$ (sum over $1 \le i \le p$ and $p+1 \le j \le n$).

Proof. By definition the vector $\lambda = \sum x_i\lambda_i$ satisfies $g(\lambda) = t\lambda$, and hence $\sum x_i\alpha_i = \sum tx_i\lambda_i$. Taking the inner product with α_j for $j \ge p+1$ gives

$$\sum_{i=1}^{p} x_i(\alpha_i, \alpha_j) = (t-2)x_j$$

which in turn implies

$$\begin{aligned} (t-2)\nu &= \sum_{j=p+1}^{n} (t-2)x_j\lambda_j = \sum_{j=p+1}^{n} \{\sum_{i=1}^{p} x_i(\alpha_i, \alpha_j)\}\lambda_j \\ &= \sum_{i=1}^{p} x_i\{\sum_{j=p+1}^{n} (\alpha_i, \alpha_j)\lambda_j\} \\ &= \sum_{i=1}^{p} x_i\{-2\lambda_i + \sum_{j=1}^{n} (\alpha_i, \alpha_j)\lambda_j\} = -2\mu + \sum_{i=1}^{p} x_i\alpha_i \ . \end{aligned}$$

Hence the vector $2\mu + (t-2)\nu$ is mapped under r to its negative. The relation $r(\nu) = \nu$ is obvious. The second formula follows from the first and the symmetry $r \leftrightarrow s$ and $\mu \leftrightarrow \nu$. ☐

Corollary 2.46. *The elements r and s leave the plane V' spanned by μ and ν invariant. Put r' and s' for the restrictions of r and s to V' respectively, and let W' be the subgroup of $\mathrm{O}(V')$ generated by the reflections r' and s'. Then W' is a dihedral group of order $2h$, where h is the Coxeter number of R. The set $V'_+ = V' \cap V_+$ is a Weyl chamber for the corresponding dihedral root system $R' \subset V'$. Each mirror in V' for R' is the intersection of V' with exactly p or $q = (n-p)$ mutually orthogonal mirrors in V for R (p in case the mirror in V' is conjugated under W' to the line $\mathbb{R}\nu$ and q in case the mirror in V' is conjugated under W' to the line $\mathbb{R}\mu$). In particular the number $2N$ of roots in R is equal to $2N = nh$.*

Proof. The element $c = rs$ is a Coxeter element for R leaving the plane V' invariant. Moreover the restriction $c' = r's'$ of c to V' has also order h, because the stabilizer in W of any vector in V_+ is trivial. This result has not been proved yet, but follows from Theorem 4.17 in the next chapter, which is valid in the greater generality of arbitrary Coxeter groups. ☐

Remark 2.47. *The relation $m_1 = 1$ or equivalently $m_n = h - 1$ which we verified in the previous section using the classification of irreducible root systems and a case by case calculation of the exponents now also follows from the calculation of the exponents in the dihedral case (which is easy).*

Remark 2.48. *The Coxeter number h and the number N of mirrors in V for R is given by*

	A_n	B_n	D_n	E_6	E_7	E_8	F_4	H_3	H_4	$\mathrm{I}_2(m)$
h	$n+1$	$2n$	$2n-2$	12	18	30	12	10	30	m
N	$\frac{1}{2}n(n+1)$	n^2	$n(n-1)$	36	63	120	24	15	60	m

using $N = nh/2$ and the actual computation of h in the previous section.

The results of this section were obtained by Robert Steinberg [54].

2.7 Integral Root Systems

In the title of this section we speak of integral root systems in order to distinguish them from the normalized root systems we have discussed so far. However in common mathematical language integral root systems are just called root systems, and that is what we shall call them here. However in the later chapter on Coxeter groups we shall use the word root system in a much more general setting, and either for normalized or for integral forms.

Definition 2.49. *Let V be a vector space over the field $\mathbb{Q}$ of dimension n with a positive definite symmetric bilinear form $(\cdot,\cdot)$ defined over $\mathbb{Q}$. A root system R in V is defined by the following four conditions:*

1. *The set R is a finite maximal rank subset of nonzero vectors in V.*
2. *For each $\alpha \in R$ the orthogonal reflection s_α in $\mathrm{O}(V)$ defined by*

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha$$

   leaves R invariant. Here $\alpha^\vee = 2\alpha/(\alpha,\alpha) \in V$ is the coroot of α.

3. *For each $\alpha, \beta \in R$ we have $(\beta, \alpha^\vee) \in \mathbb{Z}$.*
4. *If $\alpha \in R$ and $r\alpha \in R$ for some $r \in \mathbb{Q}$ then $r = \pm 1$.*

Elements of R are called roots. Clearly $R^\vee = \{\alpha^\vee; \alpha \in R\}$ is again a root system in V, called the coroot system. Elements of $R^\vee$ are called coroots. Clearly $R^{\vee\vee} = R$. The essential difference with a normalized root system is that roots need not have norm 2, but instead $(\beta, \alpha^\vee) \in \mathbb{Z}$ for each $\alpha, \beta \in R$. This is called the crystallographic restriction. If two roots $\alpha, \beta \in R$ have an angle θ then

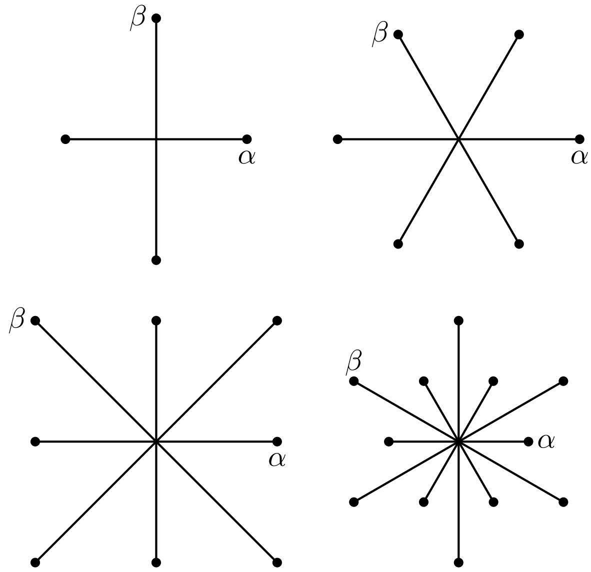
$$(\beta, \alpha^\vee)(\alpha, \beta^\vee) = \frac{4(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = 4 \cos^2 \theta \in \mathbb{Z}$$

leading to angles $\theta = \pi/2$, $\pi/3$ or $2\pi/3$, $\pi/4$ or $3\pi/4$, $\pi/6$ or $5\pi/6$, 0 or π corresponding to the values $4 \cos^2 \theta = 0, 1, 2, 3, 4$ respectively. These angles are called the crystallographic angles.

If $\beta = r\alpha$ for some $r \in \mathbb{Q}$ then the crystallographic restriction implies that $(\beta, \alpha^\vee) = 2r \in \mathbb{Z}$. Hence $r = \pm 1/2, \pm 1, \pm 2$ and the last condition implies that $r = \pm 1$ is the only possibility.

The theory of root systems can be developed in a similar way as our discussion for normalized root systems. We have V° for the complement of all mirrors, a connected component V_+ for a positive Weyl chamber and R_+ for the corresponding set of positive roots. A root $\alpha \in R_+$ is simple if it is not of the form $\beta + \gamma$ with $\beta, \gamma \in R_+$. The simple roots $\alpha_1, \cdots, \alpha_n$ are again an obtuse basis of V but this time it can be shown that each positive root $\alpha \in R_+$ is of the form $\alpha = \sum n_i \alpha_i$ with $n_i \in \mathbb{N} = \{0, 1, 2, \cdots\}$ for all i. The Weyl group W is defined as the subgroup of $\mathrm{O}(V)$ generated by the s_α for $\alpha \in R$ just as before.

There are just 4 root systems in rank two, and here are their pictures.



Let $\{\alpha, \beta\}$ be a basis of simple roots and say $(\alpha, \alpha) \leq (\beta, \beta)$. Either $(\beta, \alpha^\vee) = (\alpha, \beta^\vee) = 0$ and R is reducible and nothing about the norm ratio of β and α can be said. Or R is irreducible and

$$(\beta, \alpha^\vee) = -1, -2, -3 \quad \text{and} \quad (\alpha, \beta^\vee) = -1$$

(since their product takes values $1, 2, 3$) and the norm ratio of β and α is $1, 2, 3$ respectively.

There is a variation of the notion of Coxeter diagram for roots systems called the Dynkin diagram. It is a graph with n nodes corresponding to the simple roots $\{\alpha_1, \cdots, \alpha_n\}$. If $(\alpha_i, \alpha_i) \leq (\alpha_j, \alpha_j)$ and $(\alpha_j, \alpha_i^\vee) = -m$ for $i \neq j$ then the nodes with number i and j are connected by m bonds with an arrow pointing to the shorter root for $m = 2, 3$.

The above diagrams are the Dynkin diagrams of the rank 2 root systems of type $2\mathrm{A}_1$, A_2, B_2 and G_2 respectively. In particular for simply laced root systems the notion of Coxeter diagram and Dynkin diagram coincides. The root system can be recovered from its Dynkin diagram, apart from a scale factor in the inner product for each irreducible component of R or equivalently for each connected component of the Dynkin diagram of R. This scale factor in the inner product is usually considered irrelevant.

Renormalization of root lengths gives for each root system a normalized root system. The classification of irreducible root systems (up to a scale factor in the inner product) is now a direct consequence of the classification of connected elliptic Coxeter diagrams as given in Theorem 2.22. Clearly a root system is irreducible if and only if the Dynkin diagram is connected. The list of connected Dynkin diagrams together with their Coxeter numbers is given below in Theorem 2.53.

Example 2.50. *The normalized root system of type* B_n *for* $n \geq 3$ *gives rise to two root systems of type* B_n *and* C_n. *They are given by*

$$R(\mathrm{B}_n) = R(\mathrm{D}_n) \sqcup \{\pm\varepsilon_i ; 1 \leq i \leq n\}\ ,\ R(\mathrm{C}_n) = R(\mathrm{D}_n) \sqcup \{\pm 2\varepsilon_i ; 1 \leq i \leq n\}$$

with $R(\mathrm{D}_n) = \{\pm\varepsilon_i \pm \varepsilon_j ; 1 \leq i < j \leq n\}$ *the root system of type* D_n. *The bases of simple roots can be taken* $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ *for* $1 \leq i \leq n-1$ *in both cases while* $\alpha_n = \varepsilon_n$ *for type* B_n *and* $\alpha_n = 2\varepsilon_n$ *for type* C_n.

Example 2.51. *Using Example 2.34 the integral root system of type* F_4 *can be given in* $\mathbb{R}^4$ *as*

$$R(\mathrm{F}_4) = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j, (\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2; 1 \leq i, j \leq 4, i < j\}$$

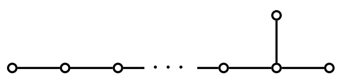
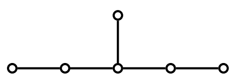
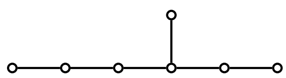
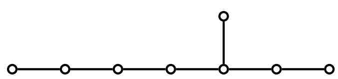
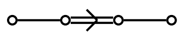
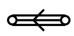
with $\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4, \alpha_4 = (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2$ *as a basis of simple roots.*

Example 2.52. *The integral root system of type* G_2 *can be given as subset of the plane* $x_1 + x_2 + x_3 = 0$ *in* $\mathbb{R}^3$ *as*

$$R(\mathrm{G}_2) = \{\pm(\varepsilon_i - \varepsilon_j), \pm(-\varepsilon_i - \varepsilon_j + 2\varepsilon_k); 1 \leq i, j, k \leq 3, i < j, k \neq i, j\}$$

with $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ *as a basis of simple roots.*

Theorem 2.53. *The Dynkin diagrams of the irreducible root systems are*

name	Dynkin diagram	h
A_n		$n+1$
B_n		$2n$
C_n		$2n$
D_n		$2n-2$
E_6		12
E_7		18
E_8		30
F_4		12
G_2		6

with h *the Coxeter number.*

The classification of the connected Dynkin diagram is a central result in mathematics, because they classify both the simple Lie algebras (a result of Wilhelm Killing from 1888 and Élie Cartan from 1894) and the simple linear algebraic groups (a result of Claude Chevally from 1955 and Robert Steinberg from 1959).

2.8 The Poincaré Dodecahedral Space

This section is meant as an exercise section for this chapter. The reader is invited to go through the text of this section and make the problems, or if necessary to look things up in the literature. Our goal is to gain more geometric insight in the reflection groups of type H_n for $n = 2, 3, 4$. For $n = 2$ the group $W(\mathrm{H}_2)$ is just the dihedral group of order 10, which appears as the symmetry group of the regular pentagon. For $n = 3$ the group $W(\mathrm{H}_3)$ is the symmetry group of the regular icosahedron amd dodecahedron. Both these Platonic solids have 30 edges, and the 15 lines through midpoints of opposite edges constitute 5 orthogonal triples. The index 2 subgroup I of rotations in $W(\mathrm{H}_3)$ acts on these 5 triples by even permutations, giving an isomorphism between the icosahedral rotation group I and $\mathcal{A}_5$. Therefore $W(\mathrm{H}_3)$ is isomorphic to $\mathcal{A}_5 \times \mathcal{C}_2$ with the order 2 group $\mathcal{C}_2$ acting by the central inversion. Below we shall explain a geometric construction of the root system $R(\mathrm{H}_4)$ using the concept of quaternions and some group theory.

Definition 2.54. *Let $\{1, i, j, k\}$ denote the standard basis of $\mathbb{R}^4$. A vector in $\mathbb{R}^4$ of the form $q = u_0 + u_1 i + u_2 j + u_3 k$ with $u_0, u_1, u_2, u_3 \in \mathbb{R}$ is called a quaternion. The quaternions form a real vector space denoted by $\mathbb{H}$. A product rule $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$ with unit 1 is defined by*

$$i^2 = j^2 = k^2 = -1 \ , \ ij = -ji = k \ , \ jk = -kj = i \ , \ ki = -ik = j$$

and extended in a real bilinear way.

Exercise 2.55. *For $q = u_0 + u_1 i + u_2 j + u_3 k \in \mathbb{H}$ we write $q = a + bj$ with $a = u_0 + u_1 i, b = u_2 + u_3 i \in \mathbb{C}$. Check that via the identification of $q \in \mathbb{H}$ with the matrix*

$$q = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}$$

the product rule on $\mathbb{H}$ corresponds to matrix multiplication.

Hence the quaternions $\mathbb{H}$ as matrix algebra form an associative algebra. The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are natural subalgebras of $\mathbb{H}$. The multiplication on $\mathbb{R}$ and $\mathbb{C}$ is commutative, but multiplication on $\mathbb{H}$ is no longer commutative. The center of $\mathbb{H}$ is just the real numbers $\mathbb{R}$. The quaternions were introduced by the Irish mathematician William Rowan Hamilton in 1843, and the letter $\mathbb{H}$ honours his contribution.

Let $q = u_0 1 + u_1 i + u_2 j + u_3 k = a + bj \in \mathbb{H}$ be a quaternion. The number $u_0 \in \mathbb{R}$ is called the real part of q, and is denoted $\Re q$. If $\Re q = 0$ then q is called a purely imaginary quaternion. Clearly $q = u_0 + u$ with real part u_0 and $u = u_1 i + u_2 j + u_3 k$ a purely imaginary quaternion. The quaternion $\overline{q} = u_0 - u$ is called the conjugate quaternion of q. Check that

$$q\overline{q} = u_0^2 + u_1^2 + u_2^2 + u_3^2 = a\overline{a} + b\overline{b} = \det(q)$$

and we denote $|q| = \sqrt{q\overline{q}}$ for the norm of q. If $\mathbb{H}^\times$ denotes the set of nonzero quaternions then the norm map

$$|\cdot| : \mathbb{H}^\times \to \mathbb{R}^\times$$

is a multiplicative homomorphism. Any $q \in \mathbb{H}^\times$ has an inverse, namely $q^{-1} = \overline{q}/|q|^2 \in \mathbb{H}^\times$ and therefore $\mathbb{H}$ is called an associative division algebra. Since $\overline{q} = q^{-1}|q|^2$ it follows that $\overline{p\,q} = \overline{q}\,\overline{p}$ for all $p, q \in \mathbb{H}$.

Definition 2.56. *The unit sphere of quaternions*

$$\{q \in \mathbb{H}; |q| = 1\} \cong \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \in \mathbb{H}; |a|^2 + |b|^2 = 1 \right\}$$

is called the group $\mathrm{U}_1(\mathbb{H}) \cong \mathrm{SU}_2(\mathbb{C})$.

The purely imaginary quaternions are identified with the Cartesian space $\mathbb{R}^3$ with standard basis $\{i, j, k\}$, and denoted by $\mathbb{H}^{\mathrm{im}}$.

Exercise 2.57. *Check that the multiplication of* $u, v \in \mathbb{H}^{\mathrm{im}}$ *is given by*

$$uv = -(u, v) + u \times v$$

with (u, v) *the scalar product and* $u \times v$ *the vector product of* u *and* v.

Define a map $\pi : \mathrm{SU}_2(\mathbb{C}) \to \mathrm{GL}_3(\mathbb{R})$ by

$$\pi(q)u = qu\overline{q} = quq^{-1}$$

for $u \in \mathbb{H}^{\mathrm{im}}$. Clearly π is a homomorphism.

Exercise 2.58. *Check that for* $u, v \in \mathbb{H}^{\mathrm{im}}$ *and* $q \in \mathrm{SU}_2(\mathbb{C})$ *we have*

$$(\pi(q)u, \pi(q)v) = (u, v)\ ,\ (\pi(q)u) \times (\pi(q)v) = \pi(q)(u \times v)$$

which in turn implies that $\pi(q) \in \mathrm{SO}_3(\mathbb{R})$.

Definition 2.59. *The homomorphism* $\pi : \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R})$ *is called the spin homomorphism.*

Let $\{u, v, w\}$ be a positively oriented orthonormal basis of $\mathbb{R}^3$. We denote by $r(w, \theta)$ the rotation of $\mathbb{R}^3$ with directed axis $\mathbb{R}w$ over an angle θ, hence with matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

relative to the basis $\{u, v, w\}$. Clearly we have $r(-w, -\theta) = r(w, \theta)$.

Exercise 2.60. *Prove the theorem of Euler saying that every element in* $\mathrm{SO}_3(\mathbb{R})$ *is of the form* $r(w, \theta)$ *for some* $w \in \mathbb{R}^3$ *with* $|w| = 1$ *and some* $\theta \in \mathbb{R}$.

Exercise 2.61. *Let us take* $w \in \mathbb{H}^{\mathrm{im}}$ *with* $|w| = 1$ *and* $\theta \in \mathbb{R}$. *Check that for* $q = (\cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta \cdot w) \in \mathrm{SU}_2(\mathbb{C})$ *we have* $\pi(q) = r(w, \theta)$.

Corollary 2.62. *The spin homomorphism* $\pi : \mathrm{SU}_2(\mathbb{C}) \twoheadrightarrow \mathrm{SO}_3(\mathbb{R})$ *is an epimorphism with kernel* $\{\pm 1\}$, *and so* $\mathrm{SU}_2(\mathbb{C})/\{\pm 1\} \cong \mathrm{SO}_3(\mathbb{R})$.

Definition 2.63. *The inverse image of a finite subgroup* $G < \mathrm{SO}_3(\mathbb{R})$ *under the spin homomorphism* $\pi : \mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{R})$ *is called the binary cover of* G, *and will be denoted by* $\mathbb{G}$.

Exercise 2.64. *The finite subgroups* $G < \mathrm{SO}_3(\mathbb{R})$ *have been classified:* G *is isomorphic to the cyclic group* $\mathcal{C}_n$ *of order* $n \geq 1$, *or the dihedral group* $\mathcal{D}_n$ *of order* $2n \geq 4$, *or the rotation groups* $\mathrm{T}, \mathrm{O}, \mathrm{I}$ *of a regular tetrahedron, octahedron and icosahedron respectively. Check the proof, as given in the appendix of the beautiful little book "Symmetry" by the Grand Old Master Hermann Weyl [67].*

Exercise 2.65. *For* K *a conjugacy class in a finite subgroup* $G < \mathrm{SO}_3(\mathbb{R})$ *denote by* $\mathbb{K}$ *the inverse image in* $\mathbb{G}$ *of* K *under the spin homomorphism* $\pi : \mathbb{G} \twoheadrightarrow G$. *If the elements of* K *have odd order* n *show that* $\mathbb{K} = \mathbb{K}_1 \sqcup \mathbb{K}_2$ *is a disjoint union of two conjugacy classes in* $\mathbb{G}$ *with elements of order* n *and* $2n$ *respectively, and* $\pi : \mathbb{K}_i \to K$ *is a bijection for* $i = 1, 2$. *For the*

Klein four group $\mathrm{V}_4 = \{e, a, b, c\}$ *viewed as subgroup of* $\mathrm{SO}_3(\mathbb{R})$ *acting by rotations over an angle* π *around the three coordinate axes the binary Klein four group* $\mathbb{V}_4 = \{\pm 1, \pm i, \pm j, \pm k\}$ *is also called the quaternion group. Work out the conjugacy classes for* V_4 *and* $\mathbb{V}_4$ *and their relations under the spin homomorpism* $\pi : \mathbb{V}_4 \twoheadrightarrow \mathrm{V}_4$.

Exercise 2.66. *Consider the spin homomorphism* $\pi : \mathbb{I} \twoheadrightarrow \mathrm{I}$ *for the binary icosahedral group. Show that the icosahedral group* $\mathrm{I} \cong \mathcal{A}_5$ *has* 5 *conjugacy classes, one with elements of order* $1, 2, 3$ *and two with elements of order* 5*, having cardinality* $1, 15, 20, 12, 12$ *respectively. Show that the binary icosahedral group* $\mathbb{I}$ *has* 9 *conjugacy classes with representative, order, cardinality and trace as elements of* $\mathrm{SU}_2(\mathbb{C})$ *given by the table*

representative	e	z	a	b	bz	c	d	dz	cz
order	1	2	4	6	3	10	5	10	5
cardinality	1	1	30	20	20	12	12	12	12
trace	2	-2	0	1	-1	τ	$\tau - 1$	$1 - \tau$	$-\tau$

with $\tau = (1 + \sqrt{5})/2$ *the golden section. Moreover* $e, z \in \mathbb{I}$ *are the two central elements and* $a, b, c \in \mathbb{I}$ *satisfy* $a^2 = b^3 = c^5 = z$ *and* $d = c^2$*. Note that* trace $= 2\cos(2\pi/\text{order})$ *except on the conjugacy classes of* cz *and* dz*. Observe that* trace *distinguishes the conjugacy classes in* $\mathbb{I}$.

We are finally in the position to link the above story to the root system $R(\mathrm{H}_4)$.

Theorem 2.67. *After scalar multiplication by* $\sqrt{2}$ *the elements of the binary icosahedral group* $\mathbb{I}$*, viewed as a collection of* 120 *vectors in* $\mathbb{H} \cong \mathbb{R}^4$*, form a normalized root system of type* H_4.

Proof. Put $R = \sqrt{2}\mathbb{I}$. We will show that $s_\alpha(\beta) \in R$ for all $\alpha, \beta \in R$, and so R is a normalized root system in $\mathbb{R}^4$ with 120 roots. From the classification it follows that R is of type H_4. By the homogeneous action of $\mathbb{I}$ on R it suffices to consider the case $\alpha = \sqrt{2}$ and $\beta = \sqrt{2}x$ for some $x \in \mathbb{I}$. From the table and the notation in Exercise 2.66 we know that x lies in a maximal cyclic subgroup C of $\mathbb{I}$ generated by an element y equal to a, b or c with $a^2 = b^3 = c^5 = z$. Clearly C is the intersection of $\mathbb{I}$ with the real plane V spanned by $\{1, y\}$ and under the natural identification $V \cong \mathbb{C}$ the cyclic group C is just the set $\sqrt[2m]{1}$ with $m = 2, 3$ or 5 respectively. After multiplication by $\sqrt{2}$ this is just the dihedral root system $R(\mathrm{I}_2(m))$ with $m = 2, 3$ or 5. $\square$

The unit sphere S^3 is a group and a smooth manifold. The factor space $S^3/\mathbb{I}$ is again a smooth manifold, called the Poincaré dodecahedral space. Since S^3 is simply connected the fundamental group $\Pi_1(S^3/\mathbb{I},*)$ is isomorphic to $\mathbb{I}$. For any compact smooth manifold M one can define homology groups $H_k(M)$ for $k \in \mathbb{Z}$, which are finitely generated Abelian groups and topological invariants of M. By general principles $H_k(M)$ is equal to 0 for $k \neq 0, 1, \cdots, n = \dim M$. Moreover for M a connected oriented manifold $H_0(M) = H_n(M) = \mathbb{Z}$ and one also has

$$H_1(M) = \Pi_1(M,*)^{\mathrm{Ab}} = \Pi_1(M,*)/[\Pi_1(M,*), \Pi_1(M,*)]$$

with $[\cdot,\cdot]$ denoting the commutator subgroup. Using that I is a simple group it is easy to see that $[\mathbb{I},\mathbb{I}] = \mathbb{I}$. Hence

$$H_1(S^3) = H_1(S^3/\mathbb{I}) = 0$$

and again by general principals (so called Poincaré duality for a compact connected oriented manifold) we conclude from this that

$$H_2(S^3) = H_2(S^3/\mathbb{I}) = 0$$

as well. The conclusion is that S^3 and $S^3/\mathbb{I}$ have the same homology.

This example was constructed by Henri Poincaré in 1904 and destroyed his original hope that a compact connected smooth threefold with the same homology as the sphere S^3 is in fact homeomorphic to S^3. In that same article Poincaré asked the question if a compact connected smooth simply connected threefold is always homeomorphic to the threesphere? In other words, not just homology but the more refined homotopy suffices to characterize S^3. This question became known under the name Poincaré conjecture.

The Poincaré conjecture was finally settled by Grisha Perelman in 2003 building on earlier work by Richard Hamilton from the nineteen eighties. Perelman received both the Fields medals and the Clay prize of one million dollars, but rejected them both.

We finally comment on the name Poincaré dodecahedral space. The Weyl chamber decomposition of the mirror complement $\mathbb{H}^\circ$ for $R(\mathrm{H}_4) = \sqrt{2}\mathbb{I}$ gives rise to a tessellation of the three sphere S^3 by 14400 closed spherical simplices. For each element $x \in \mathbb{I} \subset S^3$ there are 120 of these simplices that contain x and together these form a spherical dodecahedron with center x. The dihedral angle of a Euclidean dodecahedron is equal to 116.565° which is slightly smaller than the dihedral angle $2\pi/3 = 120^\circ$ of these spherical dodecahedra. A tessellation of Euclidean space by Euclidean dodecahedra is impossible, but we do arrive at a tessellation of S^3 by 120 spherical dodecahedra. Moreover each of them is a fundamental domain for the action of $\mathbb{I}$ on S^3. The Poincaré dodecahedral space can be thought of as one of these spherical dodecahedra with a particular glueing along the faces. For a discussion of the glue procedure we refer to the book by Thurston [57].

Exercise 2.68. *Explain this tessellation of S^3 by 120 spherical dodecahedra from the perspective of the 120–cell.*

3 Invariant Theory for Reflection Groups

3.1 Polynomial Invariant Theory

Let K be a field of characteristic 0 and V a vector space over K of dimension n. Let $\xi_1, \cdots, \xi_n$ be a basis of V. If we write $\xi \in V$ as $\xi = x_1\xi_1 + \cdots + x_n\xi_n$ with $x_1, \cdots, x_n \in K$ then $V \ni \xi \mapsto x_j = x_j(\xi) \in K$ are the linear coordinate functions relative to the given basis of V. The smallest subalgebra of the commutative algebra of all functions on V with values in K containing the linear coordinate functions $\xi \mapsto x_j(\xi)$ is called the algebra of polynomial functions on V, and is denoted $P(V)$. It is easy to check that $P(V)$ is independent of the given basis of V, and a choice of basis in V gives an isomorphism $P(V) \cong K[x_1, \cdots, x_n]$. A vector space basis of $P(V)$ is given by the monomials $x_1^{k_1} \cdots x_n^{k_n}$ with $k_1, \cdots, k_n$ running over the set of nonnegative integers. The elements of the subspace

$$P_k(V) = \{p \in P(V); p(t\xi) = t^k p(\xi) \;\forall\, t \in K, \xi \in V\}$$

are called homogeneous polynomials of degree k ($k = 0, 1, 2, \cdots$). This turns $P(V)$ into a graded commutative algebra, that is

$$P(V) = \oplus_{k \geq 0} P_k(V) \quad , \quad P_j(V)P_k(V) \subset P_{j+k}(V)$$

for all $j, k = 0, 1, 2, \cdots$.

The general linear group $\mathrm{GL}(V)$ acts on $P(V)$ in a natural way by

$$(g \cdot p)(\xi) = p(g^{-1}\xi)$$

for $g \in \mathrm{GL}(V)$, $p \in P(V)$ and $\xi \in V$. Clearly $g \cdot (h \cdot p) = (gh) \cdot p$ for $g, h \in \mathrm{GL}(V)$ and $p \in P(V)$. Moreover $g \cdot P_k(V) \subset P_k(V)$ for $g \in \mathrm{GL}(V)$ and $k \geq 0$. If $G < \mathrm{GL}(V)$ is a subgroup we say that $p \in P(V)$ is G-invariant if $g \cdot p = p$ for all $g \in G$. The collection of all G-invariants in $P(V)$ form a graded subalgebra $P^G(V)$. The goal of invariant theory is to understand the structure of such algebras $P^G(V)$.

Example 3.1. *The symmetric group $\mathcal{S}_n$ acts on $K[x_1, \cdots, x_n]$ by permutations of the indices, that is*

$$\sigma \cdot p = \sum a_{k_1, \cdots, k_n} x_{\sigma(1)}^{k_1} \cdots x_{\sigma(n)}^{k_n} = \sum a_{j_{\sigma(1)}, \cdots, j_{\sigma(n)}} x_1^{j_1} \cdots x_n^{j_n}$$

for $\sigma \in \mathcal{S}_n$ and $p = \sum a_{k_1, \cdots, k_n} x_1^{k_1} \cdots x_n^{k_n}$. The $\mathcal{S}_n$-invariant polynomials are also called symmetric polynomials. The elementary symmetric polynomials are defined by

$$(t + x_1) \cdots (t + x_n) = t^n + p_1(x_1, \cdots, x_n)t^{n-1} + \cdots + p_n(x_1, \cdots, x_n)$$

or equivalently

$$p_k(x_1, \cdots, x_n) = \sum_{1 \le j_1 < \cdots < j_k \le n} x_{j_1} \cdots x_{j_n}$$

for $k = 1, \cdots, n$. It is a classical theorem of algebra that each symmetric polynomial is a uniquely determined polynomial in the elementary symmetric polynomials, and so

$$K[x_1, \cdots, x_n]^{\mathcal{S}_n} = K[p_1, \cdots, p_n] \, .$$

In fact this result remains valid with the field K replaced by any commutative ring R.

The following theorem is a classical result due to Hilbert.

Theorem 3.2. *For $G < \mathrm{GL}(V)$ a finite subgroup the invariant algebra $P^G(V)$ is a finitely generated algebra.*

The proof will cover the rest of this section.

Definition 3.3. *A commutative ring R is called Noetherian if any ideal I of R is of finite type, that is $I = Rp_1 + \cdots + Rp_m$ for suitable $p_1, \cdots, p_m \in I$.*

Theorem 3.4 (Hilbert basis theorem)**.** *If the commutative ring R is Noetherian then the polynomial ring $R[t]$ is Noetherian as well.*

This result is called Hilbert basis theorem. For the proof we refer to [65]. The next corollary is what we use for proving finite generation of the algebra of invariants for a finite group.

Corollary 3.5. *The algebra $P(V)$ is Noetherian.*

For the proof of the Theorem 3.2 we need one more concept, which is averaging over the finite group G. Let us denote

$$\rho : P(V) \to P(V) \, , \; \rho(p) = \frac{1}{|G|} \sum_{g \in G} g \cdot p$$

for the averaging operator. Clearly ρ is a linear operator with image $P^G(V)$. Moreover $\rho^2 = \rho$, that is ρ is a projection operator and we have

$$\rho(pq) = \rho(p)q$$

for all $p \in P(V)$ and $q \in P^G(V)$. The proof of Theorem 3.2 goes as follows.

Proof. Let $Q = P^G(V)$ denote the algebra of invariants, and let us write $Q_+ = \sum_{k\geq 1} Q_k$ for the invariants that vanish at the origin. Then the ideal $I = P(V)Q_+$ is of finite type. Hence we can choose homogeneous invariants $q_1, \cdots, q_m \in Q_+$ of positive degree such that $I = P(V)q_1 + \cdots + P(V)q_m$. We claim that Q is generated as algebra by $q_1, \cdots, q_m$, that is we have to express each $q \in Q$ as polynomial in $q_1, \cdots, q_m$. It is sufficient to do this for $q \in Q_k$ homogeneous of degree k. Proceed by induction on k, the case $k = 0$ being obvious. For $k \geq 1$ we have $q \in I$, and hence

$$q = p_1 q_1 + \cdots p_m q_m$$

for suitable polynomials $p_1, \cdots, p_m \in P(V)$. After removing redundant terms we may assume that p_j is homogeneous of degree k minus the degree of q_j, and so the degree of p_j is strictly smaller than k. Averaging over G yields

$$q = \rho(q) = \rho(p_1 q_1) + \cdots + \rho(p_m q_m) = \rho(p_1)q_1 + \cdots + \rho(p_m)q_m$$

Now the $\rho(p_j)$ are homogeneous elements of Q of degree less than k. By induction the $\rho(p_j)$ are polynomials in $q_1, \cdots, q_m$ and then so is q. □

The next result is called the Molien formula and will be needed in the next section [42].

Theorem 3.6. *For $G < \mathrm{GL}(V)$ a finite subgroup we have*

$$\sum_{k\geq 0} \dim(P_k^G(V))t^k = \frac{1}{|G|} \sum_{g\in G} \frac{1}{\det(1 - tg)} .$$

Proof. Let $P_k(g)$ denote the linear operator on $P_k(V)$ induced by the element $g \in G$. Since $P_k(gh) = P_k(g)P_k(h)$ for $g, h \in G$ it follows as before that the linear operator $|G|^{-1} \sum_{g\in G} P_k(g)$ is a projection operator from $P_k(V)$ onto $P_k^G(V)$. Note that the trace of a linear projection operator is equal to the dimension of its image. Hence we get

$$\dim(P_k^G(V)) = \frac{1}{|G|} \sum_{g\in G} \mathrm{tr}(P_k(g))$$

and the result follows from $1/\det(1 - tg) = \sum_{k\geq 0} \mathrm{tr}(P_k(g))t^k$. □

Exercise 3.7. *Let $\xi_1, \cdots, \xi_n$ be a basis of V and write $\xi \in V$ as $\xi = x_1\xi_1 + \cdots + x_n\xi_n$ with $x_1, \cdots, x_n \in K$. Show that under the identification $P(V) \cong K[x_1, \cdots, x_n]$ the Euler operator $\mathcal{E} = \sum x_j \partial/\partial x_j$ acts on $P_k(V)$ by multiplication with k.*

3.2 The Chevalley Theorem

Suppose V is a Euclidean space of dimension n with scalar product $(\cdot, \cdot)$ and $W < \mathrm{O}(V)$ a finite reflection group. The next result was obtained by Chevalley in 1955 with a uniform proof, that is a proof independent of the classification. However the result had been obtained a year before by Shephard and Todd through case by case analysis and classification in the larger generality of finite unitary reflection groups [53]. In fact most of the cases had already been dealt with in the older literature.

Theorem 3.8 (Chevalley theorem)**.** *If $W < \mathrm{O}(V)$ is a finite reflection group then $P^W(V) = \mathbb{R}[p_1, \cdots, p_n]$ with p_j a homogeneous invariant of degree d_j and $p_1, \cdots, p_n$ algebraically independent.*

Apparently the phenomenon that symmetric polynomials are unique polynomials in the elementary symmetric polynomials, corresponding to the symmetric group $\mathcal{S}_n$ acting on $\mathbb{R}^n$ by permutation matrices as discussed in Example 3.1, can be generalized to the context of arbitrary finite reflection groups.

Lemma 3.9. *Let $s \in \mathrm{O}(V)$ be a reflection and let l a homogeneous linear polynomial whose zero locus is the mirror of that reflection. Then for each $q \in P_k(V)$ there exists $r \in P_{k-1}(V)$ with $q - s \cdot q = lr$.*

Proof. Choosing suitable coordinates we may assume that $l = x_n$ is the last coordinate function. Now $q - s \cdot q$ is a homogeneous polynomial in $x_1, \cdots, x_n$ of degree k, which is equal to zero if $x_n = 0$. Hence $q - s \cdot q = lr$ for some homogeneous polynomial r of one degree less. $\square$

Lemma 3.10. *Let $W < \mathrm{O}(V)$ be a reflection group. Let $p_j, q_j \in P(V)$ be homogeneous polynomials for $j = 1, \cdots, m$ with p_j invariants, and q_j not all equal to zero, and $\sum p_j q_j = 0$. Then there exist homogeneous invariant polynomials r_j for $j = 1, \cdots, m$, not all equal to zero, with degree of r_j at most equal to the degree of q_j, and $\sum p_j r_j = 0$.*

Proof. If all q_j are invariants for W then we simply take $r_j = q_j$. Now suppose $q_i \notin P^W(V)$ for some index i. Then there exists a reflection $s \in W$ with $s \cdot q_i \neq q_i$. By the previous lemma we can write $q_j - s \cdot q_j = l r_j$ for suitable homogeneous polynomials r_j, not all equal to zero (indeed $s_i \neq 0$). From $\sum p_j q_j = 0$ we get $\sum p_j (s \cdot q_j) = 0$ and by subtraction $\sum p_j (q_j - s \cdot q_j) = 0$. Since $P(V)$ has no zero divisors we arrive at $\sum p_j r_j = 0$ with not all r_j equal to 0. Proceeding by induction on the maximal degree of the q_j yields the result. $\square$

Lemma 3.11. *Let $W < \mathrm{O}(V)$ be a finite reflection group. Let $p_1, \cdots, p_j$ be homogeneous invariants of degrees $1 \leq d_1 \leq \cdots \leq d_j$ respectively. Let $d \geq d_j$ and assume that all homogeneous invariants of degree $< d$ belong to $\mathbb{R}[p_1, \cdots, p_j]$. If p is a homogeneous invariant of degree d and p is algebraic over $\mathbb{R}[p_1, \cdots, p_j]$ then $p \in \mathbb{R}[p_1, \cdots, p_j]$.*

Proof. Suppose we have an algebraic relation

$$a_0 p^r + a_1 p^{r-1} + \cdots a_r = 0$$

with $a_0, a_1, \cdots, a_r \in \mathbb{R}[p_1, \cdots, p_j]$ homogeneous (in $P(V)$) polynomials and $a_0 \neq 0$. We suppose that r is minimal and for this r the degree of a_0 is minimal among all such equations satisfied by p. We will show that $a_0 \in \mathbb{R}^\times$ has degree 0 and $r = 1$.

Let $x_1, \cdots, x_n$ be linear coordinates on V with respect to some basis of V. Differentiation of the above relation with respect to x_i gives

$$b_0 p^r + b_1 p^{r-1} + \cdots b_r = 0$$

with coefficients b_k given by

$$b_k = \frac{\partial a_k}{\partial x_i} + (r+1-k) a_{k-1} \frac{\partial p}{\partial x_i} = \frac{\partial a_k}{\partial p_1}\frac{\partial p_1}{\partial x_i} + \cdots + \frac{\partial a_k}{\partial p_j}\frac{\partial p_j}{\partial x_i} + (r+1-k) a_{k-1} \frac{\partial p}{\partial x_i}$$

by the chain rule (put $a_{-1} = 0$). Suppose that for some i not all coefficients b_k are equal to 0. From the previous lemma (with its proof in mind) we obtain a new polynomial relation

$$c_0 p^r + c_1 p^{r-1} + \cdots c_r = 0$$

with coefficients $c_k \in P^W(V)$, not all equal to 0, and of the form

$$c_k = \frac{\partial a_k}{\partial p_1} c_{k,1} + \cdots + \frac{\partial a_k}{\partial p_j} c_{k,j} + (r + 1 - k) a_{k-1} c_{k,j+1}$$

with $\deg(c_{k,l}) < d$ for all k, l. Averaging over W (at this point we use the finiteness of W) yields

$$c_k = \frac{\partial a_k}{\partial p_1} d_{k,1} + \cdots + \frac{\partial a_k}{\partial p_j} d_{k,j} + (r + 1 - k) a_{k-1} d_{k,j+1}$$

with $d_{k,l}$ homogeneous invariants of degree $< d$. By the assumptions of the lemma $d_{k,l} \in \mathbb{R}[p_1, \cdots, p_j]$, and therefore $c_k \in \mathbb{R}[p_1, \cdots, p_j]$ for all k.

Since $\deg(c_0) < \deg(a_0)$ we arrive at a contradiction with the minimality assumptions of the original algebraic equation for p.

Hence $b_k = 0$ for all k and all i. In particular for $k = 0, 1$ we get

$$b_0 = \frac{\partial a_0}{\partial x_i} = 0 \ , \ b_1 = \frac{\partial a_1}{\partial x_i} + r a_0 \frac{\partial p}{\partial x_i} = 0$$

for all i. The first equation implies $a_0 \in \mathbb{R}^\times$. Application of Euler's formula $\sum_i x_i \partial q/\partial x_i = \deg(q) q$ for a homogeneous polynomial $q \in P(V)$ to the second set of equations yields

$$\deg(a_1) a_1 + r a_0 \deg(p) p = 0 \ .$$

Hence $r = 1$ which proves the lemma. $\square$

We can now prove the Chevalley theorem.

Proof. Select homogeneous polynomials $p_1, \cdots, p_m \in P^W(W)$ as follows: p_1 has least degree $d_1 \geq 1$ and inductively p_{j+1} has least degree d_{j+1} subject to $p_{j+1} \notin \mathbb{R}[p_1, \cdots, p_j]$. The previous lemma implies that p_{j+1} is not algebraic over $\mathbb{R}[p_1, \cdots, p_j]$. Say this process stops after m steps, that is $P^W(V) = \mathbb{R}[p_1, \cdots, p_m]$. Since $p_1, \cdots, p_m$ are algebraically independent Molien's formula gives

$$\frac{1}{|W|} \sum_{w \in W} \frac{1}{\det(1 - tw)} = \prod_{j=1}^{m} \frac{1}{1 - t^{d_j}} \ .$$

The left hand side has a pole of order n at $t = 1$ corresponding to the identity element $w = 1$ in W, and so does the right hand side. Hence $m = n$ and the theorem follows. $\square$

Remark 3.12. *Note that the degrees $1 \leq d_1 \leq \cdots \leq d_n$ of a collection of homogeneous generators are uniquely determined by $W < \mathrm{O}(V)$. However for the choice of the homogeneous generators $p_1, \cdots, p_n$ there is a good deal of freedom.*

Exercise 3.13. *Using the formula in the proof of the Chevalley theorem show that the order of W is given by $|W| = d_1 \cdots d_n$ and the number N of reflections in W is given by $N = \sum_j (d_j - 1)$.*

Exercise 3.14. *Let W be the dihedral group of order $2m$ acting on $\mathbb{R}^2 \cong \mathbb{C}$ generated by a rotation of order m and complex conjugation. Show that in polar coordinates $x = r\cos\theta, y = r\sin\theta$ the Chevalley generators p_1, p_2 can be taken $p_1 = r^2$ and $p_2 = r^m \cos m\theta$.*

Proposition 3.15. *Let $R \subset V$ be a normalized root system with Weyl group W and let $R_+ \subset R$ be a set of positive roots. Let $\alpha_1, \cdots, \alpha_n$ be the simple roots in R_+ and $s_1, \cdots, s_n$ the corresponding simple reflections in W. Then the simple reflection s_i permutes the set $R_+ - \{\alpha_i\}$.*

Proof. If $\alpha > 0$ is a positive root distinct then $\alpha = \sum x_j \alpha_j$ with $x_j = 0$ or $x_j \geq 1$ for all j. If $\alpha \neq \alpha_i$ then $x_k \geq 1$ for some $k \neq i$. Hence

$$s_i(\alpha) = \alpha - (\alpha, \alpha_i)\alpha_i = \sum y_j \alpha_j$$

with $y_j = x_j$ for all $j \neq i$. In particular $y_k = x_k \geq 1$ and so $s_i(\alpha) > 0$. $\square$

Corollary 3.16. *If $\pi \in P(V)$ be the polynomial defined by*

$$\pi(\xi) = \prod_{\alpha > 0} (\alpha, \xi)$$

then $w \cdot \pi = \varepsilon(w)\pi$ for all $w \in W$ with $\varepsilon(w) = \det(w)$ the sign character of W.

Proof. By the previous proposition we have $s_i \cdot \pi = -\pi$ and the statement follows because the sign character $\varepsilon : W \to \{\pm 1\}$ is the unique homomorphism with $\varepsilon(s_i) = -1$ for all i. $\square$

A polynomial $p \in P(V)$ is called skew invariant for W if $w \cdot p = \varepsilon(w)p$ for all $w \in W$. Skew invariant polynomials p satisfy $s_\alpha \cdot p = -p$ for all $\alpha \in R$. Hence skew invariant polynomials p are of the form $p = \pi q$ with $q \in P^W(V)$. For this reason π is called the elementary skew invariant polynomial. Note that π is independent of the choice of R_+ except for a sign. The square π^2 is an invariant and hence of the form $D(p_1, \cdots, p_n)$. The polynomial D is called the discriminant (with respect to the Chevalley generators $p_1, \cdots, p_n$).

Exercise 3.17. *In the notation of Exercise 3.14 show that the elementary skew invariant becomes $\pi = 2^{3m/2} r^m \sin m\theta$ and so the discriminant is given by $D(p_1, p_2) = 2^{3m}(p_1^m - p_2^2)$.*

Proposition 3.18. *The Jacobian $J(p_1, \cdots, p_n) = \det(\partial p_i / \partial x_j)$ is equal to $c\pi$ for some $c \in \mathbb{R}^\times$.*

Proof. The Jacobian is skew invariant, and therefore $J(p_1, \cdots, p_n) = \pi q$ for some invariant q. The degree of the Jacobian $J(p_1, \cdots, p_n)$ is equal to $\sum(d_j - 1) = N = \deg(\pi)$ by Exercise 3.13, and hence $q \in \mathbb{R}$ is a constant. Finally, since $p_1, \cdots, p_n$ are algebraically independent the Jacobian $J(p_1, \cdots, p_n)$ does not vanish identically, and so $q = c \in \mathbb{R}^\times$. $\square$

Theorem 3.19. *If $R \subset V$ is an irreducible normalized root system with* $\mathrm{rk}(R) = \dim(V) = n$ *and $d_1 \leq \cdots \leq d_n$ are the degrees of the Chevalley generators then $d_j = m_j + 1$ with $m_1 \leq \cdots \leq m_n$ the exponents of R.*

Proof. If $P^W(V) = \mathbb{R}[p_1, \cdots, p_n]$ then $P^W(V_c) = \mathbb{C}[p_1, \cdots, p_n]$ with V_c the complexification of V. Let $c \in W$ be a Coxeter element, and $\xi_1, \cdots, \xi_n$ a basis of eigenvectors of c with eigenvalues ζ^{m_j} with $\zeta = \exp(2\pi i/h)$ and h the Coxeter number. Write $\xi \in V_c$ as $\xi = x_1\xi_1 + \cdots + x_n\xi_n$ and identify $P(V_c) = \mathbb{C}[x_1, \cdots, x_n]$. The action of c on the linear coordinate functions is given by $c \cdot x_j = \zeta^{-m_j} x_j$. From Corollary 2.46 it follows that $\pi(\xi_1) \neq 0$. By the above proposition this implies that $J(p_1, \cdots, p_n)(\xi_1) \neq 0$. After a possible renumbering of $p_1, \cdots, p_n$ (thereby possibly loosing the ordering $d_1 \leq \cdots \leq d_n$) we get $\partial p_1/\partial x_n(\xi_1) \neq 0, \cdots, \partial p_n/\partial x_1(\xi_1) \neq 0$. But the relation

$$\partial p_j/\partial x_{n+1-j}(1, 0, \cdots, 0) \neq 0$$

implies that the monomial $x_1^{d_j-1} x_{n+1-j}$ occurs in p_j with some nonzero coefficient. Using $c \cdot p_j = p_j$ we find $d_j - 1 + m_{n+1-j} \equiv 0$ modulo h. Since $m_j + m_{n+1-j} = h$ we get $d_j - 1 \equiv m_j$ modulo h. Finally $\sum(d_j - 1) = N$ and since $\sum m_j = nh/2 = N$ we conclude $d_j - 1 = m_j$. $\square$

The sequence of exponents $1 \leq m_1 \leq \cdots \leq m_n \leq h - 1$ for the various irreducible root systems have been tabulated in Theorem 2.22 based on the calculations in Section 2.5. Hence the degrees of the Chevalley generators can be read of from that theorem.

Corollary 3.20. *The orders $|W| = d_1 \cdots d_n$ of the finite irreducible real reflection groups W are*

X_n	A_n	B_n	D_n	E_6	E_7
$\lvert W\rvert$	$(n+1)!$	$2^n n!$	$2^{n-1} n!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5. \cdot 7$
X_n	E_8	F_4	H_3	H_4	$\mathrm{I}_2(m)$
$\lvert W\rvert$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^2$	$2^3 \cdot 3 \cdot 5$	$2^6 \cdot 3^2 \cdot 5^2$	$2m$

For example the order of $W(\mathrm{E}_8)$ is equal to 696 729 600.

3.3 Exponential Invariant Theory

Throughout this section $R \subset V$ will be an integral root system. Let R_+ be a set of positive roots, and $\alpha_1, \cdots, \alpha_n$ the corresponding basis of simple roots. Any positive root $\alpha > 0$ is of the form $\alpha = \sum n_i \alpha_i$ with $n_i \in \mathbb{N} = \{0, 1, 2, \cdots\}$. For $\alpha \in R$ let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ be the corresponding coroot, and

$R^\vee = \{\alpha^\vee; \alpha \in R\}$ the coroot system. This is again an integral root system, with positive subset $R_+^\vee = \{\alpha^\vee; \alpha \in R_+\}$ and simple coroots $\alpha_1^\vee, \cdots, \alpha_n^\vee$.

The dual basis λ_i of the basis $\alpha_j^\vee$ of simple coroots is called the basis of fundamental weights, and the weight lattice $P = \sum_i \mathbb{Z}\lambda_i$ is the dual lattice of the coroot lattice $Q^\vee = \mathbb{Z}R^\vee = \sum_i \mathbb{Z}\alpha_i^\vee$. It is clear from the definition of integral root system that the root lattice $Q = \mathbb{Z}R = \sum_i \mathbb{Z}\alpha_i$ is a sublattice of the weight lattice P of R. The cone $P_+ = \sum_i \lambda_i$ is called the cone of dominant weights.

Definition 3.21. *The Abelian group $T = V_c/2\pi i Q^\vee$ with $V_c = \mathbb{C} \otimes_{\mathbb{R}} V$ the complexification of V is called the complex torus with character lattice P. For $\lambda \in P$ we denote by e^λ the holomorphic character of T given by*

$$e^\lambda : T \to \mathbb{C}^\times,\ e^\lambda(t) = t^\lambda = e^{(\lambda, \log t)}$$

with $\xi = \log t \in V_c$ and $t = \xi + 2\pi i \mathbb{Q}^\vee \in T$. We denote by

$$\mathbb{Z}[T] = \{f = \sum a_\lambda e^\lambda; a_\lambda \in \mathbb{Z}, \sum a_\lambda^2 < \infty\}$$

the ring of Laurent polynomials on T with integral coefficients. Of course the product structure on $\mathbb{Z}[T]$ amounts to $e^\lambda e^\mu = e^{\lambda+\mu}$ and $e^0 = 1$.

The Weyl group W acts on $P \subset V$ and hence also on $\mathbb{Z}[T]$ by $w \cdot e^\lambda = e^{w\lambda}$. Equivalently this action can be defined by the usual formula $(w \cdot f)(t) = f(w^{-1}t)$ for the action on a function space. The following theorem is the exponential analogue of the Chevalley theorem, but the proof is in fact much easier.

Theorem 3.22. *For $\lambda \in P_+$ the monomial symmetric function m_λ defined by $m_\lambda = \sum_{\mu \in W\lambda} e^\mu$ form an integral basis for $\mathbb{Z}^W[T]$. Moreover the ring of symmetric Laurent polynomials $\mathbb{Z}^W[T]$ is a polynomial ring of the form*

$$\mathbb{Z}^W[T] = \mathbb{Z}[m_{\lambda_1}, \cdots, m_{\lambda_n}]$$

with $m_{\lambda_1}, \cdots, m_{\lambda_n}$ the fundamental monomial symmetric functions.

Proof. We shall need that P_+ is a strict fundamental domain for the action of W on P, that is $W \cdot P_+ = P$ and $\{W \cdot \lambda\} \cap P_+ = \lambda$ for all $\lambda \in P_+$. This will be proved in the next chapter as Theorem 4.17 in the greater generality of the action of a Coxeter group on the Tits cone. Hence the first statement is obvious.

Let $\leq$ be the partial ordering on P defined by $\mu \leq \lambda$ if $\lambda - \mu \in \mathbb{N}R_+$. This is the integral analogue of the partial ordering introduced in Section 2.3. Since $w\lambda \leq \lambda$ for $\lambda \in P_+$ and all $w \in W$ we get

$$m_{\lambda_1}^{k_1} \cdots m_{\lambda_n}^{k_n} = m_{k_1\lambda_1+\cdots+k_n\lambda_n} + \cdots$$

with $\cdots$ denoting an integral combination of those m_μ with $\mu \in P_+$ and $\mu < k_1\lambda_1 + \cdots + k_n\lambda_n$. On the one hand we have the standard basis m_λ for $\mathbb{Z}^W[T]$ with $\lambda \in P_+$. On the other hand we have a candidate basis $m_{\lambda_1}^{k_1} \cdots m_{\lambda_n}^{k_n}$ for $\mathbb{Z}^W[T]$ with $k_1, \cdots, k_n \in \mathbb{N}$. The latter candidate basis is expressed in the standard basis by a unipotent upper triangular matrix with integral coefficients. But those matrices have an inverse of the same kind. Hence $m_{\lambda_1}^{k_1} \cdots m_{\lambda_n}^{k_n}$ with $k_1, \cdots, k_n \in \mathbb{N}$ is also a basis of $\mathbb{Z}^W[T]$. But this amounts to the statement $\mathbb{Z}^W[T] = \mathbb{Z}[m_{\lambda_1}, \cdots, m_{\lambda_n}]$. □

Definition 3.23. *The vector $\rho = \sum_i \lambda_i$ is called the Weyl vector and the Laurent polynomial $\Delta = e^\rho \prod_{\alpha>0}(1 - e^{-\alpha})$ is called the Weyl denominator.*

Exercise 3.24. *Show that $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha$. Show that Δ transforms under W according to the sign character.*

Exercise 3.25. *Show that the Laurent polynomials $\sum_{w\in W} \varepsilon(w)e^{w(\lambda+\rho)}$ with $\lambda \in P_+$ form an integral basis for the collection of skew invariant Laurent polynomials in $\mathbb{Z}[T]$. Conclude that*

$$\Delta = \sum_{w\in W} \varepsilon(w)e^{w\rho}$$

which is called the Weyl denominator formula. Show that transition from the set of skew invariant Laurent polynomials Δm_λ with $\lambda \in P_+$ to the previous basis of skew invariant Laurent polynomials is given by a unipotent upper triangular matrix. Conclude that the quotients

$$\chi_\lambda = \frac{\sum_{w\in W} \varepsilon(w)e^{w(\lambda+\rho)}}{\Delta}$$

form an integral basis for the set of invariant Laurent polynomials $\mathbb{Z}^W[T]$. The functions χ_λ are called the Weyl characters.

Exercise 3.26. *Show that*

$$\lim_{t\to 1} \chi_\lambda(t) = \prod_{\alpha>0} \frac{(\lambda+\rho, \alpha^\vee)}{(\rho, \alpha^\vee)}$$

which is called the Weyl dimension formula. Hint: Evaluate χ_λ on the one dimensional subtorus of T corresponding to $\mathbb{C}\rho^\vee$ with $\rho^\vee = \frac{1}{2}\sum_{\alpha>0}\alpha^\vee$ the dual Weyl vector and use the Weyl denominator formula for the dual root system $R^\vee$.

Note that the left hand side of the Weyl dimension formula is an integer, while the right hand side is a positive rational number.

Exercise 3.27. *Define complex conjugation on $\mathbb{Z}[T]$ by $\overline{e^\lambda} = e^{-\lambda}$ for all $\lambda \in P$. Define a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{Z}[T]$ by*

$$\langle f, g\rangle = \frac{1}{|W|}\mathrm{CT}(f\overline{g}\Delta\overline{\Delta})$$

with $\mathrm{CT}(\sum a_\lambda e^\lambda) = a_0$ the constant term of a Laurent polynomial. Show that the Weyl characters χ_λ are the unique orthonormal basis of $\mathbb{Z}^W[T]$ for the above scalar product, whose values at the identity element of T are all positive.

A complex reductive group is a linear algebraic subgroup $G < \mathrm{GL}_n(\mathbb{C})$ obtained as the complexification of a compact Lie group $U < G$, which is called a compact real form of G. For example $\mathrm{GL}_n(\mathbb{C})$ itself is reductive with compact real form the unitary group $\mathrm{U}_n(\mathbb{C})$. Hermann Weyl observed that restriction induces an equivalence of categories between finite dimensional holomorphic representations of a reductive group G and finite dimensional smooth representations of a compact real form U. In case G (or equivalently U) is connected and has finite center the complexification of a maximal Abelian subgroup of U is a complex torus T in G equipped with an integral root system R in the character lattice L of T with $Q = \mathbb{Z}R < L < P$. In fact the isomorphism class of such G is determined by the isomorphism class of such R, and moreover all integral root systems R occur this way.

By the Schur orthonormality relations for the irreducible characters of a compact topological group Weyl was able to show that the restriction of an irreducible character of G to T is of the form χ_λ for some $\lambda \in P_+$. The fact that irreducible representations of G were classified by their highest weight $\lambda \in P_+$ was obtained before by Élie Cartan by Lie algebraic methods. But for answering refined representation theoretic questions (like computing branching multiplicities for the restriction from G to a connected reductive subgroup $H < G$) the Weyl character formula is an indispensable tool.

4 Coxeter Groups

4.1 Generators and Relations

A subset S of a group G is called a set of generators, if the smallest subgroup of G containing S is equal to G, or equivalently if every element in G is a finite word $s_1^{\epsilon_1} \cdots s_p^{\epsilon_p}$ with $s_1, \cdots, s_p$ not necessarily distinct elements from S and $\epsilon_1, \cdots, \epsilon_p \in \{\pm 1\}$.

Definition 4.1. *A free group on a set X is a group $F(X)$ with $X \subset F(X)$ as set of generators, such that for each group G each set map $\varphi : X \to G$ extends to a homomorphism $\varphi : F(X) \to G$.*

Because $F(X)$ is generated by X this extension is clearly unique. Indeed we have $\varphi(x_1^{\epsilon_1} \cdots x_p^{\epsilon_n}) = \varphi(x_1)^{\epsilon_1} \cdots \varphi(x_p)^{\epsilon_p}$ for each $x_1, \ldots, x_p \in X$ and each $\epsilon_1, \cdots, \epsilon_p \in \{\pm 1\}$. Clearly a set map $\varphi : X \to Y$ extends uniquely to a homomorphism $\varphi : F(X) \to F(Y)$, which in turn implies that a bijection $\varphi : X \to Y$ entends to an isomorphism $\varphi : F(X) \to F(Y)$. Hence, if the free group $F(X)$ on a set X exists, it is unique. However, the existence of $F(X)$ is not trivial.

Theorem 4.2. *For any set X the free group $F(X)$ on X exists.*

Proof. Let X be a set. Consider two sets $X^{\pm}$ both in bijection with X via $x \mapsto x^{\pm}$. Let $W(X)$ be the set of all (finite) words with letters from the alphabet $X^+ \sqcup X^-$. Hence $w \in W(X)$ is of the form $w = x_1^{\epsilon_1} \cdots x_p^{\epsilon_p}$ with $x_1, \cdots, x_p \in X$ and $\epsilon_1, \cdots, \epsilon_p \in \{\pm\}$. Concatenation $u * v$ of words $u, v \in W(X)$ is a product rule on $W(X)$ which is clearly associative and has the empty word as unit element.

A word $w = x_1^{\epsilon_1} \cdots x_p^{\epsilon_p}$ with $x_1, \cdots, x_p \in X$ and $\epsilon_1, \cdots, \epsilon_p \in \{\pm\}$ for which $x_i = x_{i+1}$ and $\epsilon_i = -\epsilon_{i+1}$ is said to have the word

$$\rho_i w = x_1^{\epsilon_1} \cdots x_{i-1}^{\epsilon_{i-1}} x_{i+2}^{\epsilon_{i+2}} \cdots x_p^{\epsilon_p}$$

as the reduction at the place i. A word $w \in W(X)$ is called reduced if at any place it has no further reductions.

Each word $w = x_1^{\epsilon_1} \cdots x_p^{\epsilon_p} \in W(X)$ gives rise by iterated reductions to a reduced word and we claim that the outcome, denoted by ρw, is independent of the reduction procedure. This will be proved by induction on the length p of w. If w is already reduced then the statement is trivial. In particular the case $p = 0, 1$ is clear. Suppose $p \geq 2$ and suppose $\rho_{i_k} \cdots \rho_{i_1} w$ and $\rho_{j_l} \cdots \rho_{j_1} w$ are both reduced for some $k, l \geq 1$. If $i_1 = j_1$ then both reduced words are reductions of the word $\rho_{i_1}w = \rho_{j_1}w$ of length $p-2$, and the statement follows from the induction hypothesis. If $i_1 \neq j_1$ then we may assume that $i_1 < j_1$ by symmetry. If $j_1 = i_1 + 1$ then

$$w = x_1^{\epsilon_1} \cdots x_{i_i-1}^{\epsilon_{i_1-1}} x^{\epsilon} x^{-\epsilon} x^{\epsilon} x_{i_1+3}^{\epsilon_{i_1+3}} \cdots x_p^{\epsilon_p}$$

and so again $\rho_{i_1}w = \rho_{j_1}w$, and we are done by the induction hypothesis. Finally, if $j_1 > i_1 + 1$ then

$$w = x_1^{\epsilon_1} \cdots x_{i_1-1}^{\epsilon_{i_1-1}} x_{i_1}^{\epsilon_{i_1}} x_{i_1}^{-\epsilon_{i_1}} x_{i_1+2}^{\epsilon_{i_1+2}} \cdots x_{j_1-1}^{\epsilon_{j_1-1}} x_{j_1}^{\epsilon_{j_1}} x_{j_1}^{-\epsilon_{j_1}} x_{j_1+2}^{\epsilon_{j_1+2}} \cdots x_p^{\epsilon_p}$$

and so $\rho_{j_1}\rho_{i_1}w = \rho_{i_1}\rho_{j_1}w$. Hence

$$\rho_{i_k} \cdots \rho_{i_1}w = \rho\rho_{i_1}w = \rho\rho_{j_1}\rho_{i_1}w = \rho\rho_{i_1}\rho_{j_1}w = \rho\rho_{j_1}w = \rho_{j_l} \cdots \rho_{j_1}w$$

using the induction hypothesis at all equality signs except the middle one. Hence the reduction ρw is a uniquely defined reduced word.

We can now construct the free group $F(X)$ on the set X. Take for $F(X)$ the set of all reduced words in $W(X)$, and for $u, v \in F(X)$ define the product $uv = \rho(u * v)$ with $u * v$ the concatenation of the two words u, v. It is easily checked that this product is a group structure on $F(X)$ with the empty word as unit element. If $\varphi : X \to G$ is a set map to a group G then we first define $\varphi : X^+ \sqcup X^- \to G$ on the alphabet $X^+ \sqcup X^-$ by $\varphi(x^+) = x$ and $\varphi(x^-) = x^{-1}$ for $x \in X$. Subsequently, if $w = x_1^{\epsilon_1} \cdots x_p^{\epsilon_p} \in F(X)$ then we define $\varphi(w) = \varphi(x_1^{\epsilon_1}) \cdots \varphi(x_p^{\epsilon_p})$ for the desired extension $\varphi : F(X) \to G$, which is easily checked to be a homomorphism. □

Definition 4.3. *Suppose G is a group with $S \subset G$ a set of generators. A surjection $\varphi : X \twoheadrightarrow S$ of sets extends to an epimorphism $\varphi : F(X) \twoheadrightarrow G$ of groups. The elements of* $\ker\varphi$ *are called relators for the set map $\varphi : X \twoheadrightarrow S$. A subset $R \subset \ker(\varphi)$ is called a set of defining relators for the pair (G, S) and the set map $\varphi : X \twoheadrightarrow S$ if the smallest normal subgroup $N(R)$ of $F(X)$ containing R is equal to* $\ker\varphi$*. In this case we say that $(\varphi : X \twoheadrightarrow S, R)$ is a presentation of the pair (G, S) by generators and relations.*

Suppose G is a group with $S = \{s_i; i \in I\} \subset G$ a set of generators with index set I. If we denote $X = \{x_i; i \in I\}$ a set with the same index set I then the bijection $\varphi : X \to S$ defined by $\varphi(x_i) = s_i$ for all $i \in I$ extends to a epimorphism $\varphi : F(X) \twoheadrightarrow G$. For R a subset of $\ker\varphi$ let $N(R)$ be the smallest normal subgroup of G containing R. Clearly $(\varphi : X \twoheadrightarrow S, R)$ is a presentation of the pair (G, S) if and only if the natural epimorphism $F(X)/N(R) \twoheadrightarrow G$ is an isomorphism. In particular, if G is a finite group of order $\#G$ then this happens if and only if the order of the abstract group $F(X)/N(R)$ is at most $\#G$.

For any set X and any subset R of $F(X)$ let $N(R)$ be the smallest normal subgroup of $F(X)$ containing R. The factor group $F(X)/N(R)$ has the factor space $Y = XN(R)$ as set of generators, and R is a set of defining relations for $(F(X)/N(R), Y)$ and $\varphi : X \twoheadrightarrow Y$ the natural map. Note that $X \twoheadrightarrow Y$ need not be a bijection.

Example 4.4. *The dihedral group $\mathcal{D}_m$ of order $2m$ has the set $S = \{s_1, s_2\}$ of simple reflections as generators, and the relations*

$$s_1^2 = 1 \ , \ s_2^2 = 1 \ , \ (s_1 s_2)^m = 1$$

trivially hold in $\mathcal{D}_m$ and give a presentation of $(\mathcal{D}_m, S)$. Indeed, let $F(X)$ be the free group on $X = \{x_1, x_2\}$ and let $N(R)$ be the smallest normal subgroup of $F(X)$ containing $R = \{x_1^2, x_2^2, (x_1x_2)^m\}$. The group $F(X)/N(R)$ has the set $Y = \{y_1 = x_1N(R), y_2 = x_2N(R)\}$ as generators, and the relations

$$y_1^2 = 1 \ , \ y_2^2 = 1 \ , \ (y_1 y_2)^m = 1$$

hold in $F(X)/N(R)$ by construction. Hence $F(X)/N(R)$ has minimal length words in y_1, y_2 of length at most m. For length 0 there is only the identity element 1, for each length i with $1 \le i \le m-1$ there are two (either $y_1y_2y_1 \cdots y_2$ and $y_2y_1y_2 \cdots y_1$ for i even, or $y_1y_2y_1 \cdots y_1$ and $y_2y_1y_2 \cdots y_2$ for i odd), and finally for length m there is just one word $y_1y_2y_1 \cdots = y_2y_1y_2 \cdots$ making altogether at most $2m$ elements.

Example 4.5. *The symmetric group $\mathcal{S}_{n+1}$ on $(n+1)$ letters is the Weyl group $W(A_n)$ with the set $S = \{s_1 = (1\ 2), \cdots, s_n = (n\ n+1)\}$ of simple reflections as generators. The relations (with $1 \le i, j \le n$ and $|i-j| \ge 2$)*

$$s_i^2 = 1 \ , \ (s_i s_j)^2 = 1 \ , \ (s_i s_{i+1})^3 = 1$$

trivially hold in $\mathcal{S}_{n+1}$ and we claim that this is a presentation of $(\mathcal{S}_{n+1}, S)$. Let $F(X)$ be the free group on $X = \{x_1, \cdots, x_n\}$ and $N(R)$ be the smallest normal subgroup of $F(X)$ containing the set

$$R = \{x_i^2, (x_ix_j)^2, (x_ix_{i+1})^3; 1 \le i, j \le n, |i-j| \ge 2\} \ .$$

Let H_k be the subgroup of $F(X)/N(R)$ generated by the elements

$$y_1 = x_1N(R), \cdots, y_k = x_kN(R)$$

for $k = 1, \cdots, n$. We will show that $\#H_k \leq (k+1)!$ by induction on k. This proves our claimed presentation of $(\mathcal{S}_{n+1}, S)$ because $F(X)/N(R) = H_n$ has order at most $(n+1)!$ as should.

For $k = 1$ we have $H_1 = \{1, y_1\}$ and $\#H_1 \leq 2$ is clear. For $p \leq k, l \leq k$ we have

$$y_p y_l y_{l+1} \cdots y_k H_{k-1} = \begin{cases} y_l y_{l+1} \cdots y_k H_{k-1} & \text{if } p \leq l-2 \\ y_{l-1} y_l \cdots y_k H_{k-1} & \text{if } p = l-1 \\ y_{l+1} y_{l+2} \cdots y_k H_{k-1} & \text{if } p = l \\ y_l y_{l+1} \cdots y_k H_{k-1} & \text{if } p > l \end{cases}$$

and also

$$y_p H_{k-1} = \begin{cases} H_{k-1} & \text{if } p < k \\ y_k H_{k-1} & \text{if } p = k \end{cases}$$

using $y_i y_j = y_j y_i, y_i y_{i+1} y_i = y_{i+1} y_i y_{i+1}$ for all $1 \leq i, j \leq n$ and $|i-j| \geq 2$. Hence the set

$$y_1 y_2 \cdots y_k H_{k-1} \cup y_2 y_3 \cdots y_k H_{k-1} \cup \cdots \cup y_k H_{k-1} \cup H_{k-1}$$

is invariant under left multiplication by $y_1, \cdots, y_k$, and hence equals H_k. Therefore we get

$$\#H_k \leq (k+1)\#H_{k-1} \leq (k+1)k! = (k+1)!$$

by the induction hypothesis. This completes the proof.

Definition 4.6. *Let I be an index set and $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix, that is $m_{ii} = 1$ for all i and $m_{ij} = m_{ji} \in \mathbb{Z} \sqcup \infty$ and ≥ 2 for all $i \neq j$. Let $X = \{x_i; i \in I\}$ be a set whose elements are indexed by I and let*

$$R = \{(x_i x_j)^{m_{ij}}; i, j \in I, m_{ij} < \infty\}$$

as subset of the free group $F(X)$ on X. Let $N(R)$ be the smallest normal subgroup of $F(X)$ containing R and let Y be the corresponding generator set $\{y_i = x_i N(R); i \in I\}$ of the factor group $W_a = F(X)/N(R)$. The pair (W_a, Y) is called the Coxeter system associated with the Coxeter matrix M. The group W_a is called the abstract Coxeter group and the elements of Y are called the Coxeter generators.

The dihedral group $\mathcal{D}_m \cong W(I_2(m)) \subset \mathrm{O}(\mathbb{R}^2)$ of order $2m$ and the symmetric group $\mathcal{S}_{n+1} \cong W(A_n) \subset \mathrm{O}(V)$ of order $(n+1)!$ (with V the zero coordinate sum hyperplane in $\mathbb{R}^{n+1}$) are examples of Coxeter groups with the simple reflections as Coxeter generators by the two examples above. A nice text book on generators and relations with an abundance of examples is the book by Coxeter and Moser [22].

4.2 The Tits Theorem

Suppose I is an index set and $M = (m_{ij})_{i,j\in I}$ is a Coxeter matrix. Let V be the real vector space with basis $B = \{\alpha_i; i \in I\}$ of simple roots equiped with the symmetric bilinear form $(\cdot,\cdot)$ given by

$$(\alpha_i, \alpha_j) = -2\cos(\pi/m_{ij})$$

for $i, j \in I$. In case $m_{ij} = \infty$ the number $(\alpha_i, \alpha_j) \leq -2$ can in fact be freely prescribed. For $i \in I$ the linear transformation $s_i : V \to V$ defined by

$$s_i(\lambda) = \lambda - (\lambda, \alpha_i)\alpha_i$$

is an involution and called the reflection in the mirror $\{\lambda \in V; (\lambda, \alpha_i) = 0\}$. The elements of $S = \{s_i; i \in I\}$ satisfy inside the orthogonal group $\mathrm{O}(V) = \{g \in \mathrm{GL}(V); (g\lambda, g\mu) = (\lambda, \mu)\ \forall \lambda, \mu \in V\}$ the relations

$$(s_i s_j)^{m_{ij}} = 1\ \forall\, i, j \in I, m_{ij} < \infty$$

We denote by W_g the subgroup of $\mathrm{O}(V)$ generated by the set S of Coxeter generators, and call it the geometric Coxeter group associated with the Coxeter matrix M.

The abstract Coxeter group W_a associated with M was defined before as $W_a = F(X)/N(R)$ with $X = \{x_i; i \in I\}$ a set whose elements are indexed by I, $F(X)$ the free group on X and $N(R)$ the smallest normal subgroup of $F(X)$ containing the set

$$R = \{(x_i x_j)^{m_{ij}}; i, j \in I, m_{ij} < \infty\}\ .$$

The group W_a is generated by the set $Y = \{y_i = x_i N(R); i \in I\}$ and the natural map $X \to Y; x_i \mapsto y_i$ is a surjection. By construction there is an epimorphism

$$W_a \twoheadrightarrow W_g\ ,\ y_i \mapsto s_i\ \forall\, i \in I$$

from the abstract Coxeter group onto the geometric Coxeter group, which is called the geometric epimorphism. The representation of W_a on the vector space V is called the geometric representation of the abstract Coxeter group.

An immediate consequence of the existence of the geometric epimorphism is that the natural surjection $X \to Y, x_i \mapsto y_i$ is a bijection. Indeed, the compositions $X \to Y \to S, x_i \mapsto y_i \mapsto s_i$ are bijections, since $s_i \neq s_j$ for all $i \neq j$.

The main result of this section is the next statement, which is due to Jacques Tits and called the Tits theorem [9].

Theorem 4.7. *The geometric epimorphism $W_a \twoheadrightarrow W_g$ is an isomorphism of groups.*

Because of this theorem we simply write $W \cong W_a \cong W_g$ and call it the Coxeter group associated with the Coxeter matrix M. Likewise we write (W, S) for the Coxeter system with S the set of Coxeter generators.

Suppose W is a group with unit element 1, and Y a generating subset of W with $1 \notin Y$ and $y^2 = 1$ for all $y \in Y$. Each element $w \in W$ can be written in the form $w = y_1 \cdots y_p$ for some (not necessarily distinct) $y_1, \cdots, y_p \in Y$. If p is as small as possible it is called the length of w en we denote $p = l(w)$. Any such expression for w as product of $p = l(w)$ elements of Y is called a reduced expression. Clearly $l(1) = 0$ since by convention the product over the empty set equals 1, and $l(w) = 1$ if and only if $w \in Y$.

Lemma 4.8. *For $w, v \in W$ and $y \in Y$ we have*

1. $l(w) = l(w^{-1})$
2. $l(wv) \leq l(w) + l(v)$
3. $l(wv) \geq l(w) - l(v)$
4. $l(w) - 1 \leq l(wy) \leq l(w) + 1$

Proof. The easy arguments are given as follows.

1. This is clear since $y^{-1} = y$ for all $y \in Y$. Indeed if $w = y_1 \cdots y_p$ is a reduced expression then also $w^{-1} = y_p \cdots y_1$ is a reduced expression.
2. If $w = y_1 \cdots y_p$ and $v = x_1 \cdots x_q$ are both reduced expressions then $wv = w = y_1 \cdots y_p x_1 \cdots x_q$ and so $l(wv) \leq p + q$.
3. This follows from the previous item by replacing v by v^{-1} and subsequently w by wv, and using that $l(v^{-1}) = l(v)$.
4. This is clear from previous two items with $v = y \in Y$.

This completes the proof of the lemma. □

Now let (W, S) be an abstract Coxeter group given by generators and relations. If $\mathcal{C}_2 = \{\pm 1\}$ denotes the multiplicative group of order 2 then the map $\varepsilon(s) = -1$ for $s \in S$ extends to a homomorphism $\varepsilon : W \to \mathcal{C}_2$, which is called the sign character of the Coxeter group (W, S). Clearly $\varepsilon(w) = 1$ if

$l(w)$ is even, and $\varepsilon(w) = -1$ if $l(w)$ is odd. Hence the last item of the above lemma can be refined to

$$l(ws) = l(w) \pm 1$$

for all $w \in W$ and $s \in S$. For $I \subset S$ the subgroup of W generated by I is denoted by W_I, and is called a parabolic subgroup of (W, S). Let W^I be the set of all $w \in W$ with $l(wt) > l(w)$ for all $t \in I$. If l_I denotes the length function of the pair (W_I, I) then it is clear that $l(w) \leq l_I(w)$ for all $w \in W_I$.

Proposition 4.9. *Each $w \in W$ can be written in the form $w = uv$ with $u \in W^I, v \in W_I$ and $l(w) = l(u) + l_I(v)$. The elements of W^I are called the minimal length coset representatives for the parabolic subgroup W_I.*

Proof. If $l(w) = 0$ then $w = 1 = 1 \cdot 1$ is the required factorization. Now suppose $l(w) \geq 1$ and proceed by induction on $l(w)$. If $w \in W^I$ then $w = w \cdot 1$ is the required factorization.

Hence we may assume that $l(wt) < l(w)$ for some $t \in I$. By the induction hypothesis we can write $wt = u'v'$ with $u' \in W^I, v' \in W_I$ and $l(wt) = l(u') + l_I(v')$. Then $w = uv$ with $u = u' \in W^I$ and $v = v't \in W_I$. We claim that $l_I(v) = l_I(v') + 1$. If not then $l_I(v) \leq l_I(v')$ and therefore

$$l(w) = l(wt) + 1 = l(u') + l_I(v') + 1 \geq$$
$$l(u) + l_I(v) + 1 \geq l(u) + l(v) + 1 \geq l(w) + 1$$

which is a contradiction. Hence $l_I(v) = l_I(v') + 1$ and therefore

$$l(w) = l(wt) + 1 = l(u') + l_I(v') + 1 = l(u) + l_I(v)$$

and the proposition follows. □

Let V be the vector space with basis $\{\alpha_s; s \in S\}$ and symmetric bilinear form $(\alpha_s, \alpha_t) = -2\cos(\pi/m_{st})$ for $s, t \in S$ (except when $m_{st} = \infty$, in which case (α_s, α_t) can be any real number ≤ -2). The geometric action of $s \in S$ on $\lambda \in V$ is given by $s(\lambda) = \lambda - (\lambda, \alpha_s)\alpha_s$, and extends to the geometric action of W on V.

Definition 4.10. *The (normalized) root system $R \subset V$ of the Coxeter group (W, S) is defined as $R = \{w(\alpha_s); w \in W, s \in S\}$.*

Clearly $R = -R$. A root $\alpha \in R$ can be written as $\alpha = \sum x_s \alpha_s$ for some unique $x_s \in \mathbb{R}$. If either $x_s = 0$ or $x_s \geq 1$ for all $s \in S$ then α is called a positieve root, and we write $\alpha > 0$. If $-\alpha$ is a positive root then α is called a negative root, and we write $\alpha < 0$. The next theorem is the key technical part in the proof of the Tits theorem.

Theorem 4.11. *Let $w \in W$ and $s \in S$. If $l(ws) > l(w)$ then $w(\alpha_s) > 0$, and if $l(ws) < l(w)$ then $w(\alpha_s) < 0$.*

Proof. Observe that the second statement follows from the first applied to ws in place of w. Indeed $l(ws) < l(w)$ if and only if $l((ws)s) > l(ws)$, while $w(\alpha_s) < 0$ if and only if $ws(\alpha_s) > 0$.

We shall prove the first statement by induction on the length $l(w)$ of $w \in W$. So assume that $l(ws) > l(w)$ for some $w \in W$ and $s \in S$. If $l(w) = 0$ then $w = 1$ and there is nothing to prove. If $l(w) \geq 1$ we can find $t \in S$ with $l(wt) = l(w) - 1$, say by choosing t the last factor in a reduced expression for w. By assumption $l(ws) = l(w) + 1$, and so $s \neq t$. If we denote $I = \{s, t\}$ then W_I is a dihedral group.

By the previous proposition we can write $w = uv$ with $u \in W^I$, $v \in W_I$ and $l(w) = l(u) + l_I(v)$. By definition $u \in W^I$ implies $l(us) = l(ut) = l(u) + 1$. Hence $v \neq 1$ and so $l(u) < l(w)$. By induction $u(\alpha_s) > 0$ and $u(\alpha_t) > 0$. The statement of the theorem will follow if we can show that $v(\alpha_s) = x_s\alpha_s + x_t\alpha_t > 0$. Indeed the action of u on $v(\alpha_s)$ then implies that $w(\alpha_s) = x_s u(\alpha_s) + x_t u(\alpha_t) > 0$ as required. We claim that $l_I(vs) = l_I(v)+1$. Otherwise we have $l_I(vs) \leq l_I(v)$ and then also

$$l(ws) = l(uvs) \leq l(u) + l(vs) \leq l(u) + l_I(vs) \leq l(u) + l_I(v) = l(w)$$

which gives a contradiction.

If we have $m := m_{st} < \infty$ then $(\alpha_s, \alpha_t) = -2\cos(\pi/m) > -2$. Since $l_I(vs) = l_I(v) + 1$ the reduced expression $v = \cdots tst$ can not start with s on the right and so has length $0 \leq p = l_I(v) \leq m - 1$. We claim that

$$v(\alpha_s) = \frac{\sin(p\pi/m)}{\sin(\pi/m)}\alpha_s + \frac{\sin((p+1)\pi/m)}{\sin(\pi/m)}\alpha_t$$

if p is odd, and

$$v(\alpha_s) = \frac{\sin((p+1)\pi/m)}{\sin(\pi/m)}\alpha_s + \frac{\sin(p\pi/m)}{\sin(\pi/m)}\alpha_t$$

if p is even. This is easily proved by induction on p using the identity

$$2\cos(\pi/m)\frac{\sin((p+1)\pi/m)}{\sin(\pi/m)} - \frac{\sin(p\pi/m)}{\sin(\pi/m)} = \frac{\sin((p+2)\pi/m)}{\sin(\pi/m)}$$

which is immediate from

$$(\zeta + \zeta^{-1})(\zeta^{p+1} - \zeta^{-p-1}) - (\zeta^p - \zeta^{-p}) = \zeta^{p+2} - \zeta^{-p-2}$$

with $\zeta = \exp(\pi i/m)$. Hence $v(\alpha_s) > 0$ as should, since $0 \le p \le m-1$.

If $m = \infty$ then $(\alpha_s, \alpha_t) \le -2$. In case $(\alpha_s, \alpha_t) = -2$ one can take the limit $m \to \infty$ in the formulas for $v(\alpha_s)$ to get

$$v(\alpha_s) = p\alpha_s + (p+1)\alpha_t \ , \ v(\alpha_s) = (p+1)\alpha_s + p\alpha_t$$

if $p = l(v)$ is odd or even respectively, and the same argument works. In case $(\alpha_s, \alpha_t) = -2\cosh\theta < -2$ with $\theta > 0$ it is easily checked that the same formulas for $v(\alpha_s)$ hold as in the case $m < \infty$ with sin and cos replaced by sinh and cosh respectively and with π/m replaced by θ. Details are left to the reader. $\square$

Corollary 4.12. *We have $R = R_+ \sqcup R_-$ with $R_+ = \{\alpha \in R; \alpha > 0\}$ and $R_- = \{\alpha \in R; \alpha < 0\}$ the set of positive and negative roots in R.*

Proof. This follows immediately from the above theorem, because for $w \in W$ and $s \in S$ we have either $l(ws) > l(w)$ or $l(ws) < l(w)$. $\square$

Corollary 4.13. *The element $s \in S$ permutes the set $R_+ - \{\alpha_s\}$.*

Proof. Every $\alpha \in R_+$ different from α_s is of the form $\alpha = \sum x_t\alpha_t$ with $x_t \ge 0$ for all $t \in S$ and $x_r \ge 1$ for some $r \in S, r \ne s$. In turn we get $s(\alpha) = \alpha - (\alpha, \alpha_s)\alpha_s = \sum y_t\alpha_t$ with $y_t = x_t$ for all $t \ne s$. In particular $y_r = x_r \ge 1$ and hence $s(\alpha) > 0$ by the previous corollary. $\square$

Theorem 4.14. *For $w \in W$ the length $l(w)$ is equal to the cardinality of the set $R_+ \cap w^{-1}R_- = \{\alpha > 0; w\alpha < 0\}$.*

Proof. By induction on the length $l(w)$ of $w \in W$. If $l(w) = 0$ then $w = 1$ and $R_+ \cap w^{-1}R_- = R_+ \cap R_- = \emptyset$ has cardinality 0. Suppose $w \in W$ has length $l(w) \ge 1$ and choose $s \in S$ with $l(ws) = l(w) - 1$. By Theorem 4.11 we get $w(\alpha_s) < 0$ and therefore

$$R_+ \cap w^{-1}R_- = \{\alpha_s\} \cup [(R_+ - \{\alpha_s\}) \cap w^{-1}R_-]$$

is equal to

$$\{\alpha_s\} \cup [s(R_+ - \{\alpha_s\}) \cap s(ws)^{-1}R_-]$$

by Corollary 4.13, and equals

$$\{\alpha_s\} \cup s[(R_+ - \{\alpha_s\}) \cap (ws)^{-1}R_-] = \{\alpha_s\} \cup s[(R_+ \cap (ws)^{-1}R_-]$$

since $ws(\alpha_s) > 0$. Hence

$$\#(R_+ \cap w^{-1}R_-) = 1 + \#(R_+ \cap (ws)^{-1}R_-) = 1 + l(ws) = l(w)$$

by the induction hypothesis. $\square$

The Tits theorem, saying that the geometric epimorphism $W_a \twoheadrightarrow W_g$ is an isomorphism, is an immediate consequence. Indeed, if $w \in W_a$ lies in the kernel of the geometric epimorphism then $R_+ \cap w^{-1}R_- = R_+ \cap R_- = \emptyset$ has cardinality zero and so $l(w) = 0$, which in turn implies that $w = 1$.

Remark 4.15. *If $M = (m_{ij})_{i,j\in I}$ is a Coxeter matrix then the associated symmetric Gram matrix $G(M) = (g_{ij})_{i,j\in I}$ of the basis $\{\alpha_i, i \in I\}$ of simple roots was defined by*

$$g_{ij} = (\alpha_i, \alpha_j) = -2\cos(\pi/m_{ij})$$

for $i, j \in I$ with the convention that in case $m_{ij} = \infty$ the coefficient $g_{ij} \le -2$ can be freely chosen. Although it need not be true that the Gram matrix $G(M)$ is determined by the Coxeter matrix M we shall nevertheless call $G(M)$ the Gram matrix for the Coxeter matrix M. Likewise we shall usually write W for the associated Coxeter group, either for the abstract Coxeter group W_a defined by generators and relations or for its geometric realization W_g as a reflection group.

4.3 The Dual Geometric Representation

Let $W \to \mathrm{GL}(V)$ be the geometric representation of a Coxeter group (W, S). The dual geometric representation $W \to \mathrm{GL}(V^*)$ on the dual vector space $V^* = \mathrm{Hom}(V, \mathbb{R})$ is defined by $(w\xi, \lambda) = (\xi, w^{-1}\lambda)$ for $\lambda \in V$ and $\xi \in V^*$. Here (ξ, λ) denotes (by abuse of notation) the natural pairing between the vectors $\xi \in V^*$ and $\lambda \in V$. For $s \in S$ we put

$$\begin{aligned} H_s &= \{\xi \in V^*; (\xi, \alpha_s) = 0\} \\ H_s^+ &= \{\xi \in V^*; (\xi, \alpha_s) > 0\} \\ H_s^- &= \{\xi \in V^*; (\xi, \alpha_s) < 0\} \end{aligned}$$

and H_s is called the wall between the two half spaces $H_s^\pm$. Clearly we have a disjoint union $V^* = H_s^+ \sqcup H_s \sqcup H_s^-$, and $s(\xi) = \xi$ for $\xi \in V^*$ if and only if $\xi \in H_s$. The hyperplane H_s is also called the mirror of the reflection $s \in S$. For $I \subset S$ the set

$$C_I = (\bigcap_{s\in I} H_s) \cap (\bigcap_{s\notin I} H_s^+)$$

is a simplicial cone of codimension $\#I$, called the facet of type I. For example $C_S = \{0\}$ is just the origin, and for $I = \emptyset$ we simply write

$$C = C_\emptyset = \bigcap_{s\in S} H_s^+ \subset V^*$$

and call it the positive chamber. The closure D of the positive chamber C is the disjoint union

$$D = \bigsqcup_{I \subset S} C_I$$

of all facets. Note that D is a closed simplicial cone in V^*. The facets of D of codimension one are called the walls, and of codimension two are called the plinths of D. The next result is just a reformulation of Theorem 4.11.

Theorem 4.16. *Let $w \in W$ and $s \in S$. If $l(sw) > l(w)$ then $w(C) \subset H_s^+$, and if $l(sw) < l(w)$ then $w(C) \subset H_s^-$.*

Proof. If $l(sw) > l(w)$ for $w \in W, s \in S$ then we also have $l(w^{-1}s) > l(w^{-1})$, which by Theorem 4.11 implies that $w^{-1}(\alpha_s) > 0$. Hence we get $(w\xi, \alpha_s) = (\xi, w^{-1}(\alpha_s)) > 0$ for all $\xi \in C$, and so $w(C) \subset H_s^+$. The other statement follows similarly. $\square$

Theorem 4.17. *Let $w \in W$ and $I, J \subset S$. If $C_I \cap w(C_J)$ is not empty then $I = J$ and $w \in W_I$.*

Proof. By induction on $l(w)$. If $l(w) = 0$ then $w = 1$ and $C_I \cap C_J$ not empty implies $I = J$. Now suppose $l(w) \geq 1$ and choose $s \in S$ with $l(sw) < l(w)$. By the previous theorem we have $w(C) \subset H_s^-$, which in turn implies that $w(D) \subset H_s \sqcup H_s^-$. Since $D \subset H_s \sqcup H_s^+$ we get $D \cap w(D) \subset H_s$.

We can draw two conclusions from this inclusion. In the first place, s fixes all elements of $D \cap w(D) \supset C_I \cap w(C_J)$, which is not empty by assumption, and therefore $s \in I$ or equivalently $s(C_I) = C_I$. In the second place, the set $C_I \cap sw(C_J) = s(C_I \cap w(C_J))$ is not empty. By the induction hypothesis we conclude that $I = J$ and $sw \in W_I$. Together with $s \in I$ this implies $w \in W_I$. $\square$

Remark 4.18. *The above proof also shows that for each $w \in W_I$ and for each reduced expression $w = s_1 \cdots s_p$ in (W, S) all factors $s_1, \cdots, s_p \in I$. In particular $l_I(w) = l(w)$ for all $w \in W_I$ in the notation of Proposition 4.9.*

Corollary 4.19. *The dual geometric action of W on the Tits cone $Y \subset V^*$ defined by*

$$Y = \bigcup_{w \in W} w(D)$$

has D as a strict fundamental domain in the sense that each orbit of W in Y intersects D in a unique point.

Corollary 4.20. *The dual geometric Coxeter group W is a discrete subgroup of* $\mathrm{GL}(V^*)$*, and by duality the geometric Coxeter group is a discrete subgroup of* $\mathrm{GL}(V)$*.*

Proof. Fix $\xi \in C$. Then the set $\{g \in \mathrm{GL}(V^*); g(\xi) \in C\}$ is an open neighborhood of 1 in $\mathrm{GL}(V^*)$ containing only the element 1 from W. In turn this implies that $W \subset \mathrm{GL}(V^*)$ is a discrete group. $\square$

Corollary 4.21. *The invariant symmetric bilinear form on the vector space V with basis $\{\alpha_s; s \in S\}$ is positive definite if and only the Coxeter group W is a finite group, or equivalently if R is a finite normalized root system.*

Proof. If the invariant symmetric bilinear form $(\cdot,\cdot)$ on V is an inner product then W is finite as a discrete subgroup of the compact orthogonal group $\mathrm{O}(V)$.

Conversely, if W is a finite group then there exists an inner product $(\cdot,\cdot)$ on V that is invariant under W (by averaging over W an arbitrary inner product on V). By possibly rescaling the root lengths we may assume that $(\alpha,\alpha) = 2$ for all $\alpha \in R$. But then this inner product coincides with the standard invariant symmetric bilinear form given on a basis of simple roots by $(\alpha_s,\alpha_t) = -2\cos(\pi/m_{st})$ with m_{st} the order of the element st in W. $\square$

Remark 4.22. *We have classified the connected elliptic Coxeter diagrams in Section 2.4 and indicated in a case by case manner that they all occur of Coxeter diagrams of finite normalized root systems. The above corollary provides a uniform proof of this latter fact.*

In the proof of Corollary 2.46 we have used that the stabilizer in W of any vector in the open Weyl chamber V_+ is trivial. This follows from Theorem 4.17 applied in case the symmetric bilinear $(\cdot,\cdot)$ on V is positive definite, and so the Weyl group W and the root system R are finite. Hence $C = V_+$ is the positive chamber (under the identification $V \cong V^*$) via the inner product).

The next theorem gives a necessary and sufficient condition for a point $\xi \in V^*$ to lie in the Tits cone Y.

Theorem 4.23. *For $\xi \in V^*$ we denote $R(\xi) = \{\alpha \in R_+; (\xi,\alpha) < 0\}$. Then we have $\xi \in Y$ if and only if $R(\xi)$ is a finite set.*

Proof. If $\xi \in Y$ then $w(\xi) \in D$ for some $w \in W$. Hence we get

$$R(\xi) = \{\alpha \in R_+; (\xi,\alpha) < 0\} = \{\alpha \in R_+; (w\xi, w\alpha) < 0\} \subset R_+ \cap w^{-1}R_-$$

and by Theorem 4.14 we conclude that $\#R(\xi) \leq l(w) < \infty$.

Conversely, suppose $\xi \in V^*$ such that $R(\xi)$ is a finite set. We shall prove $\xi \in Y$ by induction on $\#R(\xi)$. If $R(\xi) = \emptyset$ then $(\xi, \alpha) \geq 0$ for all $\alpha \in R_+$, and so $\xi \in D \subset Y$. Now suppose $R(\xi)$ is not empty. Then $R(\xi)$ contains a simple root α_s for some $s \in S$. Put $\eta = s(\xi)$. Using Corollary 4.13 it is easy to see that $R(\xi) = \{\alpha_s\} \sqcup sR(\eta)$, and so $\#R(\eta) = \#R(\xi) - 1$. By the induction hypothesis we get $\eta \in Y$, and therefore also $\xi = s(\eta) \in Y$. □

Corollary 4.24. *The Tits cone Y is a convex cone.*

Proof. For $\xi, \eta \in V^*$ we denote by $[\xi, \eta] = \{(1-t)\xi + t\eta; 0 \leq t \leq 1\}$ the line segment joining ξ and η. By the action of W on Y it is sufficient to show that $[\xi, \eta] \subset Y$ for $\xi \in D$ and $\eta \in Y$. Then $R(\xi) = \emptyset$ and $R(\zeta) \subset R(\eta)$ for all $\zeta \in [\xi, \eta]$. Hence $\zeta \in Y$ for all $\zeta \in [\xi, \eta]$ by the above theorem. □

Corollary 4.25. *The Tits cone Y is equal to all of V^* if and only if W is a finite group.*

Proof. Suppose $Y = V^*$. If $\xi \in -C \subset Y$ then $R(\xi) = R_+$ is a finite set by Theorem Tits cone membership theorem. Hence also $R = R_+ \sqcup R_-$ is finite. Therefore W is finite as group of permutations of the finite set R.

Conversely, suppose W is finite. Then R is a finite set, and therefore $R(\xi) \subset R_+$ is also finite for all $\xi \in V^*$. Hence $Y = V^*$ by Theorem 4.23. □

Corollary 4.26. *The facet C_I of the closure D of the positive chamber C (and hence also wC_I for all $w \in W$) lies in the interior of the Tits cone Y if and only if W_I is a finite group.*

Proof. By Theorem 4.17 we have $C_I \subset wD$ if and only if $w \in W_I$. Hence C_I lies in the interior of the Tits cone Y if and only if the origin lies in the interior of the Tits cone $Y_I \subset V_I^*$ for the dual geometric representation of the Coxeter group (W_I, I). The statement therefore follows from Corollary 4.25 applied to the latter situation. □

4.4 The Classification of Some Coxeter Diagrams

Suppose $M = (m_{ij})$ is a Coxeter matrix and $G(M) = (-2\cos(\pi/m_{ij}))$ the corresponding Gram matrix of the basis of simple roots $\{\alpha_i\}$ in V. The classification of all connected elliptic Coxeter diagrams was done in Theorem 2.22. Using this result we shall classify in this section some other Coxeter diagrams, namely the parabolic Coxeter diagrams, and particlar hyperbolic Coxeter diagrams, the so called Lannér diagrams and the Koszul diagrams.

The corresponding Coxeter groups are affine reflection groups, discussed in the next section, and particular hyperbolic reflection groups, discussed in the next chapter.

Definition 4.27. *A connected Coxeter diagram associated with the Coxeter matrix M is called parabolic if* 0 *is an eigenvalue of the Gram matrix $G(M)$ and all other eigenvalues are positive.*

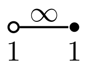
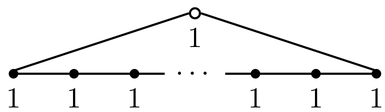
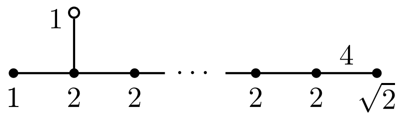
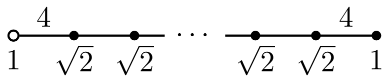
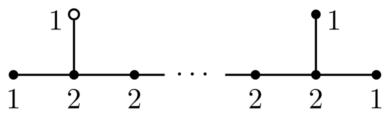
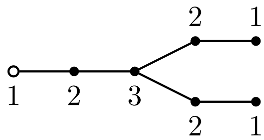
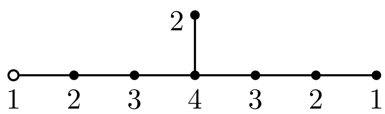
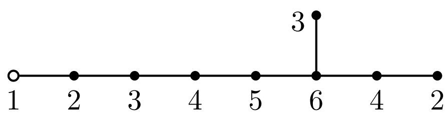
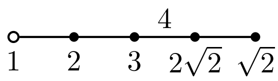
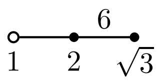
By the Perron–Frobenius theorem the smallest eigenvalue 0 of the Gram matrix $G(M)$ of a parabolic connected Coxeter diagram has multiplicity one. Moreover the kernel of $G(M)$ is spanned by a vector $(\cdots, k_i, \cdots)$ all whose coordinates are positive. The vector $\delta = \sum k_i\alpha_i$ is the up to a positive scalar unique null vector in V for the invariant symmetric bilinear form $(\cdot,\cdot)$. We may and will assume that $\min k_i = 1$. In the first classification theorem below these numbers k_i are written next to the node with index i.

All proper subdiagrams of a parabolic connected Coxeter diagram are elliptic. Indeed the kernel of the symmetric bilinear form is not contained in the real span of a proper subset of the basis of simple roots, because $k_i > 0$ for all $i \in I$. Conversely a connected Coxeter diagram all whose proper subdiagrams are elliptic is either elliptic (and these were classified in the first chapter), or parabolic or hyperbolic depending on whether $\det G(M)$ is positive, or zero or negative. These latter two Coxeter diagrams are classified by the two theorems below.

Definition 4.28. *A connected Coxeter diagram is called a Lannér diagram if* $\det G(M) < 0$ *and all proper subdiagrams are elliptic Coxeter diagrams.*

The two classification theorems below will follow from the classification of all connected Coxeter diagrams that are not elliptic, but all whose proper subdiagrams are elliptic. This classfication follows from the classification of elliptic Coxeter diagrams in a straightforward way. Subsequently one checks whether $\det G(M) = 0$ (and the Coxeter diagram is parabolic), or $\det G(M) < 0$ (and the Coxeter diagram is a Lannér diagram). These latter diagrams were classified in 1950 by Lannér [35].

Theorem 4.29. *The connected parabolic Coxeter diagrams are*

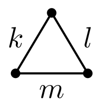
name	Coxeter diagram
$\tilde{\mathrm{A}}_1$	
$\tilde{\mathrm{A}}_n$	
$\tilde{\mathrm{B}}_n$	
$\tilde{\mathrm{C}}_n$	
$\tilde{\mathrm{D}}_n$	
$\tilde{\mathrm{E}}_6$	
$\tilde{\mathrm{E}}_7$	
$\tilde{\mathrm{E}}_8$	
$\tilde{\mathrm{F}}_4$	
$\tilde{\mathrm{G}}_2$	

with $n \geq 2$ *for* $\tilde{\mathrm{A}}_n$, $n \geq 3$ *for* $\tilde{\mathrm{B}}_n$, $n \geq 2$ *for* $\tilde{\mathrm{C}}_n$ *and* $n \geq 4$ *for* $\tilde{\mathrm{D}}_n$.

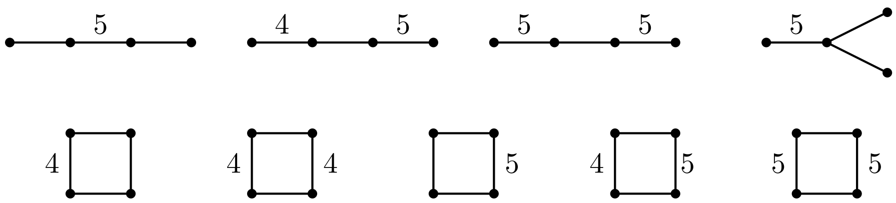
The names $\tilde{\mathrm{X}}_n$ are given for the following reason. The nodes with index

i for which $k_i = 1$ are called the special nodes. By inspection of the table the special nodes are a single orbit under the group of Coxeter diagram automorphisms. Pick one special node, and mark it with a little circle rather than a black dot. If we delete this special node then the remaining Coxeter diagram is an elliptic connected Coxeter diagram of type X_n (with the convention $\mathrm{B}_n = \mathrm{C}_n$), and all elliptic connected Coxeter diagrams occur for which there is an underlying Dynkin diagram as given in Theorem 2.53.

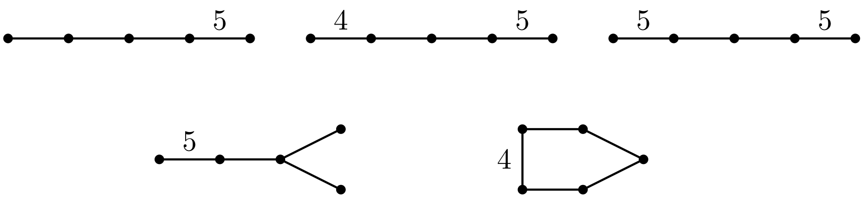
Theorem 4.30. *There are infinitely many Lannér diagrams in dimension $n = 3$ of the form*



with $2 \leq k, l, m < \infty$ and $1/k + 1/l + 1/m < 1$. In dimension $n = 4$ there are 9 Lannér diagrams



and in dimension $n = 5$ there are 5 Lannér diagrams



and in dimension $n \geq 6$ there are none.

Lemma 4.31. *If a mark m of a branch in a connected Coxeter diagram is increased to m' then the lowest eigenvalue t' of the Gram matrix $G(M')$ is strictly smaller than the lowest eigenvalue t of $G(M)$.*

Proof. By the Perron–Frobenius theorem the Gram matrix $G(M) = (g_{ij})$ has eigenvector for the lowest eigenvalue t equal to $(x_1, \cdots, x_n)$ with $x_i > 0$

for all i. Now $m' > m$, and so $-2\cos(\pi/m') < -2\cos(\pi/m)$. Hence we get with $G(M') = (g'_{ij})$

$$t' \sum x_i^2 \leq \sum g'_{ij} x_i x_j < \sum g_{ij} x_i x_j = t \sum x_i^2$$

and therefore $t' < t$. $\square$

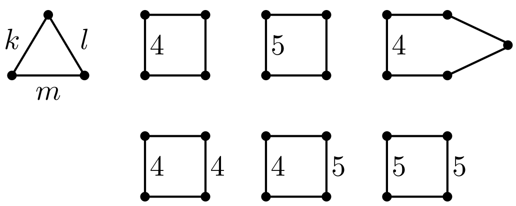
We now come to the proof of Theorem 4.29 and Theorem 4.30. It is easy to check that all diagrams listed in Theorem 4.29 are parabolic, by checking that the given vector $(\cdots, k_j, \cdots)$ lies in $\ker G(M)$. For the simply laced diagrams this means that each $2k_i$ is equal to $\sum k_j$ with the sum over all nodes with index j that are connected to the node with index i. Similarly in general $2k_i$ is equal to $\sum 2\cos(\pi/m_{ij})k_j$ with the same notation, remembering $2\cos(\pi/3) = 1$, $2\cos(\pi/4) = \sqrt{2}$ and $2\cos(\pi/6) = \sqrt{3}$.

Using the above lemma it is also clear that all Coxeter diagrams listed in Theorem 4.30 with the possible exception of the two diagrams

are Lannér diagrams. For these two exceptions $\det G(M)$ is equal to $5 - 4\tau$ and $6 - 4\tau$ by Lemma 2.30. These numbers are both negative (since $\tau = (1+\sqrt{5})/2 > 3/2$), and therefore these two diagrams are Lannér diagrams as well.

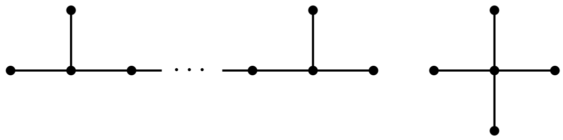
It remains to show that the diagrams listed in the two theorems exhaust the collection of all connected Coxeter diagrams that are not elliptic but whose proper subdiagrams are all elliptic. Take such a connected Coxeter diagram say with n nodes that is not elliptic but all proper subdiagrams are elliptic. Clearly the underlying graph of this diagram by ignoring the marks on the edges is either a cycle or a tree.

If this underlying graph is a cycle and unmarked then we have the parabolic diagram of type $\tilde{\mathrm{A}}_{n-1}$. If there is a mark $m \geq 4$ then $n \leq 5$. It is now easy to check that the diagrams

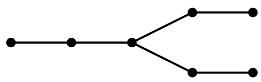


with $k, l, m \geq 3, \max\{k, l, m\} \geq 4$ exhaust all the possibilities. These are all the Lannér diagrams in Theorem 4.30 whose underlying graph is a cycle, as should.

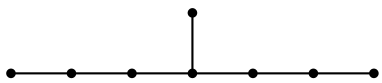
Now suppose the underlying graph is a tree. There are at most two triple nodes, or one quadruple node. Two triple nodes or one quadruple node occur for the parabolic diagrams



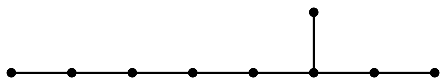
of type $\tilde{\mathrm{D}}_{n-1}$ for $n \geq 6$ and $\tilde{\mathrm{D}}_4$. It is easy to check that no Lannér diagrams with such an underlying graph are possible. If one triple node occurs then the underlying graph is of type T_{pqr} with $2 \leq p \leq q \leq r, p + q + r = n + 2$. Now $p \leq 3$ and $p = 3$ only occurs for the parabolic diagram



of type $\tilde{\mathrm{E}}_6$. Otherwise $p = 2, q \leq 4$ and $q = 4$ only occurs for the parabolic diagram

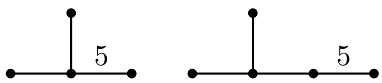


of type $\tilde{\mathrm{E}}_7$. Moreover $p = 2, q = 3$ implies that $r = 6$ and we find the parabolic diagram



of type $\tilde{\mathrm{E}}_8$. Finally we have to consider the case $p = q = 2$ and $r = n-2 \geq 2$. These cases occur for the parabolic diagram

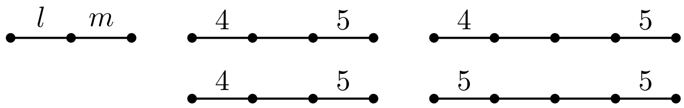
of type $\tilde{\mathrm{B}}_{n-1}$ for $n \geq 4$, and for the Lannér diagrams for $n = 4$ or 5



and these are the only such cases.

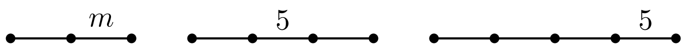
Now we are left with the case of a simple graph of type A_n and the Coxeter diagram has certain marks attached to the branches. At most two marks $l, m \geq 4$ are attached to two branches, and if there are two such marks they are attached to the two extremal branches. This case gives rise to the parabolic diagram

of type $\tilde{\mathrm{C}}_{n-1}$ for $n \geq 3$, and the Lannér diagrams for $n = 3, 4$ or 5



with $l, m \geq 4, \max\{l, m\} \geq 5$ and these are all such cases. If there is only one mark $m \geq 4$ attached to some branch then we easily find the parabolic diagrams

of type $\tilde{\mathrm{A}}_1, \tilde{\mathrm{G}}_2$ and $\tilde{\mathrm{F}}_4$ respectively, and the Lannér diagrams for $n = 3, 4$ and 5



for $m \geq 7$, and these are all possibities. This completes the proof of the two classification theorems.

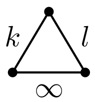
Definition 4.32. *A connected Coxeter diagram is called a Koszul diagram if* $\det G(M) < 0$ *and all proper subdiagrams are elliptic or parabolic, and at least one proper subdiagram is parabolic (so as to make Lannér diagrams and Koszul diagrams disjoint sets).*

These diagrams appeared in the work by Koszul on hyperbolic Coxeter groups [34]. In the next chapter the Coxeter groups for Lannér and Koszul diagrams will turn out to act as particular reflection groups W on hyperbolic space, namely those for which there is a simplex as fundamental domain for the action of W on hyperbolic space. The simplex is compact for the Lannér diagrams, while the simplex has some ideal vertices but still finite volume for the Koszul diagrams.

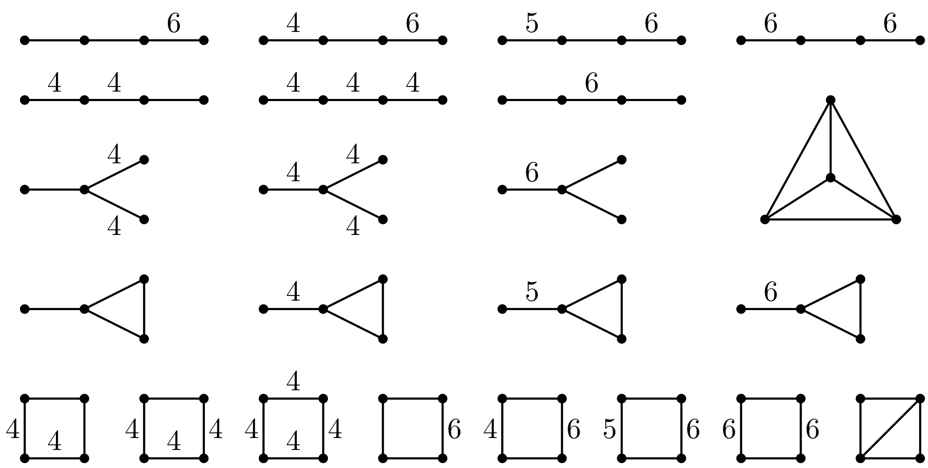
However these Coxeter groups are just the tip of the iceberg of Coxeter groups W acting as reflection groups on hyperbolic space. In general the fundamental domain is a convex polyhedron P, and our object of study in the next chapter is to understand those groups W for which P is either compact or has finite volume with some ideal vertices.

The Koszul diagrams have also been classified, and we include the result just for the record without proof. It is easy to check whether any given Coxeter diagram is a Koszul diagram, but the table of all Koszul diagrams is already quite involved, and a proof is probably best given by computer verification, so as not to overlook any possibilities. Such verification was carried out by Chein [14].

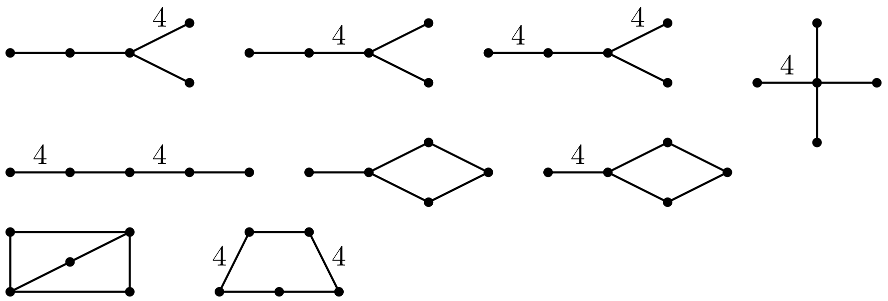
Theorem 4.33. *There are infinitely many Koszul diagrams in dimension $n = 3$ of the form*



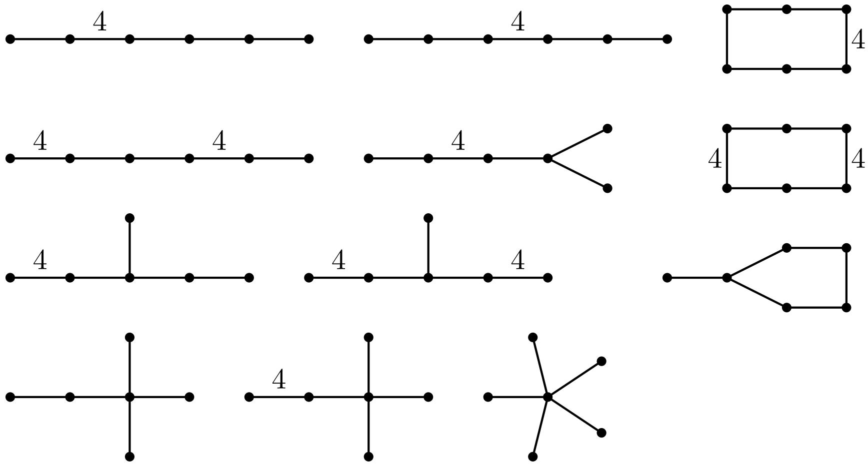
with $k \geq 2, l \geq 3$. In dimension $n = 4$ there are 23 Koszul diagrams



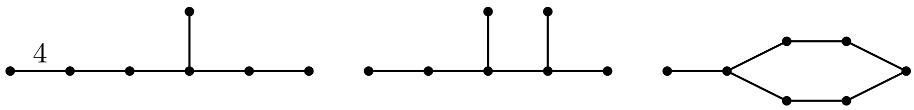
and in dimension $n = 5$ there are 9 Koszul diagrams



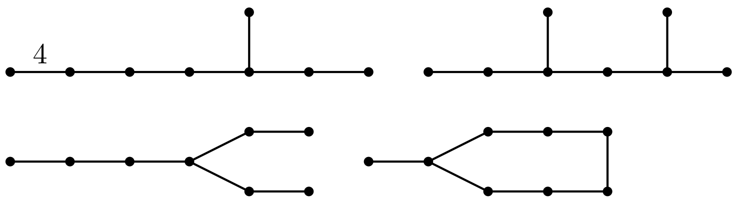
and in dimension $n = 6$ there are 12 Koszul diagrams



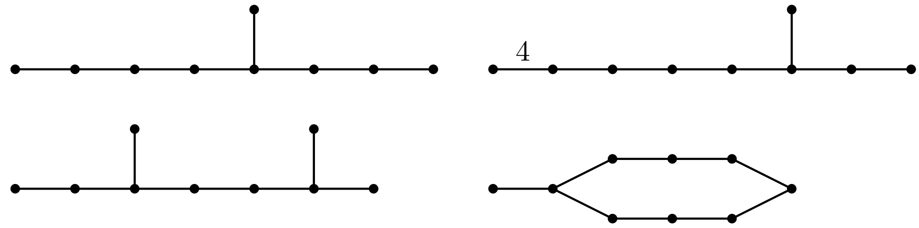
and in dimension $n = 7$ there are 3 Koszul diagrams



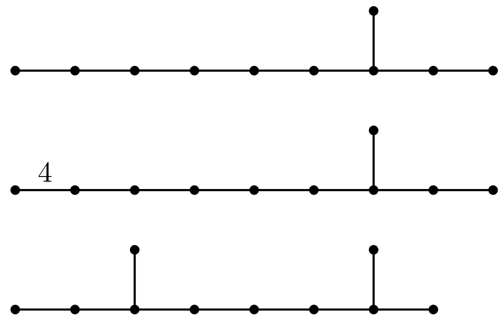
and in dimension $n = 8$ there are 4 Koszul diagrams



and in dimension $n = 9$ *there are* 4 *Koszul diagrams*



and in dimension $n = 10$ *there are* 3 *Koszul diagrams*



and in dimension $n \geq 11$ *there are none.*

4.5 Affine Reflection Groups

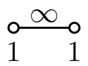
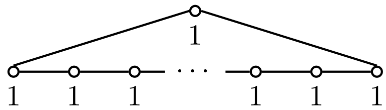
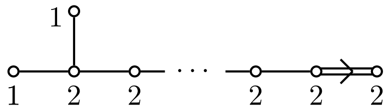
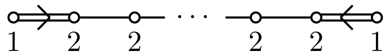
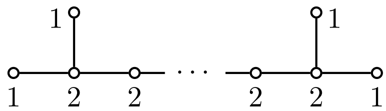
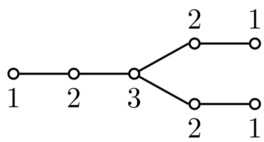
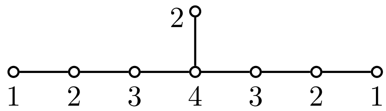
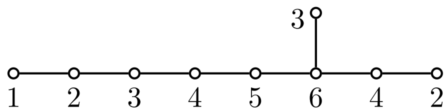
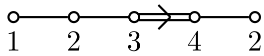
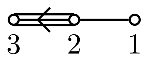
Let $M = (m_{ij})$ for $0 \leq i, j \leq n$ be the Coxeter matrix of a connected parabolic Coxeter diagram with $n + 1$ nodes. These Coxeter diagrams have been classified in Theorem 4.29. It so happens (and in fact for a good reason) that all entries of the Coxeter matrix are in $\{1, 2, 3, 4, 6, \infty\}$. By definition the vector space V of the geometric representation has a basis of simple roots $\{\alpha_i\}$ with symmetric bilinear form $(\alpha_i, \alpha_j) = -2\cos(\pi/m_{ij})$.

As in Section 2.7 we will allow the simple roots to be rescaled by positive real numbers, and denote these rescaled roots by α_i. The matrix $A = (a_{ij})$ defined by

$$a_{ij} = (\alpha_i, \alpha_j^\vee) = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

is called the Cartan matrix. Clearly $a_{ii} = 2$ and we can and will choose the rescaling in such a way that $a_{ij} \in -\mathbb{N}$ for all $i \neq j$. The corresponding Dynkin diagram, as defined in Section 2.7, is then called an affine Dynkin diagram. One parabolic Coxeter diagram might lead to several affine Dynkin diagrams, and we shall make a particular *untwisted* choice.

Theorem 4.34. *The connected untwisted affine Dynkin diagrams are*

name	Coxeter diagram
$\tilde{\mathrm{A}}_1$	
$\tilde{\mathrm{A}}_n$	
$\tilde{\mathrm{B}}_n$	
$\tilde{\mathrm{C}}_n$	
$\tilde{\mathrm{D}}_n$	
$\tilde{\mathrm{E}}_6$	
$\tilde{\mathrm{E}}_7$	
$\tilde{\mathrm{E}}_8$	
$\tilde{\mathrm{F}}_4$	
$\tilde{\mathrm{G}}_2$	

with $n \geq 2$ *for* $\tilde{\mathrm{A}}_n$, $n \geq 3$ *for* $\tilde{\mathrm{B}}_n$, $n \geq 2$ *for* $\tilde{\mathrm{C}}_n$ *and* $n \geq 4$ *for* $\tilde{\mathrm{D}}_n$.

Indeed, in case $m_{ij} = 4, 6$ there are two choices: one of the two simple roots is long and the other is short. In case $m_{ij} = \infty$ then either $a_{12} = a_{21} = -2$ and the two simple roots have equal lengths, or one is long and the other short. For the diagram $\tilde{\mathrm{A}}_1$ the untwisted choice is $a_{12} = a_{21} = -2$.

We denote by $\tilde{R}$ the untwisted affine root system with basis of simple roots $\{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ and by $\tilde{W}$ we denote the affine Weyl group generated by the simple reflections $s_0, s_1, \cdots, s_n$. The kernel op the symmetric bilinear form $(\cdot, \cdot)$ on V is spanned by a vector $\delta = \sum k_i \alpha_i$ with $k_i > 0$ for all i by the Perron–Frobenius theorem, and $\min k_i = 1$ by assumption. By our integral normalization is so happens that all k_i are positive integers. The labels k_i are written in the table of Theorem 4.34, which is easily derived from the table in Theorem 4.29. We numerate the simple roots such that $k_0 = 1$. The automorphism group of the untwisted affine Dynkin diagram is the same as the automorphism group of the corresponding parabolic Coxeter diagram, and the nodes with $k_i = 1$ form a single orbit. Let W be the finite Weyl group generated by $s_1, \cdots, s_n$ and let $R = W\{\alpha_1, \cdots, \alpha_n\}$ be the corresponding finite root system.

Proposition 4.35. *We have $\delta = \alpha_0 + \theta$ with $\theta \in R$ the unique long root with $(\theta, \alpha_i^\vee) \in \mathbb{N}$ for $1 \leq i \leq n$.*

Proof. Clearly $(\theta, \theta) = (\delta - \alpha_0, \delta - \alpha_0) = (\alpha_0, \alpha_0)$. For type ADE we know that $R = \{\alpha \in Q(R); (\alpha, \alpha) = 2\}$ if roots are normalized as norm 2 vectors by Example 2.33. Hence it follows that $\theta \in R$ for type ADE. For the remaining root systems (say without triple node) we just check that $\theta \in R$ case by case

name	$\theta = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n$
$\mathrm{A}_n, n \geq 1$	$\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_n = \varepsilon_0 + \varepsilon_n$
$\mathrm{B}_n, n \geq 3$	$\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_n = \varepsilon_1 + \varepsilon_2$
$\mathrm{C}_n, n \geq 2$	$\theta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n = 2\varepsilon_1$
F_4	$\theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \varepsilon_1 + \varepsilon_2$
G_2	$\theta = 3\alpha_1 + 2\alpha_2 = -\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3$

using the explicit descriptions of these root systems in Section 2.7.

Since δ lies in the kernel of the symmetric bilinear form we conclude that $(\theta, \alpha_i^\vee) = -(\alpha_0, \alpha_i^\vee) \in \mathbb{N}$ for $1 \leq i \leq n$. Since the long roots in R form a single orbit under W and each orbit of W intersects the closure of the positive chamber in a unique vector by Theorem 4.17 it follows that θ is the unique such root. $\square$

The affine root system $\tilde{R}$ has a simple description in terms of the finite root system R and the null vector δ.

Theorem 4.36. *We have* $\tilde{R} = R + \mathbb{Z}\delta$.

Proof. Since $\delta = \alpha_0 + \theta$ is null vector and therefore $(\alpha_0, \alpha_0) = (\theta, \theta)$ we get $(\lambda, \alpha_0^\vee) = -(\lambda, \theta^\vee)$ for all $\lambda \in V$. We claim that the element $t = s_\theta s_0 \in \tilde{W}$ acts on V by

$$t(\lambda) = \lambda + (\lambda, \theta^\vee)\delta.$$

for all $\lambda \in V$. Indeed $s_0(\lambda) = \lambda - (\lambda, \alpha_0^\vee)\alpha_0 = \lambda + (\lambda, \theta^\vee)\alpha_0$ and because $s_\theta(\alpha_0) = \alpha_0 + 2\theta$ we get

$$t(\lambda) = \lambda - (\lambda, \theta^\vee)\theta + (\lambda, \theta^\vee)(\alpha_0 + 2\theta)$$

and the claim is checked. Using conjugation by elements of the finite Weyl group W and iteration we see that for all $k \in \mathbb{Z}$ and all long roots $\beta \in R$ the linear transformation

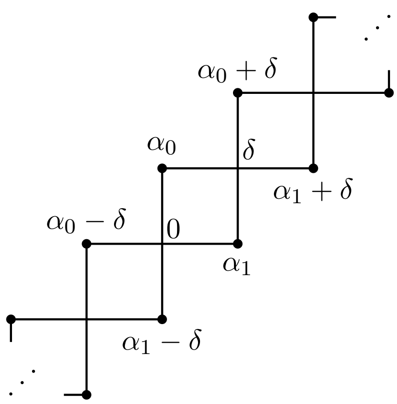
$$\lambda \mapsto \lambda + k(\lambda, \beta^\vee)\delta$$

on V belongs to the affine Weyl group $\tilde{W}$.

In case $\tilde{R}$ is not of type $\tilde{\mathrm{A}}_1$ there exists for each root $\alpha \in R$ a long root $\beta \in R$ with $(\alpha, \beta^\vee) = 1$ and we conclude that $R + \mathbb{Z}\delta \subset \tilde{R}$. The converse inclusion is easy, since $R + \mathbb{Z}\delta$ is invariant under $\tilde{W}$ and contains all simple roots. In case $\tilde{R}$ is of type $\tilde{\mathrm{A}}_1$ we have

$$s_0(\alpha_1) = \alpha_0 + \delta \ , \ s_1(\alpha_0) = \alpha_1 + \delta$$

with $\delta = \alpha_0 + \alpha_1$ and the statement of the theorem is clear. In this case there are two orbits of $\tilde{W}$ in $\tilde{R}$. $\square$



Theorem 4.37. *The Tits cone is equal to* $Y = \{0\} \sqcup \{\xi \in V^*; (\xi, \delta) > 0\}$.

Proof. The basis of V^* of fundamental coweights ξ_i for $i = 0, 1, \cdots, n$ is characterized by $(\xi_i, \alpha_j) = \delta_{ij}$ with $(\cdot, \cdot)$ by abuse of notation also denoting the natural pairing of V^* and V. The closure D of the positive chamber is just the set $\{\sum x_i \xi_i; x_i \geq 0\}$. If we denote $Y = \{0\} \sqcup \{\xi \in V^*; (\xi, \delta) > 0\}$ then $D \subset Y$ and Y is invariant under $\tilde{W}$. Hence the Tits cone is contained in Y.

We claim that the mirrors H_α for $\alpha \in \tilde{R}$ form a locally finite set of hyperplanes on $\{\xi \in V^*; (\xi, \delta) > 0\}$. This means that any compact subset K of $\{\xi \in V^*; (\xi, \delta) > 0\}$ intersects only finite many mirrors. If we put $m = \min\{(\xi, \delta); \xi \in K\} > 0$ and $M = \max\{|(\xi, \alpha)|; \xi \in K, \alpha \in R\} < \infty$ then

$$|(\xi, \alpha + k\delta)| \geq |k|(\xi, \delta) - |(\xi, \alpha)| \geq |k|m - M > 0$$

for all $\xi \in K$ if $|k| \geq M/m$. Hence K meets only finitely many mirrors. In particular for $\eta \in C$ and $(\xi, \delta) > 0$ the line segment $[\eta, \xi]$ meets only finitely many mirrors, and so $R(\xi)$ is finite, and ξ lies in the Tits cone by Theorem 4.23. Hence the Tits cone contains Y. $\square$

Definition 4.38. *For* $\alpha \in R$ *we have defined* $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ *as vector in* V. *We also define the coroot* $\alpha^\vee$ *as vector in* V^* *by*

$$(\alpha^\vee, \delta) = 0 \ , \ (\alpha^\vee, \beta) = (\beta, \alpha^\vee)$$

for all $\beta \in R$. *Here the first and second bracket* $(\cdot, \cdot)$ *denote the pairing of* V^* *and* V *while the last bracket* $(\cdot, \cdot)$ *is just the canonical symmetric bilinear form on* V.

Observe that the dual action of $t = s_\theta s_0$ is given by

$$t(\xi) = \xi - (\delta, \xi)\theta^\vee$$

for all $\xi \in V^*$. Indeed we have

$$(t(\xi), \lambda) = (\xi, t^{-1}(\lambda)) = (\xi, \lambda - (\lambda, \theta^\vee)\delta) = (\xi - (\xi, \delta)\theta^\vee, \lambda)$$

for all $\lambda \in V$ and all $\xi \in V^*$.

We can now explain the name affine reflection group for $\tilde{W}$. Since $w\delta = \delta$ for all $w \in \tilde{W}$ the affine hyperplane

$$A^n = \{\xi \in V^*; (\xi, \delta) = 1\}$$

of dimension n is invariant under the action of $\tilde{W}$. Clearly A^n is an affine space with the linear hyperplane $\{\xi \in V^*; (\xi, \delta) = 0\}$ acting on A^n as group of translations. Since $\{\xi \in V^*; (\xi, \delta) = 0\} \cong V/\mathbb{R}\delta$ using the symmetric bilinear form the space A^n becomes a Euclidean affine space. Note that t acts on A^n as a translation over the coroot $-\theta^\vee$.

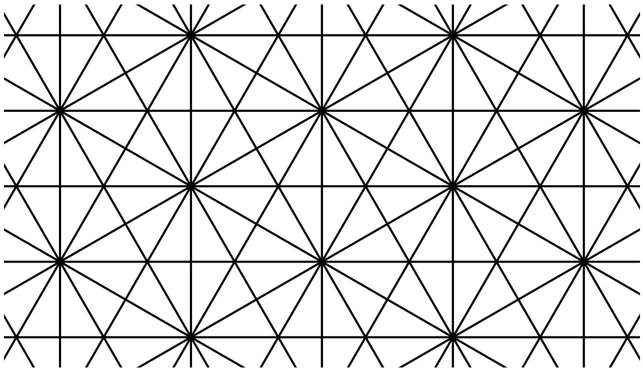
The intersection $D \cap A^n$ of the closed fundamental chamber D with A^n is a strict fundamental domain for the faithful action of $\tilde{W}$ on A^n and is called the fundamental alcove. Note that $D \cap A^n$ is a simplex with (n+1) vertices ξ_i/k_i with $\xi_i \in V^*$ the fundamental coweights dual to the simple roots. All $n(n+1)/2$ inner angles along the plinths of $D \cap A^n$ are dihedral angles π/m_{ij}. The theory of affine reflection groups essentially boils down to the study of black and white tessellations of Euclidean space by congruent dihedral simplices, with colours black and white to distinguish the two mirror images.

The affine Weyl group $\tilde{W}$ is generated by the reflections $s_0, s_1, \cdots, s_n$ through the Coxeter presentation. But now we find a different realization of $\tilde{W}$ as a semidirect product $\tilde{W} \cong Q^\vee \rtimes W$. The coroot lattice $Q^\vee$ is viewed as subset of $\{\xi \in V^*; (\xi, \delta) = 0\}$ and hence acts on A^n via translations. The finite Weyl group generated by $s_1, \cdots, s_n$ acts on A^n leaving the point ξ_0 fixed, and so acts through linear orthogonal transformations when A^n is viewed as Euclidean vector space with origin ξ_0. For $\xi \in Q^\vee$ we denote by t_ξ the translation of A^n over ξ. The relation between translations and finite Weyl group elements is given by the push relations

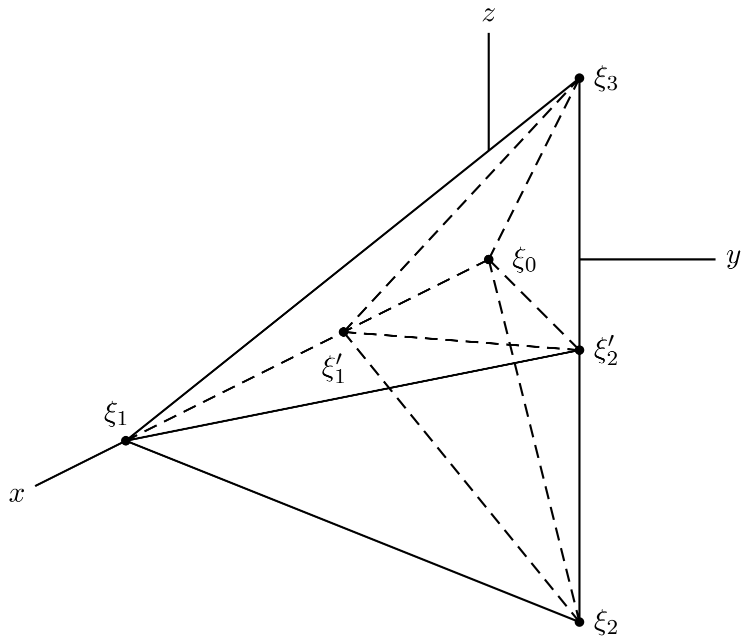
$$wt_\xi = t_{w\xi}w$$

for all $w \in W$ and $\xi \in Q^\vee$.

Here is a picture of the tessellation for $\tilde{\mathrm{G}}_2$ with fundamental alcove a triangle with vertices $\xi_0, \xi_1/2, \xi_2/3$ with angles $\pi/6, \pi/2, \pi/3$ respectively. The coroot lattice $Q^\vee$ is spanned by $\xi_1 - 2\xi_0$ and $\xi_2 - 3\xi_0$.



Exercise 4.39. *Let* $\xi_0 = (0,0,0), \xi_1 = (2,0,0), \xi_2 = (1,1,-1), \xi_3 = (1,1,1)$ *be the vertices of the polytope* A*, let* $\xi_0, \xi_1' = (1,0,0), \xi_2, \xi_3$ *be the vertices of the polytope* B *and let* $\xi_0, \xi_1', \xi_2' = (1,1,0), \xi_3$ *be the vertices of the polytope* C*. Show that all inner dihedral angles on the edges of* A, B *and* C *are equal to* π/m *for some integers* $m \geq 2$*.*



Check that A, B, C *are fundamental alcoves of types* $\tilde{A}_3, \tilde{B}_3, \tilde{C}_3$ *respectively, and conclude that* $W(\tilde{C}_3) < W(\tilde{B}_3) < W(\tilde{A}_3)$ *are both inclusions of index two subgroups.*

4.6 Crystallography

In this section we shall discuss the basic theory of crystallographic space groups. These results are not very relevant for understanding affine Coxeter groups, which just appear as a particular example. The purpose is to place things in more general perspective. Proofs that are given are sketchy and details are left to the reader.

Let V be a Euclidean vector space, and A its underlying Euclidean affine space by forgetting the origin. For $\xi \in V$ we denote by $t_\xi : A \to A$ the translation of A over $\xi \in V$. This action of V on A is simply transitive, meaning transitive and the stabilizer in V of a point in A is trivial. Hence $V \cong A$ but this identification is not canonical and only possible after choice of an origin in A.

A distance preserving transformation of A is called a Euclidean motion, and the group of all Euclidean motions of A is denoted $\mathrm{M}(A)$ and called the Euclidean motion group of A.

Theorem 4.40. *The motion group* $\mathrm{M}(V)$ *is equal to the semidirect product* $\mathrm{T}(V) \rtimes \mathrm{O}(V)$ *of the translation group* $\mathrm{T}(V) = V$ *and the orthogonal group* $\mathrm{O}(V)$*. For* $\xi \in V$ *and* $g \in \mathrm{O}(V)$ *we have the push relation*

$$gt_\xi = t_{g\xi}g$$

and together with the additive structure on $\mathrm{T}(V)$ *and the multiplicative structure on* $\mathrm{O}(V)$ *this determines the group structure on* $\mathrm{M}(V)$*.*

Proof. The crucial step is to show that a motion $m : V \to V$ with $m(0) = 0$ is in fact an orthogonal linear transformation: $m \in \mathrm{O}(V)$. First show that $(m\xi, m\eta) = (\xi, \eta)$ and so m preserves the inner product on V. Subsequently deduce that $m : V \to V$ is a linear transformation. $\square$

We conclude that $\mathrm{M}(A) \cong \mathrm{T}(A) \rtimes \mathrm{O}(V)$ with $V = T(A)$ the group of translations of A and $\mathrm{O}(V)$ the orthogonal group of V. But this isomorphism is not canonical, and depends on an identification $V \cong A$. The quotient map $\pi : \mathrm{M}(A) \to \mathrm{O}(V)$ has kernel $V = \mathrm{T}(A)$. The orthogonal group $\mathrm{O}(V)$ has two connected components with $\mathrm{SO}(V)$ the connected component of the identity. In turn $\mathrm{M}(A)$ has also two connected components with $\mathrm{T}(V) \rtimes \mathrm{SO}(V)$ the group of proper Euclidean motions as connected component of the identity.

For any two points ξ, η in the affine space A and any two numbers $x, y \in \mathbb{R}$ with $x + y = 1$ the affine combination $x\xi + y\eta \in A$ is well defined, independently of the choice of a possible origin. An invertible transformation $a : A \to A$ is called affine if $a(x\xi + y\eta) = xa(\xi) + ya(\eta)$ for all $\xi, \eta \in A$ and $x, y \in \mathbb{R}$ with $x + y = 1$. The affine transformations $a : A \to A$ form a group under composition, called the affine group and denoted $\mathrm{Aff}(A)$, and as for the Euclidean motion group we have $\mathrm{Aff}(A) \cong T(V) \rtimes \mathrm{GL}(V)$. The index two subgroup $\mathrm{Aff}_+(A)$ of proper affine transformations of A is isomorphic to $T(V) \rtimes \mathrm{GL}_+(V)$ with $\mathrm{GL}_+(V) = \{a \in \mathrm{GL}(V); \det a > 0\}$ the proper general linear group.

Definition 4.41. *A subgroup* G *of* $\mathrm{M}(A)$ *is called a crystallographic space group on* A *if* $G \cap \mathrm{T}(A)$ *is just a lattice* $L = \mathbb{Z}B$ *in* V *with* B *a vector* *space basis of V. The lattice L in V is called the translation lattice of G. The image $P = \pi(G) < \mathrm{O}(V)$ of G under the natural homomorphism $\mathrm{M}(A) \to \mathrm{O}(V)$ is called the point group of G.*

Suppose $G < \mathrm{M}(A)$ is a crystallographic space group on A with translation lattice $L < V$ and point group $P < \mathrm{O}(V)$.

Theorem 4.42. *The group L is the maximal Abelian normal subgroup of the group G.*

Proof. Let N be a normal Abelian subgroup of G. For $n \in N$ we shall write $p = \pi(n) \in P$ and we have

$$t_{pl-l} = nt_l n^{-1} t_{-l} = t_l n^{-1} t_{-l} n = t_{l-p^{-1}l}$$

for all $t_l \in L$. Hence $pl - l = l - p^{-1}l$ for all $l \in L$, and so $p - 1 = 1 - p^{-1}$ as identity in $\mathrm{End}(V)$. Therefore $p + p^{-1} = 2$ and using $p \in \mathrm{O}(V)$ this implies $p = 1$. Hence $n \in L$ and $N < L$. □

The conclusion is that both the translation lattice L and the point group $P = G/L$ as abstract groups are invariants of the abstract group G. Isomorphic crystallographic space groups have isomorphic translation lattices and isomorphic point groups. In particular the rank of L is an invariant of the abstract group G.

Theorem 4.43. *The point group P is in fact a subgroup of orthogonal group $\mathrm{O}(L) = \{g \in \mathrm{O}(V); gL = L\}$ of the lattice L. In particular P is a finite group as discrete subgroup of the compact group $\mathrm{O}(V)$.*

The easy proof is left to the reader.

Definition 4.44. *Two crystallographic space groups $G_1, G_2 < \mathrm{M}(A)$ are called equivalent if there exists a proper affine transformation $a \in \mathrm{Aff}_+(A)$ such that $G_2 = aG_1a^{-1}$.*

Equivalence between crystallographic space groups is an equivalence relation. There are up to equivalence 17 crystallographic plane groups in rank 2, also called the wall paper groups. In rank 3 there are 230 equivalence classes of crystallographic space groups. This result was obtained independently by Barlow (in 1894 in England), Fedorov (in 1891 in Russia) and Schönflies (in 1891 in Germany). This marks the first important application of group theory in physics. Each of the 230 equivalence classes occurs as the symmetry group of a crystal appearing in nature!

For arbitrary $n \in \mathbb{N}$ the number of equivalence classes of crystallographic space groups of rank n is finite, a theorem due to Bieberbach in 1911 and conjectured by Hilbert in 1900 as Hilbert Problem 18. The correct number 4894 for $n = 4$ is due to Neubüser, Souvignier and Wondratschek and was obtained in 2002 using computer calculations. A classification for large n is both hopeless and useless. Affine reflection groups are a very special class of crystallographic space groups for which a classification in arbitrary dimension is possible and even quite easy.

5 Hyperbolic Reflection Groups

5.1 Hyperbolic Space

Let V be a Lorentzian vector space of dimension $n+1$, which means that there is given a nondegenerate symmetric bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$ of signature $(n,1)$. The quadratic hypersurface

$$\{\lambda \in V; (\lambda,\lambda) = -1\}$$

is a hyperboloid of two sheets, and one connected component H^n will be called hyperbolic space of dimension n. The tangent space of H^n at $\lambda \in H^n$ is equal to the orthogonal complement of λ in V and therefore inherits a natural Euclidean structure from $(\cdot,\cdot)$ by restriction. This turns H^n into a Riemannian manifold of dimension n. The geodesics are the nonempty intersections of H^n with a linear plane in V. Likewise the hyperplanes in H^n are the nonempty intersections of H^n with a linear hyperplane in V. All this is analoguous to the situation of the unit sphere S^n of dimension n in a Euclidean vector space V of dimension $n+1$.

We shall discuss the three familiar models of hyperbolic geometry: the projective model of Klein, the ball model of Riemann and the upper half space model of Poincaré. The ball model was discribed by Riemann in his famous Habilitationsvortrag in 1854. The projective model is due to Klein in 1878 and the upper half space model is due to Poincaré in 1882. However both the projective and the upper half space model had been given before by the Italian geometer Eugenio Beltrami in 1868. The attribution of results and theorems in mathematics need not be to the person who found it first. Even the ball model is usually attributed to Poincaré for his rediscovery 30 years after Riemann, while the Habilitationsvortrag of Riemann is one of the most influential and well read papers on differential geometry in the nineteenth century!

Let us denote by $\mathbb{R}^{n,1}$ the standard Lorentzian space with vectors $y = (y_0, y_1, \cdots, y_n)$ and quadratic form $(y,y) = -y_0^2 + y_1^2 + \cdots + y_n^2$. We shall take for $H^n = \{y \in \mathbb{R}^{n,1}; (y,y) = -1, y_0 > 0\}$ the upper sheet of the hyperboloid of two sheets. Let $x = (x_1, \cdots, x_n)$ be a vector in the standard Euclidean space $\mathbb{R}^n$ with norm $x^2 = (x,x) = x_1^2 + \cdots + x_n^2$ and denote

$$\mathbb{B}^n = \{x \in \mathbb{R}^n; x^2 < 1\}\ ,\ \mathbb{H}^n_+ = \{x \in \mathbb{R}^n; x_n > 0\}$$

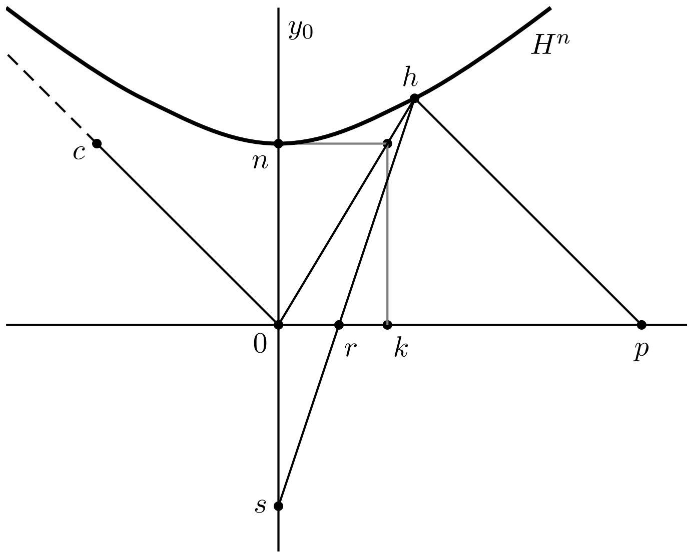
for the unit ball and the upper half space in $\mathbb{R}^n$. For $U \subset \mathbb{R}^n$ an open set and $U \ni x \mapsto y(x) \in H^n$ local coordinates on H^n we denote by

$$ds^2 = \sum_{i,j} g_{ij} dx_i dx_j \ , \ dV = \sqrt{g} dx_1 \cdots dx_n$$

the length element (a Riemannian metric) and the volume element with

$$g_{ij} = \sum_k \frac{\partial y_k}{\partial x_i} \frac{\partial y_k}{\partial x_j} \ , \ g = \det g_{ij}$$

the coefficients of the Riemannian metric and its determinant. The three models of hyperbolic geometry can be conveniently drawn in the following picture



with $c = (1, 0, \cdots, 0, -1)$ an isotropic vector (that is $(c, c) = 0$) and with $n = (1, 0, \cdots, 0) \in H^n$ and $s = (-1, 0, \cdots, 0)$ the north and south pole. The points $k, r \in \mathbb{B}^n$ and $p \in \mathbb{H}^n$ in the above picture all represent the point $h \in H^n$ in the Klein model, the Riemann model and the Poincaré model respectively.

For the Klein model one takes $k = x$ and

$$\pi_K : \mathbb{B}^n \to H^n \ , \ \pi_K(x) = (1/\sqrt{1 - x^2}, x/\sqrt{1 - x^2})$$

with π_K the linear projection with center the origin from $(\mathbb{B}^n, 1)$ onto H^n. The Riemannian metric for the Klein model becomes

$$ds^2 = (1 - x^2)^{-1} \sum_i dx_i^2 + (1 - x^2)^{-2} \sum_{i,j} x_i x_j dx_i dx_j$$

with

$$dV = (1 - x^2)^{-(n+1)/2} dx_1 \cdots dx_n$$

the associated volume element. The advantage of the Klein model is that geodesics become straight lines for the Euclidean geometry on $\mathbb{B}^n$. Likewise the hyperbolic hyperplanes are just Euclidean hyperplanes inside $\mathbb{B}^n$.

For the Riemann model one takes $r = x$ and

$$\pi_R : \mathbb{B}^n \to H^n \ , \ \pi_R(x) = (1 + x^2, 2x)/(1 - x^2)$$

with π_R the stereographic projection with center the south pole s from $\mathbb{B}^n$ onto H^n. The Riemannian metric for the Riemann model becomes

$$ds^2 = 4(1 - x^2)^{-2} \sum_i dx_i^2$$

with

$$dV = 2^n (1 - x^2)^{-n} dx_1 \cdots dx_n$$

the associated volume element. The geodesics are circular arcs perpendicular to the boundary of $\mathbb{B}^n$. The Riemann model is conformal in the sense that angles between curves in this model coincide with the Euclidean angles.

For the Poincaré model one takes $p = x$ and

$$\pi_P : \mathbb{H}^n_+ \to H^n \ , \ \pi_P(x) = ((1+x^2)/(2x_n), x_1, \cdots, x_{n-1}, x_n - (1+x^2)/(2x_n))$$

with π_P the parallel projection along the vector c from $\mathbb{H}^n_+$ onto H^n. The Riemannian metric for the Poincaré model becomes

$$ds^2 = x_n^{-2} \sum_i dx_i^2$$

with

$$dV = x_n^{-n} dx_1 \cdots dx_n$$

the associated volume element. The geodesics are half circles and half lines perpendicular to the boundary of $\mathbb{H}^n_+$. Likewise the hyperbolic hyperplanes are just hemispheres with center on the boundary $\partial\mathbb{H}^n_+$ including vertical half planes perpendicular to the boundary. Like the Riemann model the Poincaré model is also a conformal model.

The Lorentz group $\mathrm{O}(V)$ has four connected components and we shall denote by $\mathrm{O}^+(V)$ the index two subgroup leaving H^n invariant. The group $\mathrm{O}^+(V)$ has still two connected components separated by the value of the determinant. The spherical geometry $S^n \subset \mathbb{R}^{n+1}$ with orthogonal group

$O_{n+1}(\mathbb{R})$, the (affine) Euclidean geometry $E^n \cong \mathbb{R}^n$ with Euclidean motion group $\mathrm{M}(E^n) \cong \mathbb{R}^n \rtimes \mathrm{O}_n(\mathbb{R})$ and the hyperbolic geometry $H^n \subset \mathbb{R}^{n,1}$ with Lorentz group $\mathrm{O}^+_{n,1}(\mathbb{R})$ are three geometries for which the action of the the symmetry group is transitive on the space with stabilizer of a point acting on the tangent space at that point equal to the full orthogonal group of that tangent space. This optimal transitivety property of the symmetry group characterizes these three classical geometries up to a scale factor.

If $\Gamma < \mathrm{O}^+(V)$ is a discrete subgroup then the orbit space H^n/Γ is a metric space and called a hyperbolic space form. A deep and remarkable result is the Mostow–Prasad rigidity theorem obtained by Daniel Mostow in 1968 in the case of compact hyperbolic space forms and extended by Gopal Prasad in 1973 to finite volume hyperbolic space forms. The proof has been simplified by Michael Gromov in 1981 using ideas of William Thurston.

Theorem 5.1. *If $n \geq 3$ and H^n/Γ_1 and H^n/Γ_2 are hyperbolic space forms of finite volume and $\Gamma_1 \cong \Gamma_2$ are isomorphic as abstract groups then there exists an isometry $g \in \mathrm{O}^+(V)$ of H^n with $\Gamma_2 = g\Gamma_1 g^{-1}$.*

The group Γ is the so called orbifold fundamental group of H^n/Γ and orbifold homotopy equivalence of two finite volume hyperbolic space forms means that they are in fact isometric, at least for $n \geq 3$. For $n = 2$ there is a good deal of flexibility in hyperbolic surface forms. Examples will be given in the next section with hyperbolic Coxeter groups. Note that rigidity evidently fails for spherical or Euclidean compact space forms.

Exercise 5.2. *Check the formulas for the length element and the volume element for the above three models of hyperbolic geometry.*

Exercise 5.3. *Suppose the natural numbers $2 \leq m_1, \cdots, m_k \leq \infty$ satisfy $\sum m_i^{-1} < (k-2)$. Show that the space of convex polygons with successive angles π/m_i in the hyperbolic plane H^2 has $k-3$ real moduli. Check that the area of this polygon is equal to $(k-2-\sum m_i^{-1})\pi$. The group generated by the reflections in the sides of such a polygon is a finite covolume Coxeter group. For $k \geq 4$ these groups are examples of discrete finite covolume groups on hyperbolic space for which rigidity fails in dimension $n = 2$.*

5.2 Hyperbolic Coxeter Groups

Let $M = (m_{ij})$ be a Coxeter matrix with index set I corresponding to a connected Coxeter diagram. We will allow that $m_{ij} = \infty$ for certain $i \neq j$. The indecomposable Gram matrix $G(M) = (g_{ij})$ has been defined by

$$g_{ij} = -2\cos(\pi/m_{ij})$$

for all i, j with $m_{ij} < \infty$, but in case $m_{ij} = \infty$ we shall allow $g_{ij} = g_{ji} \leq -2$. Let W_a be the abstract Coxeter group with generators $S = \{s_i\}$ the set of simple involutions. Let V be the real vector space with basis $\{\alpha_i\}$ and symmetric bilinear form $(\cdot, \cdot)$ with Gram matrix entries $(\alpha_i, \alpha_j) = g_{ij}$ for all $i, j \in I$. If we define by abuse of notation the reflection $s_i \in \mathrm{GL}(V)$ by

$$s_i(\lambda) = \lambda - (\lambda, \alpha_i)\alpha_i$$

then the geometric homomorphism $W_a \to W_g$ and the dual representation have all the same properties derived in the previous chapter. The abstract Coxeter group W_a depends only on the Coxeter matrix M but the geometric Coxeter group W_g depends on both the Coxeter matrix M and a compatible choice of the Gram matrix $G(M)$.

Definition 5.4. *The geometric Coxeter group W_g is called a hyperbolic Coxeter group if the Gram matrix $G(M) = (g_{ij})$ has one negative eigenvalue and all other eigenvalues are ≥ 0.*

Exercise 5.5. *Show that the Coxeter group $W(\mathrm{T}_{pqr})$ associated with the Coxeter graph T_{pqr} with $n = p+q+r-2$ nodes and legs of lengths $p, q, r \geq 2$ is hyperbolic if $1/p + 1/q + 1/r < 1$. In particular, hyperbolic Coxeter groups exist in arbitrary high dimension.*

In the rest of this section we will assume that W_g is a hyperbolic Coxeter group. We denote by K the kernel of the the form $(\cdot, \cdot)$ on V and

$$V' = \{\xi \in V^*; (\xi, \lambda) = 0 \;\forall\; \lambda \in K\}$$

is called the restricted dual vector space. Clearly V/K inherits a natural Lorentzian structure: $(\lambda + K, \mu + K) = (\lambda, \mu)$ for all $\lambda, \mu \in V$. Likewise $V' \simeq (V/K)^*$ becomes a Lorentzian vector space by transport of structure. The geometric Coxeter group W_g acts trivially on K, which in turn implies that the dual action leaves the restricted dual vector space V' invariant. We shall study the action of W_g on the restricted Tits cone $Y' = Y \cap V'$ with its fundamental domain $D' = D \cap V'$. The main question is to understand the structure of $C_J \cap V'$ for $J \subset I$ and notably to decide whether $C_J \cap V'$ is empty or not. We denote by G_J the submatrix of G with indices taken from $J \subset I$.

Proposition 5.6. *Let $J \subset I$ such that G_J is positive definite. Then there exists a vector $\xi_J \in C_J \cap V'$ with $(\xi_J, \xi_J) < 0$ and $C_J \cap V'$ is a facet of the restricted fundamental chamber D' of codimension $|J|$. In particular the* *chamber D' is a closed convex polyhedral cone in V' with nonempty interior $C' = C \cap V'$ and $|I|$ faces and therefore no longer simplicial unless the kernel K of the symmetric bilinear form $(\cdot, \cdot)$ is zero.*

Proof. Let $J \subset I$ such that G_J is positive definite. Let 1_J denote the matrix with 1 on the place ii for $i \notin J$ and 0 elsewhere. For $t \in \mathbb{R}$ sufficiently large the matrix $G + t1_J$ is positive definite, and let t_J be the infimum of these t. Indeed, if for $\lambda = \sum x_i \alpha_i \in V$ we write $y = (x_j) \in \mathbb{R}^J$ and $z = (x_k) \in \mathbb{R}^{I-J}$ then $G_J > 0$ implies

$$(\lambda, \lambda) \geq \epsilon |y|^2 - 2M|y||z| - N|z|^2 = \epsilon(|y| - M|z|/\epsilon)^2 - (N + M^2/\epsilon)|z|^2$$

for some $\epsilon > 0$ and some $M, N \in \mathbb{R}$. Hence

$$(\lambda, \lambda) + t|z|^2 > 0$$

for $t > N + M^2/\epsilon$ unless $y = 0, z = 0$.

Clearly $t_J > 0$ and $G + t_J 1_J$ is positive semidefinite with nonzero kernel. By the Perron–Frobenius theorem this kernel is one dimensional and spanned by a vector x_J with coordinates $x_{Ji} > 0$ for all $i \in I$. Now put

$$\lambda_J = \sum_{i \in I} x_{Ji} \alpha_i \in V \ , \ \xi_J = \sum_{i \notin J} x_{Ji} \xi_i \in V^*$$

with ξ_i the basis of V^* defined by $(\xi_i, \alpha_j) = \delta_{ij}$ for all $i, j \in I$. Then we have on the one hand

$$(\lambda_J, \alpha_j) = 0 \text{ for } j \in J \ , \ (\lambda_J, \alpha_j) = -t_J x_{Jj} \text{ for } j \notin J$$

with the bracket the bilinear form on V. On the other hand

$$(\xi_J, \alpha_j) = 0 \text{ for } j \in J \ , \ (\xi_J, \alpha_j) = x_{Jj} \text{ for } j \notin J$$

with the bracket the pairing between V^* and V. Hence

$$(\lambda_J, \lambda) + (t_J \xi_J, \lambda) = 0$$

for all $\lambda \in V$, which in turn implies that $\xi_J \in V'$ and

$$(\xi_J, \xi_J) = -t_J^{-1}(\xi_J, \lambda_J) = -t_J^{-1} \sum_{i \notin J} x_{Ji}^2 < 0$$

as required. Finally the codimension of C_J as facet of D in V^* is equal to the codimension of $C_J \cap V'$ as facet of D' in V', because the intersection $C_J \cap V'$ is transversal. Indeed the intersection of the span of $\{\alpha_j; j \in J\}$ with the kernel K is zero. $\square$

Let us keep the notation of the proof of the above proposition. In the Lorentzian space V' the set $\{\xi \in V'; (\xi,\xi) < 0\}$ of time like vectors has two connected components and the one containing the point $\xi_\emptyset = \sum x_{\emptyset i}\xi_i$ will be denoted V'_+ and is called the forward time like cone. If $\lambda_\emptyset = \sum x_{\emptyset i}\alpha_i$ then

$$(\xi, \lambda_\emptyset) + t_\emptyset(\xi, \xi_\emptyset) = 0$$

for all $\xi \in V'$ and we conclude that

$$(\xi, \lambda_\emptyset) > 0$$

for all $\xi \in V'_+$. Moreover $(\xi_J, \lambda_\emptyset) = \sum_{i \notin J} x_{Ji}x_{\emptyset i} > 0$ and so $\xi_J \in V'_+$ for all $J \subset I$ with G_J positive definite. Hence $C_J \cap V'_+$ is not empty for $G_J > 0$.

Corollary 5.7. *The intersection $C_J \cap V'_+$ is not empty if and only if the matrix G_J is positive definite and in that case $C_J \cap V'_+$ is a facet of $D \cap V'_+$ of codimension $|J|$.*

Proof. It remains to prove that if $C_J \cap V'_+$ is not empty then G_J is positive definite. The stabilizer of $\xi \in V'_+$ in the Lorentz group $\mathrm{O}^+(V')$ is a compact group. Hence the stabilizer W_J of $\xi \in C_J \cap V'_+$ in the discrete subgroup W_g of $\mathrm{O}^+(V')$ is a finite group, which in turn implies that G_J is positive definite. □

Proposition 5.8. *The forward time like cone V'_+ is contained in the Tits cone $Y' = Y \cap V'$.*

Proof. Since W_g is a discrete subgroup of the Lorentz group $\mathrm{O}^+(V')$ the mirror arrangement is locally finite in V'_+. Hence V'_+ is contained in Y' by Theorem 4.23. □

For every subset $J \subset I$ let V_J be the span of $\{\alpha_j; j \in J\}$. Let us denote

$$Z(J) = \{i \in I; g_{ij} = 0 \,\forall\, j \in J\}\ ,\ N(J) = J \sqcup Z(J)$$

and so V_J and $V_{Z(J)}$ are orthogonal subspaces of V. If the matrix G_J is positive semidefinite but not definite then $G_{N(J)}$ is still positive semidefinite. If G_J is indefinite then $G_{Z(J)}$ is positive definite.

Proposition 5.9. *Let $J \subset I$ be the set of nodes of a connected parabolic subdiagram of the Coxeter diagram. Then $C_{N(J)} \cap V'$ contains a vector $\xi_{N(J)}$ with $(\xi_{N(J)}, \xi_{N(J)}) = 0$ and this isotropic vector $\xi_{N(J)}$ is unique up to a positive scalar.*

Proof. The Coxeter subdiagram with nodes from $N(J)$ is a disjoint union of elliptic and parabolic connected Coxeter diagrams. Now there exists a vector $\lambda_J = \sum k_j \alpha_j \in V_J$ with $(\lambda_J, \alpha_j) = 0$ and $k_j < 0$ for all $j \in J$ by the Perron–Frobenius theorem. Hence

$$(\lambda_J, \alpha_k) = 0 \ , \ (\lambda_J, \alpha_l) > 0$$

for all $k \in N(J)$ and all $l \notin N(J)$. If $\{\xi_i\}$ is the basis of V^* dual to the basis $\{\alpha_i\}$ of V then the vector $\xi_{N(J)} = \sum (\lambda_J, \alpha_i)\xi_i \in C_{N(J)}$ satisfies

$$(\lambda_J, \alpha_i) = (\xi_{N(J)}, \alpha_i)$$

for all i and so $\xi_{N(J)}$ lies in V'. Hence $0 = (\lambda_J, \lambda_J) = (\xi_{N(J)}, \lambda_J)$ and therefore $(\xi_{N(J)}, \xi_{N(J)}) = (\xi_{N(J)}, \lambda_J) = 0$. The conclusion is that $\xi_{N(J)} \in C_{N(J)}$ lies in the closure of the forward time like cone V'_+.

If both $\xi_{N(J)}$ and $\eta_{N(J)}$ are two independent vectors in the intersection of $C_{N(J)}$ with the closure of V'_+ then $\xi_{N(J)} + \eta_{N(J)}$ lies in $C_{N(J)} \cap V'_+$. By Corollary 5.7 the matrix $G_{N(J)}$ is positive definite, which is a contradiction with the submatrix G_J being not positive definite. Hence $\xi_{N(J)}$ and $\eta_{N(J)}$ are positive multiples. $\square$

The intersection $\mathbb{B}^n = V'_+ \cap A'$ of the forward time like cone V'_+ with the affine hyperplane $A' = \{\xi \in V'; (\xi, \lambda_\emptyset) = 1\}$ will be considered the Klein model of hyperbolic space. Since the vector $\lambda_\emptyset = \sum x_{\emptyset i}\alpha_i$ in V has all coefficients $x_{\emptyset i} = (\xi_i, \lambda_\emptyset) > 0$ we conclude that the intersection $D \cap A$ with $A = \{\xi \in V^*; (\xi, \lambda_\emptyset) = 1\}$ is a simplex with vertices $\xi_i / x_{\emptyset i}$ for $i \in I$. In turn this implies that $D \cap A' = D \cap A \cap V'$ a compact convex polytope, with nonempty interior in A' and with $|I|$ facets.

Corollary 5.10. *Suppose $J \subset I$ is the index set of the nodes of a connected parabolic subdiagram of the Coxeter diagram. Suppose that the isotropic vector $\xi_{N(J)}$ is normalized to lie in the boundary $\partial\mathbb{B}^n$ of hyperbolic space $\mathbb{B}^n = V'_+ \cap A'$. Then the intersection of the compact convex polytope $D \cap A'$ with $\mathbb{B}^n$ has locally near $\xi_{N(J)}$ finite volume if and only if the Coxeter subdiagram with nodes from $N(J)$ is a disjoint union of m connected parabolic Coxeter diagrams with $|N(J)| = n + m - 1$.*

Proof. This is immediate in the Poincaré upper half space model $\mathbb{H}^n_+$ of hyperbolic space H^n. Clearly the condition $|N(J)| = n + m - 1$ is equivalent to the rank of $G_{N(J)}$ being $n - 1$. $\square$

Proposition 5.11. *Suppose $J \subset I$ with G_J indefinite. Then the interstion of $C_J \cap A'$ with the closed ball $\mathbb{B}^n \sqcup \partial\mathbb{B}^n$ is empty.*

Proof. If $C_J \cap A'$ contains points of the hyperbolic ball $\mathbb{B}^n$ then G_J is positive definite by Corollary 5.7.

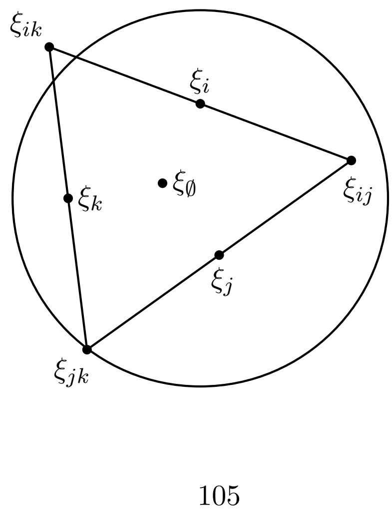
Suppose the intersection $C_J \cap A'$ contains a boundary point $\xi \in \partial\mathbb{B}^n$. The stabilizer of ξ in the Lorentz group $\mathrm{O}^+(V')$ is isomorphic to the motion group of a Euclidean space of dimension $n-1$. Hence the stabilizer W_J of ξ in the discrete subgroup $W_g < \mathrm{O}^+(V')$ is a discrete reflection subgroup of the Euclidean motion group in dimension $n-1$. Hence G_J is positive semidefinite.

The conclusion is that for $J \subset I$ with G_J indefinite the intersection of $C_J \cap A'$ with $\mathbb{B}^n \sqcup \partial\mathbb{B}^n$ is empty. $\square$

If the entry m_{ij} of the Coxeter matrix M is equal to infinity then the entry g_{ij} of the Gram matrix $G(M)$ is smaller or equal to -2. Following Vinberg we distinguish this in the Coxeter diagram by a dashed branch for $g_{ij} < -2$ and a branch with mark ∞ if $g_{ij} = -2$. The connected Coxeter diagram of a hyperbolic Coxeter group W_g with this additional notation will be called a Vinberg diagram.

A dashed branch in a Vinberg diagram means that the two simple mirrors are ultraparallel in hyperbolic space. The left diagram with a dashed branch will be considered a Lannér diagram and as such it should be added in the list of Lannér diagrams in Theorem 4.30.

Below we have drawn a picture for the Vinberg diagram with three nodes with index set $I = \{i, j, k\}$ and $g_{ij} > -2, g_{jk} = -2, g_{ik} < -2$ of the disc $\mathbb{B}^2$ and the compact convex triangle $D \cap A'$ inside the affine plane A'.



Suppose $J \subset I$ such that $C_J \cap A'$ is a facet of $D \cap A'$. Then the conclusion of this section is that we have the following three possibilities. The intersection $C_J \cap \mathbb{B}^n$ is not empty for all J with G_J positive definite. The intersection $C_J \cap (\mathbb{B}^n \sqcup \partial\mathbb{B}^n)$ is empty for all J with G_J indefinite. Finally if G_J is positive semidefinite but not definite then the intersection $C_{N(J)} \cap (\mathbb{B}^n \sqcup \partial\mathbb{B}^n)$ consists of a unique ideal boundary point $\xi_{N(J)}$.

The next result is a direct consequence of the propositions of this section, and is called the Vinberg criterion.

Theorem 5.12. *The hyperbolic Coxeter group W_g acting on hyperbolic space*

$$H^n = \{\xi \in V'; (\xi, \xi) = -1, (\xi, \lambda_\emptyset) > 0\}$$

of dimension n has fundamental domain $D \cap H^n$ of finite hyperbolic volume if and only if the following two conditions on the Vinberg diagram are satisfied:

- *For each connected parabolic subdiagram with index set J the subdiagram with index set $N(J) = J \sqcup Z(J)$ is a disjoint union of $m \geq 1$ connected parabolic subdiagrams with $|N(J)| = n + m - 1$.*
- *For each proper Lannér subdiagram with index set J the intersection $C_{N(J)} \cap V'$ is empty.*

The fundamental domain $D \cap V'_+$ is compact if and only if the Vinberg diagram has no parabolic subdiagrams and the second condition still holds.

Proof. The fundamental domain $D \cap H^n$ has finite volume if and only if in the Klein model the intersection $D \cap A'$ is contained in $\mathbb{B}^n \sqcup \partial\mathbb{B}^n$. Indeed, the convex hull of a finite set of (possibly ideal) points in hyperbolic space has finite volume as can be easily checked in the Poincaré model $\mathbb{H}^n_+$.

Hence each parabolic subdiagram with index set J should be contained in a subdiagram with index set $N(J)$ that is a disjoint union of parabolic diagrams, whose corresponding reflection group $W_{N(J)}$ is cocompact in the stabilizer in $\mathrm{O}^+(V')$ of the the ideal point $\xi_{N(J)} \in \partial\mathbb{B}^n$. Since this stabilizer is the motion group of a Euclidean space of dimension $n-1$ (clear in the Poincaré model) we have to require $|N(J)| - m = n - 1$.

Likewise, for each proper $J \subset I$ with G_J indefinite we should have $C_J \cap A' = \emptyset$. For $J \subset N \subset I$ the intersection $C_N \cap A'$ lies in the closure of $C_J \cap A'$. Therefore the condition that $C_J \cap A'$ is empty for all J with G_J indefinite is satisfied as soon as $C_J \cap A'$ is empty for all minimal J with G_J indefinite. The minimality of such J implies that it suffices to check that $C_J \cap A'$ is empty for all Lannér and Koszul subdiagrams. But for the Koszul subdiagrams with index set J we have that $C_J \cap A'$ is empty because of the first condition. Hence the theorem follows from the next lemma. ∎

Lemma 5.13. *Suppose that $J \subset I$, $L \subset Z(J)$ and the matrix G_L is positive definite. Then $C_J \cap V' = \emptyset$ whenever $C_{J \sqcup L} \cap V' = \emptyset$.*

Proof. By induction on the number of connected components of the subdiagram with index set L the proof is reduced to the case that this subdiagram is connected. Hence we can and will assume that the Gram matrix G_L is indecomposable.

The condition $C_{J \sqcup L} \cap V' = \emptyset$ means that there exist $c_i \in \mathbb{R}$ with $c_i > 0$ for all $i \notin J \sqcup L$ such that $\sum c_i \alpha_i$ lies in the kernel K of the symmetric bilinear form $(\cdot, \cdot)$ on V. For each $l \in L$ we get

$$\sum_{k \in L} c_k g_{kl} = \sum_{k \in J \sqcup L} c_k g_{kl} = - \sum_{i \notin J \sqcup L} c_i g_{il} \geq 0$$

and the inequality is strict for at least one $l \in L$. Otherwise G_L would be a direct summand of the indecomposable Gram matrix $G(M)$.

Because G_L is positive definite and indecomposable all entries of its inverse G_L^{-1} are positive, and by the above inequality we get $c_k > 0$ for all $k \in L$. Hence $c_i > 0$ for all $i \notin J$, and therefore $C_J \cap V' = \emptyset$. $\square$

Exercise 5.14. *Let $\{\alpha_i\}$ be an obtuse indecomposable basis of a Euclidean vector space V, so $(\alpha_i, \alpha_j) \leq 0$ for all $i \neq j$ and there is no nontrivial partition $I \sqcup J$ of the index set such that $(\alpha_i, \alpha_j) = 0$ for all $i \in I, j \in J$. Show that the dual basis ξ_i of V defined by $(\xi_i, \alpha_j) = \delta_{ij}$ is strictly sharp, so $(\xi_i, \xi_j) > 0$ for all i, j.*

Remark 5.15. *The first condition of the Vinberg criterion is easy to check from the Vinberg diagram. But the second condition for proper Lannér subdiagrams with index set J might be difficult to check. However, it is always verified if*

$$|N(J)| = n + 1$$

for that Lannér index set J. Indeed, then $V_{N(J)} = \mathbb{R}N(J)$ has dimension $n + 1$. Since the symmetric bilinear $(\cdot, \cdot)$ is nondegenerate Lorentzian on $V_{N(J)}$ the intersection of $V_{N(J)}$ with the kernel K of $(\cdot, \cdot)$ on V is zero. Hence we get $C_{N(J)} \cap V' = \emptyset$, because $V_{N(J)} + K = V$ and so $\{\xi \in V'; (\xi, \lambda) = 0 \; \forall \lambda \in V_{N(J)}\} = \{0\}$.

In the rest of this chapter we will study hyperbolic Coxeter groups acting on hyperbolic space H^n with a fundamental domain $D \cap H^n$ of finite volume. In the next section we shall give plenty examples of such groups.

5.3 Examples of Hyperbolic Coxeter Diagrams

Suppose first that the symmetric bilinear form $(\cdot,\cdot)$ on V is nondegenerate with Lorentzian signature. The intersection $D \cap A$ of the closed fundamental chamber with the affine hyperplane $A = \{\xi \in V^*; (\xi, \lambda_\emptyset) = 1\}$ is a simplex with vertices corresponding to the maximal proper subsets $J \subset I$. The nature of the subdiagram with nodes from J gives information of the location of that vertex. Indeed, a vertex lies inside $\mathbb{B}^n = V_+ \cap A$ if the subdiagram is elliptic, lies on the boundary $\partial\mathbb{B}^n$ if the subdiagram is parabolic and lies outside $\mathbb{B}^n \sqcup \partial\mathbb{B}^n$ if the subdiagram is hyperbolic.

The conclusion is that $D \cap A$ lies inside $\mathbb{B}^n$ if the Vinberg diagram is a Lannér diagram, and $D \cap A$ lies inside $\mathbb{B}^n \sqcup \partial\mathbb{B}^n$ if the Vinberg diagram is a Koszul diagram. The classification of Lannér and Koszul diagrams amounts to the classification of reflection groups W acting on hyperbolic space H^n with a simplex as fundamental domain of finite volume.

For the rest of this section we shall look at Vinberg diagrams with $\det G(M) = 0$. The first few examples are Vinberg diagrams of compact type, which means that the compact convex polytope $D \cap A'$ in A' is contained in the open ball $\mathbb{B}^n = V'_+ \cap A'$. These examples are due to Vinberg, Makarov and Bugaenko [61],[37],[10].

The first example has Vinberg diagram

$$\begin{array}{ccccccccccc} & 8 & & & & 4 & & & & 8 & \\ \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet \\ -z & & -y & & -x & & x & & y & & z \end{array}$$

with 6 nodes numbered from left to right. The coefficients $g_{12} = g_{56}$ are equal to $-2\cos(\pi/8) = -\sqrt{2+\sqrt{2}}$ using $4\cos^2\phi = 2 + 2\cos(2\phi)$. Hence the Gram matrix G has a one dimensional kernel with coefficients indicated below the corresponding nodes with $x, y, z \in \mathbb{R}$ solutions of the three linear equations

$$2x = -x\sqrt{2} + y \ , \ 2y = x + z\sqrt{2+\sqrt{2}} \ , \ 2z = y\sqrt{2+\sqrt{2}}$$

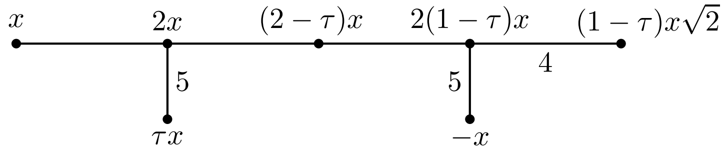
which reduce to two independent linear equations

$$2x = y(2-\sqrt{2}) \ , \ 2z = y\sqrt{2+\sqrt{2}}$$

with one dimensional solution space. We claim that $D \cap H^4$ is compact. First remark that the Vinberg diagram has no parabolic subdiagrams as should. For the two Lannér subdiagrams with index set J equal to the three left or the three right nodes we have $C_J \cap V' = \emptyset$ by the Vinberg criterion $|N(J)| = 5$ in Remark 5.15.

The vertices of $D \cap A'$ arise as $C_J \cap A'$ for $J \subset I$ with G_J positive definite of rank 4. These 9 subsets J are the complement of $\{k, l\}$ with k from the left three and l from the right three nodes. The conclusion is that the combinatorial type of $D \cap A'$ is a product of two triangles.

Exercise 5.16. *Consider the Vinberg diagram*



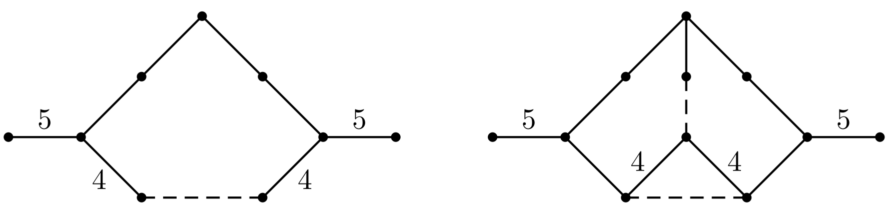
with 7 nodes. Check that the Gram matrix $G(M)$ has a one dimensional kernel (see the above diagram). Show that the combinatorial type of the compact convex polytope $D \cap A'$ is a product of a tetrahedron with a triangle. Show that of its 12 vertices 9 lie inside and 3 lie outside hyperbolic space $\mathbb{B}^5 = V'_+ \cap A'$. We refer to Vinberg [61] for a further discussion of this example, and how to arrive by excision of these 3 vertices at a compact convex polytope in $\mathbb{B}^5$ with only dihedral angles π/m for $m = 2, 3, 4, 5$.

Exercise 5.17. *Suppose that a Vinberg diagram has a Gram matrix $G(M)$ with a one dimensional kernel and a compact fundamental polytope $D \cap \mathbb{B}^n$ with $\mathbb{B}^n = V'_+ \cap A'$ in the Klein model of hyperbolic space. Show that the combinatorial type of $D \cap \mathbb{B}^n$ is a product of two simplices.*

The following two Vinberg diagrams were discovered by Bugaenko in 1984 in his study of reflection groups associated with the quadratic form

$$-\tau x_0^2 + x_1^2 + \cdots + x_n^2$$

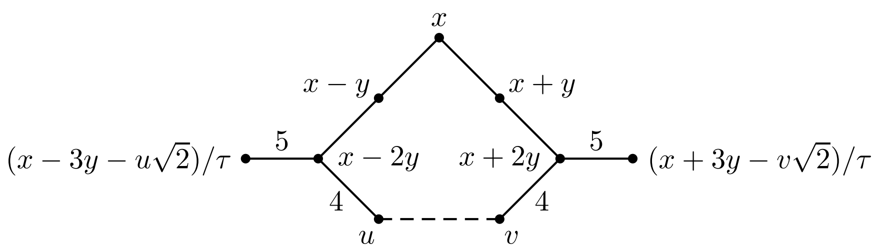
with the usual notation $\tau = (1 + \sqrt{5})/2$ for the golden ratio [10].



with 9 and 11 nodes respectively.

Theorem 5.18. *For the above two Vinberg diagrams the compact polytope $D \cap A'$ is contained in $\mathbb{B}^n = V'_+ \cap A'$ with $n = 6, 7$ respectively.*

Proof. For the first diagram



we compute the coefficients x_j of $\sum x_j\alpha_j$ in the kernel of $(\cdot,\cdot)$ and place them in the diagram next to the corresponding node. The equations become

$$2(x-3y-u\sqrt{2})/\tau = \tau(x-2y)\ ,\ 2(x+3y-v\sqrt{2})/\tau = \tau(x+2y)$$

or equivalently

$$2u\sqrt{2} = (1-\tau)x - 2(2-\tau)y\ ,\ 2v\sqrt{2} = (1-\tau)x + 2(2-\tau)y$$

and

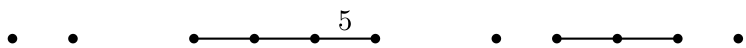
$$2u = (x-2y)\sqrt{2} + 2gv\ ,\ 2v = (x+2y)\sqrt{2} + 2gu$$

with the Gram matrix entry of the dashed branch equal to $-2g < -2$. The solutions are

$$u = \frac{-x}{(g-1)\sqrt{2}} - \frac{y\sqrt{2}}{(g+1)}\ ,\ v = \frac{-x}{(g-1)\sqrt{2}} + \frac{y\sqrt{2}}{(g+1)}\ ,\ g = \frac{\tau+1}{\tau-1} = 2\tau+1$$

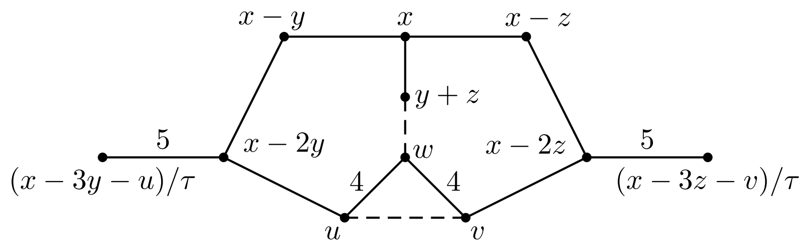
for which the kernel of $(\cdot,\cdot)$ has dimension 2 with free parameters x, y. First observe that there are no parabolic subdiagrams. Up to Vinberg diagram automorphisms there are three Lannér subdiagrams

with index set J and with Vinberg diagram of $Z(J)$ equal to



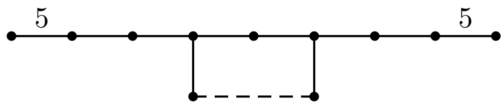
respectively. Hence $|N(J)| = 7$ in all three cases and the statement of the theorem follows from the Vinberg criterion. $\square$

Exercise 5.19. *Work out the second example of Bugaenko with the kernel of the Gram matrix $G(M)$ of dimension 3. In the above notation check that*

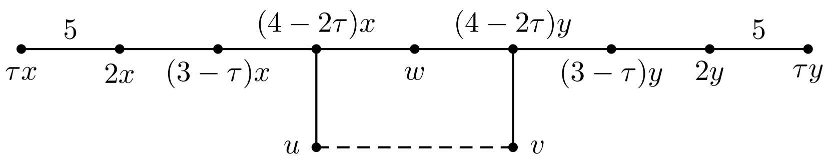


with $2u = (1-\tau)x+2(\tau-2)y, 2v = (1-\tau)x+2(\tau-2)z$ has solution $2w = \cdots$ with three free parameters x, y, z for suitable weights of the dashed branches.

For some time no examples were known of hyperblic reflection groups W_g acting on H^n with a compact fundamental domain $D \cap H^n$ for $n \geq 8$. However in 1992 Bugaenko found another such example for $n = 8$ with 11 faces [11]. Its Vinberg diagram is give by [64]



Exercise 5.20. *Show that the above diagram of Bugaenko is the Vinberg diagram of a cocompact reflection group W_g acting on hyperbolic space H^8. Hint: In case the Gram matrix $G(M)$ entry corresponding to the dashed branch is equal to -2τ check that the kernel of $G(M)$ has dimension 2 and is given by*

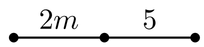


in the notation as above, with $u = (3-2\tau)x-(2-\tau)y, v = (3-2\tau)y-(2-\tau)x$ *and* $w = (2-\tau)(x+y)$, *and with* x, y *free real parameters. Subsequently apply the Vinberg criterion of Remark 5.15 to the various Lannér subdiagrams.*

The following no go result is due to Vinberg [62].

Theorem 5.21. *There are no hyperbolic reflection groups* W_g *acting on* H^n *with a compact fundamental domain* $D \cap H^n$ *for* $n \geq 30$.

The reflection group W generated by the three reflections in the sides of a hyperbolic triangle with angles $\pi/5, \pi/2, \pi/2m$ for $m = 2, 3, \cdots, \infty$ at vertices ξ_1, ξ_2, ξ_3 respectively has Coxeter diagram

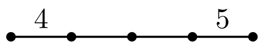


The dihedral group $W(\mathrm{I}_2(5))$ of order 10 acts with fixed point ξ_1 and 10 triangles around ξ_1 glue together to form a pentagon with angles π/m. The reflection subgroup W' generated by the 5 reflections in the sides of this pentagon is a normal subgroup of W with $W/W' \cong W(\mathrm{I}_2(5))$.

Similarly the reflection group W of the Lannér and Koszul diagrams

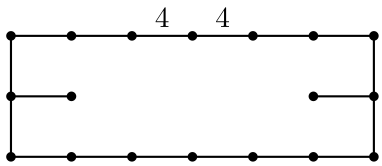
has a simplicial fundamental chamber $D \cap H^3$. In the first case $D \cap H^3$ is compact, and in the second case $D \cap H^3$ has finite volume and the vertex ξ_4 (number from left to right) is ideal. The icosahedral reflection group $W(\mathrm{H}_3)$ of order 120 acts on H^3 with fixed point ξ_1, and 120 of these simplices around ξ_1 glue together to form a hyperbolic dodecahedron with with dihedral angles $\pi/2$ and $\pi/3$ respectively. The reflection subgroup W' generated by the 12 reflections in the faces of the dodecahedron is a normal subgroup of W with $W/W' \cong W(\mathrm{H}_3)$. The dihedral angles of these two dodecahedra are $\pi/2, \pi/3$ respectively. For the first example of the hyperbolic dodecahedron with only orthogonal dihedral angles one can construct a further index 2 normal subgroup W'' of W' with fundamental polyhedron two dodecahedra glued along a common face. Iterating this procedure constructs an infinite number of cocompact reflection groups in hyperbolic space H^3. By this kind of method Allcock has shown that for each $n \leq 19$ there are infinitely many cofinite volume hyperbolic reflection groups in H^n [2].

Exercise 5.22. *Show that the hyperbolic space H^4 has a regular tessellation with congruent 120-cells with all dihedral angles $\pi/2$, and each 120-cell is obtained by gluing 14400 fundamental simplices for the Coxeter group with Lannér diagram*

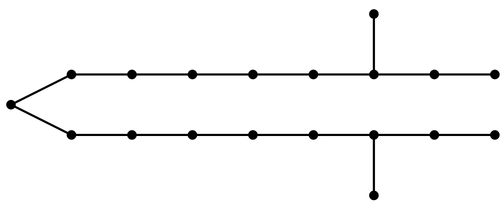


at the common node $\xi_1 \in D \cap H^4$ with stabilizer $W(\mathrm{H}_4)$.

We end this section by a discussion of two cofinite volume hyperbolic reflection groups in hyperbolic space H^n of dimension $n = 15$ and $n = 17$. Their Vinberg diagrams are given below. The first one has 18 nodes and its Gram matrix has rank 16. The second one has 19 nodes and its Gram matrix has rank 18. Both Vinberg diagrams have no Lannér subdiagrams, and the Vinberg criterion $|N(J)| = n + m - 1$ is easily applied.



Indeed the first Vinberg diagram has parabolic subdiagrams of type $\tilde{\mathrm{B}}_6 + \tilde{\mathrm{E}}_8$, $\tilde{\mathrm{C}}_2 + \tilde{\mathrm{D}}_{12}$ and $2\tilde{\mathrm{E}}_7$, and $6 + 8 = 2 + 12 = 7 + 7 = 15 - 1$ as should.



The second Vinberg diagram has parabolic subdiagrams of type $\tilde{\mathrm{D}}_{16}$ and $2\tilde{\mathrm{E}}_8$, and $16 = 8 + 8 = 17 - 1$ as should.

Examples of hyperbolic reflection groups with a fundamental domain $D \cap H^n$ of finite volume are easier to find for large n than the ones with compact fundamental domain. The largest dimension $n = 21$ known of such an example is due to Borcherds [7]. Again there is a no go theorem due to Khovanskii and Prokhorov [31],[45].

Theorem 5.23. *There are no hyperbolic reflection groups W_g acting on H^n with a fundamental domain $D \cap H^n$ of finite volume for $n \geq 996$.*

The gap in the numbers n between the examples of cocompact or finite covolume reflection groups on H^n and the bounds on $n = \dim H^n$ from the no go theorems are substantial. Maybe these bounds can be further improved, but it is unlikely to have any reasonable sort of classification of the ocean full of hyperbolic Coxeter groups acting on H^n with a compact or finite volume fundamental domain $D \cap H^n$. There is however a classification of such groups with a Vinberg diagram of $n+2$ nodes due to Esselman in the cocompact case and Tumarkin in the cofinite volume case [26],[58]. Even the case with $n+3$ faces has been settled by Tumarkin [59]. But the conclusion of the paper by Allcock [2] is that these lists are just a small tip of the iceberg.

5.4 Hyperbolic reflection groups

Let V be a Lorentzian vector space of dimension $n+1$ with scalar product $(\cdot,\cdot)$. Let H^n be a connected component of the two sheeted hyperboloid $\{\lambda \in V; (\lambda,\lambda) = -1\}$, which is a model for hyperbolic space of dimension n. For $\alpha \in V$ with $(\alpha,\alpha) > 0$ the orthogonal transformation $s_\alpha \in \mathrm{O}^+(V)$ defined by

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha$$

is called a reflection of H^n with root α and coroot $\alpha^\vee = 2\alpha/(\alpha,\alpha)$.

Definition 5.24. *A hyperbolic reflection group $W < \mathrm{O}^+(V)$ is a discrete subgroup generated by orthogonal reflections.*

We shall always assume that for each reflection $s \in W$ we have chosen two roots $\pm\alpha \in V$ with $s = s_\alpha$, in such a way that the collection of all these roots, denoted R and called a root system underlying the reflection group W, is invariant under W. A canonical way would be the normalized root system R with $(\alpha,\alpha) = 2$ for all $\alpha \in R$. But sometimes it might be possible by varying root lengths that $(\beta,\alpha^\vee) \in \mathbb{Z}$ for all $\alpha,\beta \in R$, in which case R is called an integral root system.

We can now repeat the discussion of Section 2.1. Say R is the normalized root system for W. The forward time like cone $V_+ = \mathbb{R}_+ H^n$ is open and convex. Because the reflection group $W < \mathrm{O}^+(V)$ is a discrete subgroup the collection of all mirrors $\{\alpha^\perp; \alpha \in R\}$ is locally finite on V_+. Let us denote by V° the complement in V_+ of all the mirrors. Connected components of V° are called chambers. Let us fix a chamber, called the positive chamber, and denote it by C_+. Let R_+ be the set of positive roots relative to C_+, that is $R_+ = \{\alpha \in R; (\lambda, \alpha) > 0\ \forall \lambda \in C_+\}$. A positive root $\alpha \in R_+$ is called simple if α is not of the form $\alpha = x_1\alpha_1 + x_2\alpha_2$ with $x_1, x_2 \geq 1$ and $\alpha_1, \alpha_2 \in R_+$. The next theorem is proved in the same way as Theorem 2.12 and Proposition 2.15.

Theorem 5.25. *The positive chamber C_+ is a fundamental domain for the action of W on the mirror complement V° in V_+. The walls of C_+ support the mirrors of the simple roots in R_+, and the group W is generated by the simple reflections. If two walls of C_+ corresponding to two simple roots $\alpha_i, \alpha_j \in R_+$ meet along a plinth inside V_+ then $(\alpha_i, \alpha_j) = -2\cos(\pi/m_{ij})$ for some $m_{ij} \in \mathbb{Z}, m_{ij} \geq 2$. If the two walls do not meet inside V_+ then we put $m_{ij} = \infty$.*

It follows that W is just equal to the geometric Coxeter group W_g for the Gram matrix with entries $g_{ij} = (\alpha_i, \alpha_j)$. Two walls are parallel if $g_{ij} = -2$ and are ultraparallel if $g_{ij} < -2$. In particular the closure $D_+ = \mathrm{Clos}(C_+)$ of the positive chamber is a strict fundamental domain for the action of W on the (possibly ideal) hyperbolic part $Y_+ = Y \cap \mathrm{Clos}(V_+)$ of the Tits cone Y. The main differences with the case of finite reflection groups in Chapter 2 is that two walls of V_+ can be parallel or even ultraparallel. In addition the simple roots need not be a basis of V, so D_+ need not be a simplical cone. In fact it can even happen that there are infinitely many simple roots.

Exercise 5.26. *Let W_∞ be the reflection group acting on the upper half plane $\mathbb{H}^2$ generated by the reflections s_k in the semicircles $|z-k| = 1, \Im z > 0$ with radius 1 en centers k for $k \in \mathbb{Z}$. Show that the region*

$$C_+ = \{z \in \mathbb{H}^2; |z-k| > 1\ \forall k \in \mathbb{Z}\}$$

is a positive chamber for the action of W on $\mathbb{H}^2$. Show that the group $\mathrm{Isom}(C_+)$ of isometries of C_+ is isomorphic to the infinite dihedral group $\mathcal{D}_\infty$. Show that the group $W = W_\infty \rtimes \mathrm{Isom}(C_+)$ is generated by the three reflections

$$z \mapsto 1/\overline{z},\ z \mapsto -\overline{z},\ z \mapsto 1-\overline{z}$$

and determine the Coxeter diagram of this latter group. Show that the index two subgroup $W_+ < W$ of orientation preserving isometries is generated by the two transformations $S : z \mapsto -1/z$ and $T : z \mapsto 1+z$. It is well known that the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acting on the hyperbolic plane $\mathbb{H}^2$ by fractional linear transformations is generated by S and T.

5.5 Lorentzian Lattices

Let V be a real vector space with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ of signature either $(n,0)$ or $(n,1)$ making V a Euclidean or a Lorentzian vector space respectively. A lattice L in V is the integral span of a vector space basis of V, and L is called integral if $(\lambda,\mu)\in\mathbb{Z}$ for all $\lambda,\mu\in L$. In this section all lattices will be integral, and we shall use the word lattice for integral lattice. We shall speak of Euclidean or Lorentzian lattices in case V is a Euclidean or a Lorentzian vector space.

We will review some basic results about lattices, and for proofs refer to the Cours d'aritmétique by Serre [51]. If $\{\lambda_i\}$ is a lattice basis of L then the number $d(L)=|\det(\lambda_i,\lambda_j)|$ is independent of the choice of basis, and called the discriminant of L. The dual lattice $L^*=\{\xi\in V;(\xi,\lambda)\in\mathbb{Z}\;\forall\;\lambda\in L\}$ is in general only a rational lattice, containing L as sublattice of index $d(L)$. In case $d(L)=1$ or equivalently $L^*=L$ the lattice L is called unimodular. The lattice L is called even is the norms $\lambda^2=(\lambda,\lambda)$ are even for all $\lambda\in L$. If L is not even then L is called odd.

The next theorem is due to Siegel [52],[8].

Theorem 5.27. *For a Lorentzian lattice L of rank $n+1$ the automorphism group $\mathrm{O}^+(L)$ acts on hyperbolic space H^n with a fundamental domain of finite volume. The fundamental domain is compact if and only if the zero vector is the only isotropic vector $\lambda\in L$ with $\lambda^2=0$.*

The next theorem is due to Meyer [51].

Theorem 5.28. *A Lorentzian lattice L of rank $n+1\geq 5$ has always nonzero isotropic vectors.*

The Euclidean lattice $\mathbb{Z}^n$ has basis $\{\varepsilon_i\}$ for $1\leq i\leq n$ with $(\varepsilon_i,\varepsilon_j)=\delta_{ij}$ for $1\leq i,j\leq n$. Likewise the Lorentzian lattice $\mathbb{Z}^{n,1}$ has basis $\{\varepsilon_i\}$ for $0\leq i\leq n$ with $(\varepsilon_i,\varepsilon_j)=\delta_{ij}$ for $(i,j)\neq(0,0)$ and $(\varepsilon_0,\varepsilon_0)=-1$. These are odd unimodular lattices. The root lattice of type E_8 is an even unimodular Euclidean lattice and is sometimes denoted just E_8. The root lattice D_n is the index 2 sublattice of $\mathbb{Z}^n$ of even norm vectors. The even unimodular rank 2 Lorentzian lattice with basis $\{\varepsilon_0,\varepsilon_1\}$ and $(\varepsilon_0,\varepsilon_0)=(\varepsilon_1,\varepsilon_1)=0,(\varepsilon_0,\varepsilon_1)=-1$ is denoted by U, and called the hyperbolic plane lattice.

The classification of unimodular odd or even Euclidean lattices is very subtle, while for unimodular Lorentzian lattices (or more generally for unimodular indefinite lattices) the classification has a very simple form [18], [51].

Theorem 5.29. *Every odd unimodular Lorentzian lattice is isomorphic to the lattice* $\mathbb{Z}^{n,1} = -\mathbb{Z} \oplus \mathbb{Z}^n$ *for* $n \geq 1$*, with* $-\mathbb{Z}$ *the negative definite rank one unimodular lattice. Every even unimodular Lorentzian lattice is isomorphic to the lattice* $\mathrm{U} \oplus \mathrm{E}_8 \oplus \cdots \oplus \mathrm{E}_8$ *(*n *summands* E_8*) and so exists only and uniquely in rank* $8n+2$ *for some integer* $n \geq 0$*.*

Definition 5.30. *A Lorentzian lattice* L *is called a root lattice if its root system*

$$R(L) = \{\alpha \in L; \alpha^\vee \in L^*\}$$

spans L*. Note that norm one and norm two vectors in* L *are always roots. Let* L *be a Lorentzian root lattice. The orthogonal reflections*

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha \ , \ \alpha \in R(L)$$

lie in $\mathrm{O}^+(L)$ *and generate a subgroup* $W(L) < \mathrm{O}^+(L)$*, called the reflection group of* L*. The Lorentzian root lattice* L *is called reflective if* $W(L)$ *has finite index in* $\mathrm{O}^+(L)$*.*

By the above theorem of Siegel the reflection group $W(L)$ of a reflective Lorentzian lattice L of rank $n+1$ has a fundamental domain in H^n of finite volume. Indeed the volume of $H^n/W(L)$ is equal to the volume of $H^n/\,\mathrm{O}^+(L)$ times the index $[\mathrm{O}^+(L) : W(L)]$.

Exercise 5.31. *Show that the bound* rank ≥ 5 *in the theorem of Meyer is sharp using an example from Theorem 4.30.*

Let L be a Lorentzian root lattice with root system $R = R(L)$. We shall describe an algorithm due to Vinberg for constructing a basis of simple roots [60]. We need from the start a nonzero vector $\kappa \in L^*$ in the closure of the forward time like cone V_+, called a controlling vector. For $\lambda \in L$ we denote by $\mathrm{ht}(\lambda) = (\lambda, \kappa) \in \mathbb{Z}$ the height of λ with respect to the controlling vector κ. The root subsystem

$$R' = \{\alpha \in R; \mathrm{ht}(\alpha) = 0\}$$

is of at most parabolic type, and so an orthogonal direct sum of finite and affine irreducible root systems. If R'_+ is a set of positive roots in R' then the set

$$R_+ = R'_+ \sqcup \{\alpha \in R; \mathrm{ht}(\alpha) \geq 1\}$$

is a set of positive roots in R. A root $\alpha \in R_+$ is simple if α can not be written as $\alpha = \beta_1 + \beta_2$ with $\beta_1, \beta_2 \in R_+$. The simple roots in R_+ are an obtuse set.

Let D be the unique fundamental chamber for $W(L)$ with $\kappa \in D$ and $(\xi, \alpha) \geq 0$ for all $\xi \in D$ and $\alpha \in R'_+$. Then simple roots $\alpha \in R_+$ of positive height are chosen according to the hyperbolic distance $d(\alpha)$ of $\kappa/\sqrt{-(\kappa,\kappa)}$ to the mirror corresponding to α (say $(\kappa,\kappa) < 0$). Since

$$\sinh^2 d(\alpha) = -\frac{(\alpha,\kappa)^2}{(\alpha,\alpha)(\kappa,\kappa)}$$

this amounts indeed to search for simple roots of the same norm by height.

Theorem 5.32. *Let L be a Lorentzian lattice, and write $R = R_{12}$ for the root system of all norm one and norm two vectors in L, and let $W = W_{12}$ be the subgroup of $\mathrm{O}^+(L)$ generated by all reflections in these roots. Fix a controlling vector $\kappa \in L^*$ in the closure of the forward time like cone V_+, and let $\mathrm{ht}(\alpha) = (\alpha, \kappa)$ be the corresponding height function on R with values in $\mathbb{Z}$. Let R_+ be a positive subsystem of R of roots of height ≥ 0, or equivalently let $R_+ = R'_+ \sqcup \{\alpha \in R; \mathrm{ht}(\alpha) \geq 1\}$ with R'_+ a positive subsystem of the root system R' of height zero roots.*

If $\alpha \in R_+$ with $\mathrm{ht}(\alpha) \geq 1$ and $(\alpha,\alpha) = 2$ then α is a simple root in R_+ if and only if $(\alpha, \beta) \leq 0$ for all simple roots $\beta \in R_+$ with $\mathrm{ht}(\beta) \leq \mathrm{ht}(\alpha)/2$.

Proof. Since distinct simple roots are obtuse the condition of the theorem is clearly necessary.

Conversely, let $\alpha \in R_+$ with $\mathrm{ht}(\alpha) \geq 1, (\alpha,\alpha) = 2$ and $(\alpha,\beta) \leq 0$ for all simple roots $\beta \in R_+$ with $\mathrm{ht}(\beta) \leq \mathrm{ht}(\alpha)/2$. Assume α is not simple. Then there exists a simple root $\gamma > 0$ with $(\alpha,\gamma) \geq 1$ and $s_\gamma(\alpha) > 0$. Write $\alpha = \sum n_i \alpha_i$ as a nonnegative linear combination of simple roots. If $\gamma \neq \alpha_i$ for all i with $n_i \geq 1$ then $(\alpha, \gamma) = \sum n_i(\alpha_i, \gamma) \leq 0$ because distinct simple roots are obtuse. This is a contradiction with the assumption $(\alpha,\gamma) \geq 1$ and therefore we may assume $\gamma = \alpha_1$ with $n_1 \geq 1$ after possible renumeration.

Since $(\alpha, \alpha_1) \geq 1$ we have $\mathrm{ht}(\alpha_1) > \mathrm{ht}(\alpha)/2$ by the assumptions on α. Because

$$\mathrm{ht}(s_1(\alpha)) = \mathrm{ht}(\alpha) - (\alpha, \alpha_1^\vee)\mathrm{ht}(\alpha_1) \geq 0$$

we conclude that $(\alpha, \alpha_1^\vee) = 1$ and $\mathrm{ht}(s_1(\alpha)) < \mathrm{ht}(\alpha)/2$. Since $(\alpha, \alpha_1) \geq 1$ and $(\alpha, \alpha_1^\vee) = 1$ we get $(\alpha_1, \alpha_1) = 2$ and $(\alpha, \alpha_1) = 1$. If $n_1 \geq 2$ then

$$0 \leq \mathrm{ht}(\alpha - 2\alpha_1) < 0$$

gives a contradiction. Hence $n_1 = 1$ and $\mathrm{ht}(\alpha_i) < \mathrm{ht}(\alpha)/2$ for all $i \geq 2$ with $n_i \geq 1$.

On the one hand, we have

$$(\alpha, \alpha - \alpha_1) = \sum_{i \geq 2} n_i (\alpha, \alpha_i) \leq 0$$

because $(\alpha, \alpha_i) \leq 0$ for all $i \geq 2$ with $n_i \geq 1$. On the other hand, we get

$$(\alpha, \alpha - \alpha_1) = (\alpha, \alpha) - (\alpha, \alpha_1) = 1$$

and we arrive at a contradiction. This contradiction is a consequence of the assumption that α is not a simple root, and therefore we deduce that α is simple in R_+. $\square$

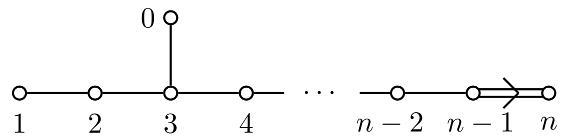
If the Vinberg algorithm produces simple roots α_i for $i \in I$, which are all of height $\leq N$ and for which the cone

$$D = \{\lambda \in V; (\lambda, \alpha_i) \geq 0 \; \forall i \in I\}$$

is contained in the closure of the forward time like cone V_+ then the Vinberg algorithm terminates at height N. If the number of simple roots of height $\leq N$ is finite then the condition $D \subset \mathrm{Clos}(V_+)$ is usually verified using the Vinberg criterion of Theorem 5.12.

A first application of the Vinberg algorithm is the following theorem of Vinberg about the case of the odd unimodular Lorentzian lattice [60].

Theorem 5.33. *The lattice $\mathbb{Z}^{n,1}$ is reflective for $n = 4 \leq n \leq 9$ with*



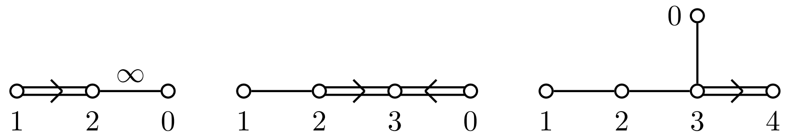
the Vinberg diagram of $W(\mathbb{Z}^{n,1}) = \mathrm{O}^+(\mathbb{Z}^{n,1})$.

Proof. Take $\kappa = -\varepsilon_0$ as controlling vector. Then the zero height root system R' is of type B_n with simple roots

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 \;, \; \cdots \;, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n \;, \; \alpha_n = \varepsilon_n$$

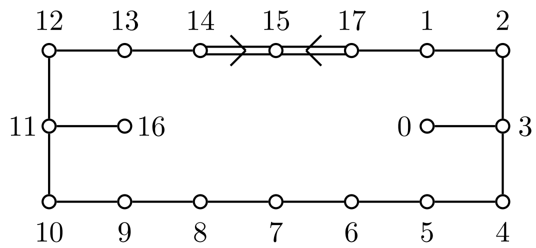
as in Example 2.50. The root $\alpha_0 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$ has height 1, norm 2 and $(\alpha_1, \alpha_i) \leq 0$ for all $i \geq 1$. Hence α_0 is a simple root. The Dynkin diagram is a Koszul diagram, and therefore there are no other roots $\alpha \in R$ with $(\alpha, \alpha_i) \leq 0$ for all $i \geq 0$. The group $W(\mathbb{Z}^{n,1})$ is a normal subgroup of $\mathrm{O}^+(\mathbb{Z}^{n,1})$ and so $\mathrm{O}^+(\mathbb{Z}^{n,1})/W(\mathbb{Z}^{n,1})$ acts as the group of Dynkin diagram automorphisms. Since the Dynkin diagrams found above have no nontrivial automorphisms we conclude that $W(\mathbb{Z}^{n,1}) = \mathrm{O}^+(\mathbb{Z}^{n,1})$. $\square$

Exercise 5.34. *Show using the controlling vector* $\kappa = -\varepsilon_0$ *that for* $n = 2, 3, 4$ *we have* $W(\mathbb{Z}^{n,1}) = \mathrm{O}^+(\mathbb{Z}^{n,1})$ *and check that*



are their Vinberg diagrams, with simple roots denoted as in the above proof, except that $\alpha_0 = \alpha_0 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2$ *is a norm* 1 *vector in case* $n = 2$.

Exercise 5.35. *Show using the controlling vector* $\kappa = -\varepsilon_0$ *that* $W(\mathbb{Z}^{15,1})$ *is an index two subgroup of* $\mathrm{O}^+(\mathbb{Z}^{15,1})$ *with Vinberg diagram with* 18 *nodes*

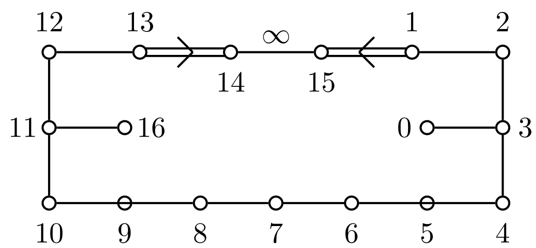


for the simple roots given by $\alpha_0 = \varepsilon_0 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$, *and* $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ *for* $1 \le i \le 14$, $\alpha_{15} = \varepsilon_{15}$, *and* $\alpha_{16} = 3\varepsilon_0 - (\varepsilon_1 + \cdots + \varepsilon_{11})$, *and finally* $\alpha_{17} = 4\varepsilon_0 - (2\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{15})$.

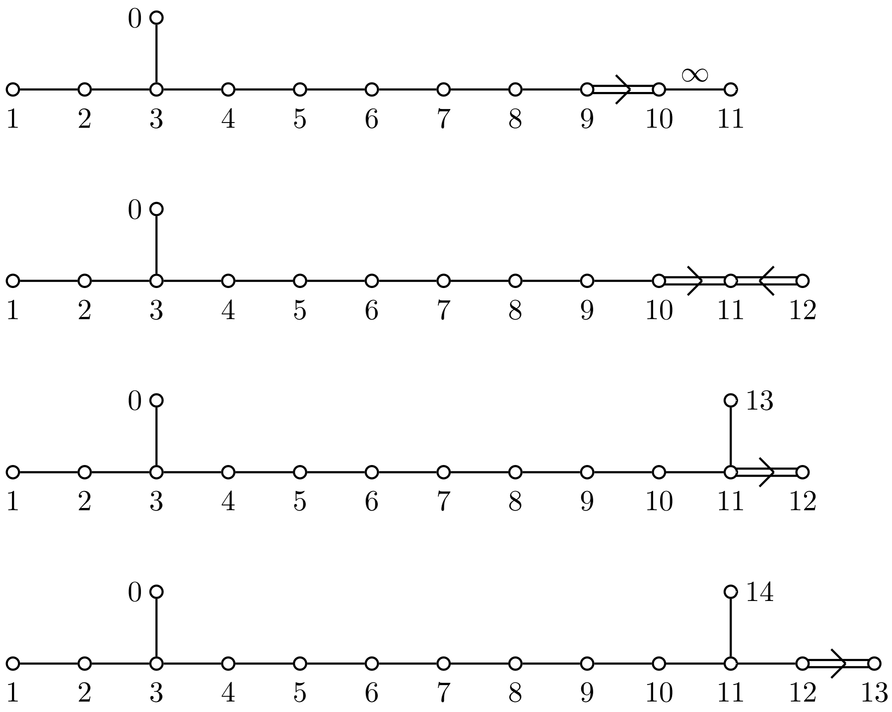
Theorem 5.36. *The odd unimodular Lorentzian lattice* $\mathbb{Z}^{n,1}$ *is reflective if and only if* $n \le 19$.

This result was obtained by Vinberg for $n \le 17$ and by Vinberg and Kaplinskaja for $n \ge 18$ [60],[63]. The Vinberg diagrams have 37 and 50 nodes for $n = 18, 19$ respectively.

Remark 5.37. *Just for the record we mention the Coxeter diagrams for* $W(\mathbb{Z}^{n,1})$ *for* $n = 14$



and for $n = 10, 11, 12, 13$



which can be derived just as in Exercise 5.35.

The odd Lorentzian lattices with discriminant $d \geq 1$ associated to the quadratic form

$$-dx_0^2 + x_1^2 + \cdots + x_n^2$$

have been investigated for $d = 2$ by Vinberg [60], for $d = 3$ by Mcleod [40] and $d = 5$ by Mark [39]. They are reflective for $d = 2$ if $n \leq 8$ or $n = 10$ or 12, for $d = 3$ if and only if $n \leq 13$ and for $d = 5$ if and only if $n \leq 8$. This ends our discussion of the odd integral Lorentzian lattices.

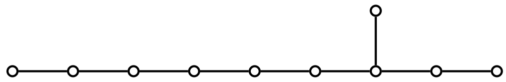
The examples found by Bugaenko came from a study of the quadratic form

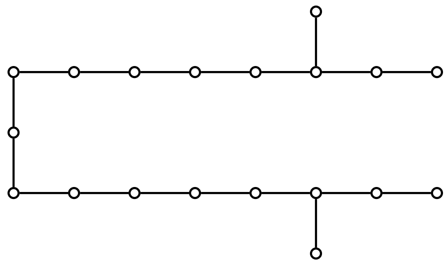
$$-\tau x_0^2 + x_1^2 + \cdots + x_n^2$$

with $\tau = (1 + \sqrt{5})/2$. Bugaenko showed that these lattices (over $\mathbb{Z}[\tau]$) have cocompact reflection groups as automorphism groups if and only if $n \leq 7$ [10],[64]. The examples for $n = 6, 7$ gave the Vinberg diagrams of Bugaenko as discussed in the previous section.

We now come to the even integral Lorentzian lattices.

Theorem 5.38. *The even unimodular Lorentzian lattice* $\mathrm{U}\oplus n\mathrm{E}_8$ *is reflective for* $n \leq 2$. *Their Vinberg diagrams are*





with 10 *and* 19 *nodes for* $n = 1, 2$ *respectively.*

Proof. For $\mathrm{U} \oplus \mathrm{E}_8$ and $\mathrm{U} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$ take κ a primitive null vector in U. The height zero root system R' is equal to $\tilde{\mathrm{E}}_8$ and $\tilde{\mathrm{E}}_8 \oplus \tilde{\mathrm{E}}_8$ respectively. The Vinberg algorithm gives a single simple root at height one: the left node in the first diagram and the middle node in the second diagram. Since these diagrams bound a finite volume fundamental chamber $D \cap H^n$ by previous discussions the Vinberg algorithm terminates after one step. $\square$

The largest number n of a reflective Lorentzian lattice L of rank $n+1$ was found by Richard Borcherds for $n = 21$ [5], [7]. He took for L the unique even sublattice of $\mathbb{Z}^{21,1}$ of vectors $(x_0, x_1, \cdots, x_{21})$ with $\sum x_i$ even. It has discrimant 4 and in fact $L = \mathrm{U} \oplus \mathrm{D}_{20}$ with D_{20} the root lattice of that type. The fundamental chamber D has $42 + 168 = 210$ faces (or equivalently the Vinberg diagram has that many nodes) corresponding to roots of norm 2 and norm 4 respectively. The group of Vinberg diagram automorphisms is isomorphic to the the group $\mathrm{PSL}_3(4) \rtimes \mathcal{D}_6$ of order $2^8 \cdot 3^3 \cdot 5 \cdot 7$. The normal subgroup $\mathrm{PSL}_3(4)$ is a simple group (also denoted M_{21}) of order $2^6 \cdot 3^2 \cdot 5 \cdot 7$ with $\mathcal{D}_6 \cong \mathcal{C}_2 \times \mathcal{S}_3$ its group of outer automorphisms. It was shown by Esselmann that this example of Borcherds gives the largest dimension $n = 21$ of a hyperbolic space H^n with an action of a cofinite volume reflection group $W(L)$ obtained from a Lorentzian lattice L of rank $n+1$ [27].

Theorem 5.39. *All reflective Lorentzian lattices have rank at most* 22*. Moreover, the example by Borcherds is the unique such lattice in rank* 22 *and all others have rank at most* 20*, with the highest rank example of Vinberg and Kaplinskaja showing that the bound* 20 *is also sharp.*

Definition 5.40. *A cofinite volume hyperbolic reflection group W acting on a Lorentzian vector space V is called an arithmetic group if there is a real number field $\mathbb{F}$ and a basis of V such that its Gram matrix G and all matrix entries of $w \in W$ are defined over the ring $\mathbb{O}$ of algebraic integers in $\mathbb{F}$ and all nontrivial Galois conjugates of G are positive definite.*

Clearly the reflection group of a reflective Lorentzian lattice is an arithmetic group with number field $\mathbb{F} = \mathbb{Q}$ and ring of integers $\mathbb{O} = \mathbb{Z}$. But the examles found by Bugaenko are arithmetic reflection groups with number field $\mathbb{F} = \mathbb{Q}(\sqrt{5})$ and ring of integers $\mathbb{O} = \mathbb{Z}[\tau]$. The next result was obtained independently by Nikulin [44] and Agol, Belolipetsky, Storm and Whyte [1].

Theorem 5.41. *There are only finitely many conjugacy classes of arithmetic maximal hyperbolic reflection groups.*

This might give some hope that at least the arithmetic maximal hyperbolic reflection groups can be classified after all. We are still far from that because the finite number of the theorem is gigantic. For those arithmetic reflection groups acting on the hyperbolic plane $\mathbb{H}^2$ with a triangle as fundamental domain the classification is known (with a list of 85 examples), and is due to Takeuchi [56].

Exercise 5.42. *Let L be a reflective Lorentzian lattice with fundamental chamber $D \subset \mathrm{Clos}(V_+)$ and corresponding set of positive roots R_+. Show that the Tits cone $Y \supset D$ is equal to*

$$Y = V_+ \sqcup \mathbb{R}_+ \partial V_+ \sqcup \{0\}$$

with ∂V_+ the set of nonzero rational isotropic vectors in $\mathrm{Clos}(V_+)$. For this reason $\partial V_+/\mathbb{Q}_+$ is called the rational boundary of $H^n = V_+/\mathbb{R}_+$.

6 The Leech Lattice

6.1 Modular Forms

We shall denote by $\mathbb{H}_+$ the upper half plane in $\mathbb{C}$, that is the set of all complex numbers z with $\Im z > 0$. The group $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}_+$ by fractional linear transformations, and the subgroup $\Gamma = \mathrm{PSL}_(\mathbb{Z})$ is called the modular group.

Definition 6.1. *For $k \in \mathbb{N}$ a holomorphic function f on $\mathbb{H}_+$ is called a weak modular form of weight $2k$ if on $\mathbb{H}_+$ we have*

$$f(\gamma z) = f(\frac{az+b}{cz+d}) = (cz+d)^{2k} f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the modular group Γ. Since Γ is generated by the two elements

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad , \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

the above relation amounts to

$$f(z+1) = f(z) \quad , \quad f(-1/z) = z^{2k} f(z)$$

for all $z \in \mathbb{H}_+$. So one can view f as a holomorphic function of $q = e^{2\pi i z}$ on the unit disc $\mathbb{D}^\times = \{0 < |q| < 1\}$, and as such it has a Laurent series expansion

$$f(z) = \sum_{m=-\infty}^{\infty} a_m q^m$$

convergent on $\mathbb{D}^\times$. If $a_m = 0$ for $m < 0$ then f is called a modular form on $\mathbb{H}_+$, and if in addition $a_0 = 0$ then f is called a cusp form.

The modular forms on $\mathbb{H}_+$ form an algebra $M = \oplus M_k$ graded by weight. Examples are the so called Eisenstein series

$$E_k(z) = 1 + (-1)^k \frac{4k}{B_k} \sum_{m=1}^{\infty} \sigma_{2k-1}(m) q^m$$

of weight $2k$, with the Bernoulli numbers B_k defined by the power series

$$\frac{x}{e^x - 1} = 1 - x/2 + \sum_{k=1}^{\infty} B_k x^{2k}/(2k)!$$

arouns $x = 0$ and $\sigma_k(m) = \sum_{d|m} d^k$. For example, we get

$$\begin{aligned} E_2(z) &= 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^m \\ E_3(z) &= 1 - 504 \sum_{m=1}^{\infty} \sigma_5(m) q^m \end{aligned}$$

and therefore (using $3 \times 240 + 2 \times 504 = 1728 = 12^3$)

$$E_2(z)^3 - E_3(z)^2 = 12^3 \Delta(z)$$

with $\Delta(z) = q + \cdots$ a cusp form of weight 12. The function

$$\Delta(z) = q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{m=1}^{\infty} \tau(m) q^m$$

is called the discriminant and τ is called the Ramanujan τ-function. In fact the algebra M of modular forms is equal to $\mathbb{C}[E_2, E_3]$. For these and many more results on modular forms we refer to Serre's great little book [51].

For L an even Euclidean lattice the theta series θ_L of L is the power series

$$\theta_L(q) = \sum_{\lambda \in L} q^{\lambda^2/2} = \sum_{m=0}^{\infty} N_{2m} q^m$$

with $N_{2m} = |\{\lambda \in L; \lambda^2 = 2m\}|$. It converges for $|q| < 1$ or if $q = e^{2\pi i z}$ for $\Im z > 0$, and defines a holomorphic function. The next theorem is a classical result of Hecke [51].

Theorem 6.2. *If L is an even unimodular Euclidean lattice of rank n (also abbreviated L is a lattice of type II_n) then $n \in 8\mathbb{N}$ and θ_L is a modular form of weight $n/2$.*

Corollary 6.3. *Let L be a lattice of type II_n.*

1. *If $n = 8$ then $\theta_L = E_2 = 1 + 240q + \cdots$.*
2. *If $n = 16$ then $\theta_L = E_4 = E_2^2 = 1 + 480q + \cdots$.*
3. *If $n = 24$ then $\theta_L = E_2^3 + (N_2 - 720)\Delta$.*

Example 6.4. *By abuse of notation we shall also write* D_n *for the root lattice of type* D_n*, that is*

$$\mathrm{D}_n = \{x \in \mathbb{Z}^n; x_1 + \cdots + x_n \in 2\mathbb{Z}\}.$$

The dual lattice D_n^* *is the weight lattice of type* D_n*, so*

$$\mathrm{D}_n^* = \mathrm{D}_n \sqcup (\varpi_1 + \mathrm{D}_n) \sqcup (\varpi_{n-1} + \mathrm{D}_n) \sqcup (\varpi_n + \mathrm{D}_n)$$

with

$$\varpi_1 = (1, 0, \cdots, 0), \varpi_{n-1} = (\tfrac{1}{2}, \cdots, \tfrac{1}{2}, -\tfrac{1}{2}), \varpi_n = (\tfrac{1}{2}, \cdots, \tfrac{1}{2}, \tfrac{1}{2})$$

having norms $\varpi_1^2 = 1, \varpi_{n-1}^2 = \varpi_n^2 = n/4$*. The fundamental weights* ϖ_i *for* $i = 1, \cdots, n$ *are defined by* $(\varpi_i, \alpha_j) = \delta_{ij}$ *with*

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \cdots, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \alpha_n = \varepsilon_{n-1} + \varepsilon_n$$

the basis of simple roots. For $n \in 2\mathbb{N}$ *put*

$$\mathrm{D}_n^+ = \mathrm{D}_n \sqcup (\varpi_n + \mathrm{D}_n)$$

which is a sublattice of D_n^* *of index* 2*, since* $2\varpi_n \in \mathrm{D}_n$*. The lattice* D_n^+ *is integral (and hence unimodular) if and only if* $\varpi_n^2 \in \mathbb{N}$*, which is equivalent to* $n \in 4\mathbb{N}$*. Likewise* D_n^+ *is even unimodular if and only if* $\varpi_n^2 \in 2\mathbb{N}$*, which is equivalent to* $n \in 8\mathbb{N}$*. Note that the lattice* $\mathrm{D}_n \sqcup (\varpi_{n-1} + \mathrm{D}_n)$ *is isomorphic to* D_n^+*, the isomorphism sending the last coordinate to its negative. In turn this implies that* D_n^+ *is the up to isomorphism unique even unimodular lattice containing* D_n *as an index* 2 *sublattice.*

The lattice D_8^+ *contains* 112 *norm 2 roots of the form* $\pm\varepsilon_i \pm \varepsilon_j$ *for* $1 \leq i < j \leq 8$*, and* 128 *norm* 2 *roots of the form* $\sum_1^8 \epsilon_i \varepsilon_i$ *with* $\epsilon_i = \pm 1$ *and* $\prod \epsilon_i = 1$*. The roots* $\alpha_1 = \varepsilon_1 - \varepsilon_2, \cdots, \alpha_7 = \varepsilon_7 - \varepsilon_8$ *and* $\alpha_8 = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2},)$ *are the simple roots for the root system of type* E_8*. Therefore the lattice* D_8^+ *is just the lattice* E_8*.*

The norm 2 *vectors in* D_{16}^+ *generate the index* 2 *sublattice* D_{16}*, and hence* D_{16}^+ *is not isomorphic to* $2\mathrm{E}_8$*. Nevertheless these two distinct lattices do have the same theta functions* $E_2^2(z) = 1 + 480q + \cdots$*. This gave the first known counterexample to the famous question of Mark Kac: Can one hear the shape of a drum? If the drum as bounded region in* $\mathbb{R}^n$ *is replaced by a compact Riemannian manifold, then the two tori* $\mathbb{R}^{16}/\mathrm{D}_{16}^+$ *and* $\mathbb{R}^{16}/2\mathrm{E}_8$ *are isospectral (for the Laplace operator) but not isometric.*

Example 6.5. *The root lattice* A_{24} *of type* A_{24} *is generated by the simple roots* $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \mathbb{Z}^{25}$ *for* $i = 1, \cdots, 24$*, and so*

$$\mathrm{A}_{24} = \{x \in \mathbb{Z}^{25}; x_1 + \cdots + x_{25} = 0\}.$$

The dual weight lattice A_{24}^* *is generated by the fundamental weights*

$$\varpi_i = \sum_{k=1}^{i} \varepsilon_k - \frac{i}{25} \sum_{k=1}^{25} \varepsilon_k = \frac{25-i}{25} \sum_{k=1}^{i} \varepsilon_k - \frac{i}{25} \sum_{k=i+1}^{25} \varepsilon_k$$

for $i = 1, \cdots, 24$*, of norms* $(i(25-i)^2 + (25-i)i^2)/625 = i(25-i)/25$*, which are even integers if and only if* $i = 5, 10, 15, 20$*. Because*

$$\varpi_{5i} = \frac{5-i}{5} \sum_{k=1}^{i} \delta_k - \frac{i}{5} \sum_{k=i+1}^{5} \delta_k \ , \ \delta_k = \sum_{i=5k-4}^{5k} \varepsilon_i \ , \ (\delta_k, \delta_l) = 5\delta_{kl}$$

for $i = 1, 2, 3, 4$ *and* $k, l = 1, 2, 3, 4, 5$ *it follows that (say* $i \leq j$*)*

$$(\varpi_{5i}, \varpi_{5j}) = ((5-i)(5-j)i - i(5-j)(j-i) + ij(5-j))/5 = i(5-j)$$

is integral for all $i, j = 1, 2, 3, 4$*. Hence*

$$\mathrm{A}_{24}^+ = \mathrm{A}_{24} \sqcup (\varpi_5 + \mathrm{A}_{24}) \sqcup (\varpi_{10} + \mathrm{A}_{24}) \sqcup (\varpi_{15} + \mathrm{A}_{24}) \sqcup (\varpi_{20} + \mathrm{A}_{24})$$

is the unique even unimodular lattice containing A_{24} *as its root sublattice.*

Remark 6.6. *For* R *a simply laced irreducible integral root system one has* $|R| = nh$ *with* n *the rank and* h *the Coxeter number of* R*, as found in Corollary 2.46 and given by*

$$h(\mathrm{A}_n) = n + 1, h(\mathrm{D}_n) = 2n - 2, h(\mathrm{E}_6) = 12, h(\mathrm{E}_7) = 18, h(\mathrm{E}_8) = 30.$$

In particular if $\mathrm{rk}(R) \leq 16$ *then* $h(R) \leq 30$ *with equality if and only if* R *is of type* E_8 *or* D_{16}*.*

The next theorem was found by Witt in 1935.

Theorem 6.7. *Up to isomorphism there is one lattice of type* II_8 *(namely* E_8*), and two lattices of type* II_{16} *(namely* D_{16}^+ *and* $2\mathrm{E}_8$*).*

Proof. For L a lattice of type II_n the set $R(L) = \{\alpha \in L; \alpha^2 = 2\}$ is a simply laced root system. It follows from Corollary 6.3 $|R(L)| = 240$ for $n = 8$, and $|R(L)| = 480$ for $n = 16$. By the above remark it follows that all irreducible components of $R(L)$ have Coxeter number $h = 30$, and so $R(L)$ is of type E_8 for $n = 8$ and of type D_{16} or $2\mathrm{E}_8$ for $n = 16$. Hence L is equal to E_8 for $n = 8$, and equal to D_{16}^+ or $2\mathrm{E}_8$ for $n = 16$. □

The isomorphism classes of lattices of type II_{24} were determined by Niemeier in 1968 and Conway in 1969 [18]. For L a lattice of type II_{8n} we denote by $R(L) = \{\alpha \in L; \alpha^2 = 2\}$ the root system of L and by $\mathbb{Z}R(L)$ the root sublattice of L.1

Theorem 6.8. *There are exactly* 24 *isomorphism classes of lattices of type* II_{24}*. Moreover* 23 *of these have roots, and there is a unique lattice* Λ *of type* II_{24} *without roots, found in* 1965 *by John Leech and called the Leech lattice.*

The next result is called the Minkowski–Siegel mass formula [51].

Theorem 6.9. *For* $n = 2k \in 8\mathbb{N}$ *we have*

$$\sum_{L} \frac{1}{|\mathrm{O}(L)|} = \frac{|B_{k/2}|}{2k} \prod_{j=1}^{k-1} \frac{|B_j|}{4j}$$

with the sum over the isomorphism classes of lattices of type II_n*.*

For $n = 32$ the right hand side of the mass formula is greater than $4 \cdot 10^7$. Because $|\mathrm{O}(L)| \geq 2$ there are at least 80 millions different isomorphism classes of lattices of type II_{32}, and the classification of lattices of type II_n for $n \geq 32$ is hopeless.

In the rest of this chapter we shall discuss the classification of the Niemeier lattices, and especially explain the key role played by hyperbolic reflection groups, notably for understanding the Leech lattice.

6.2 A Theorem of Venkov

Lattices of type II_{24} are also called Niemeier lattices, and will be usually denoted by N. As before let $R(N) = \{\alpha \in N; \alpha^2 = 2\}$ be the root system of N. Let $V = \mathbb{R} \otimes_{\mathbb{Z}} N$ be the ambient Euclidean space. The next theorem was proved by Boris Venkov using the theory of modular forms, see Chapter 18 of [18].

Theorem 6.10. *For* N *a Niemeier lattice we have*

$$\sum_{\alpha \in R(N)} (\alpha, \xi)^2 = |R(N)|(\xi, \xi)/12$$

for all $\xi \in V$*.*

Let R be a simply laced irreducible integral root system with the inner product normalized by $(\alpha, \alpha) = 2$ for all $\alpha \in R$. Let $n = \mathrm{rk}(R) = \dim(V)$ and h the Coxeter number of R. Recall from Corollary 2.46 that $|R| = nh$.

Proposition 6.11. *We have $\sum_{\alpha \in R} (\alpha, \xi)(\alpha, \eta) = 2h(\xi, \eta)$ for all $\xi, \eta \in V$.*

Proof. Let $(\cdot,\cdot)_K$ be the unique invariant inner product on V, characterized by $\sum_\alpha (\alpha, \xi)_K (\alpha, \eta)_K = (\xi, \eta)_K$ for all $\xi, \eta \in V$. This is the so called Killing form normalization, which is familiar from the theory of semisimple Lie algebras. The Gram matrix $(\alpha, \beta)_K$ with $\alpha, \beta \in R$ is idempotent of size nh by nh and has rank equal to n. Hence its trace is equal to n, and therefore $(\alpha, \alpha)_K = 1/h$. We find that $(\cdot,\cdot) = 2h(\cdot,\cdot)_K$ satisfies $(\alpha, \alpha) = 2$ as required. □

Given $R_+ \subset R$ the vector $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ is called the Weyl vector. By Corollary 4.13 a simple reflection $s_i \in W$ permutes the set $R_+ - \{\alpha_i\}$, which in turn implies that $s_i(\rho) = \rho - \alpha_i$ or equivalently $(\rho, \alpha_i) = 1$. Hence ρ lies in the weight lattice $P = Q^*$ of R.

Proposition 6.12. *We have $(\rho, \rho) = nh(h + 1)/12$.*

Proof. The strange formula of Freudenthal–de Vries $(\rho, \rho)_K = n(h + 1)/24$ gives immediately $(\rho, \rho) = nh(h + 1)/12$. The strange formula was discovered by Freudenthal–de Vries [28] from the Taylor expansion of the Weyl character formula for the adjoint representation at the identity. An elementary proof was given by Burns [12]. Another quick proof follows by direct verification using the Bourbaki tables [9]. □

Let N be a Niemeier lattice with root system $R(N) = \{\alpha \in N; \alpha^2 = 2\}$ and corresponding Weyl vector ρ.

Theorem 6.13. *The root system $R(N)$ is either empty or has full rank 24. In the latter case all irreducible components of $R(N)$ have the same Coxeter number $h = h(N)$. Moreover $|R(N)| = 24h$ and $(\rho, \rho) = 2h(h + 1)$.*

Proof. If $R(N)$ is empty then we have $h = 0$ and $\rho = 0$. If $R(N)$ is not empty then $R(N)$ has full rank 24 by the Venkov theorem. In addition all irreducible components of $R(N)$ have the same Coxeter number $h = |R(N)|/24$, which by Proposition 6.12 implies $(\rho, \rho) = 2h(h + 1)$. □

We shall call $h = h(N)$ the Coxeter number and ρ the Weyl vector of the Niemeier lattice N.

Proposition 6.14. *The Weyl vector ρ of the Niemeier lattice N lies in N.*

Proof. Clearly $2\rho \in Q \subset N$ and for all $\nu \in N$ we get

$$(2\rho, \nu)^2 = (\sum_{\alpha > 0} (\alpha, \nu))^2 \equiv \sum_{\alpha > 0} (\alpha, \nu)^2 = h(\nu, \nu) \equiv 0$$

modulo 2, using Proposition 6.11 and since N is even. Hence $(2\rho,\nu) \in 2\mathbb{Z}$ and $(\rho,\nu) \in \mathbb{Z}$. Therefore $\rho \in N$ because N is unimodular. $\square$

Proposition 6.15. *Let N be a Niemeier lattice with Coxeter number $h > 0$ and let $Q < N < P$ with Q the root lattice and $P = Q^*$ the weight lattice of $R(N)$. Then for all $\lambda \in P$ we have*

$$(\lambda - \rho/h)^2 \geq 2(1 + 1/h)$$

with equality if and only if $(\lambda,\alpha) \in \{0,1\}$ for all $\alpha > 0$.

Proof. For all λ in P we have

$$(\lambda - \rho/h)^2 - 2(1+1/h) = (\lambda - \rho/h)^2 - (\rho/h)^2 = \lambda^2 - (\lambda, 2\rho)/h =$$
$$(\sum_{\alpha>0}(\lambda,\alpha)^2 - \sum_{\alpha>0}(\lambda,\alpha))/h = \sum_{\alpha>0}(\lambda,\alpha)((\lambda,\alpha)-1)/h \geq 0$$

using Theorem 6.13, Proposition 6.11 and $(\lambda,\alpha) \in \mathbb{Z}$ for all $\lambda \in P$ and $\alpha > 0$. $\square$

Corollary 6.16. *Suppose $h = h(N) > 0$. Then we have for all $\nu \in N$*

$$(\nu - \rho/h)^2 \geq 2(1+1/h) = (\rho/h)^2$$

and the $\nu \in N$ for which equality holds are those $\nu \in N$ with $(\nu,\alpha) \in \{0,1\}$ for all $\alpha > 0$.

The action of the affine reflection group $Q \rtimes W$ on V has the alcove

$$D = \{\xi \in V; 0 \leq (\xi,\alpha) \leq 1\ \forall \alpha > 0\}$$

as fundamental domain. It is a product of d simplices with with d the number of irreducible components of $R(N)$. The points $\lambda \in P$ with $(\lambda,\alpha) \in \{0,1\}$ for all $\alpha > 0$ are the so called special vertices of D, and besides the origin 0 they are by definition the minuscule fundamental weights in P_+.

The sphere in V with center ρ/h and radius $1/h$ is the inscribed sphere for the alcove D. Indeed, if $\alpha > 0$ is a simple root then $(\rho/h,\alpha) = 1/h$, while if $\theta > 0$ is a highest root then $(\rho/h,\theta) = (h-1)/h$ as follows from the results of Section 4.5. So the sphere with center ρ/h and radius $1/h$ is tangent to all walls of D.

Remark 6.17. *For L an integral Euclidean lattice the* packing radius *is the largest number $r > 0$ such that the open balls centered at the lattice points with radius r do not overlap, and the* covering radius *is the smallest number $R > 0$ such that the closed balls centered at the lattice points cover V. The smaller the quotient R/r the better the ball packing with centers at the lattice points of L.*

Let N be a Niemeier lattice with roots, and so $h > 0$. Since the inequality $(\lambda, \lambda) \geq 2$ for all nonzero $\lambda \in N$ is sharp the packing radius of N is equal to $r = 1/\sqrt{2}$. It follows from Corollary 6.16 that the covering radius R satisfies

$$R \geq |\rho/h| = \sqrt{2(1 + 1/h)} > \sqrt{2}$$

and so $R/r > 2$. For the Leech lattice Λ without roots the packing radius $r = 1$ because of the sharp inequality $(\lambda, \lambda) \geq 4$ for all nonzero $\lambda \in \Lambda$. Later we shall prove that for the Leech lattice $R = \sqrt{2}$, and so $R/r = \sqrt{2}$. The Leech lattice is in fact the optimal ball packing in 24 dimensions. The book by Conway and Sloane [18] contains a wealth of information about the Leech lattice.

6.3 The Classification of Niemeier Lattices

Suppose in this section that $R \subset V$ is an integral normalized root system, that is a direct sum of type ADE root systems, such that

- $\mathrm{rk}(R) = 24$,
- and all irreducible components of R have the same Coxeter number h.

These root systems are easily classified and the outcome is given in the next theorem.

Theorem 6.18. *Under these conditions we have the following 23 possibilities:*

1. $24\mathrm{A}_1$, $12\mathrm{A}_2$, $8\mathrm{A}_3$, $6\mathrm{A}_4$, $4\mathrm{A}_6$, $3\mathrm{A}_8$, $2\mathrm{A}_{12}$, A_{24}
2. $6\mathrm{D}_4$, $4\mathrm{D}_6$, $3\mathrm{D}_8$, $2\mathrm{D}_{12}$, D_{24}
3. $4\mathrm{E}_6$, $3\mathrm{E}_8$
4. $4\mathrm{A}_5 + \mathrm{D}_4$, $2\mathrm{A}_7 + 2\mathrm{D}_5$, $2\mathrm{A}_9 + \mathrm{D}_6$, $\mathrm{A}_{15} + \mathrm{D}_9$
5. $\mathrm{A}_{11} + \mathrm{D}_7 + \mathrm{E}_6$, $\mathrm{A}_{17} + \mathrm{E}_7$, $\mathrm{D}_{10} + 2\mathrm{E}_7$, $\mathrm{D}_{16} + \mathrm{E}_8$

All together there are $8+5+2+4+4=23$ *possibilities. Note that a certain type is completely determined by the occurence of the last alphabetic letter together with its multiplicity.*

Proof. Say R is of type $\sum p_i\mathrm{A}_i + \sum q_j\mathrm{D}_j + \sum r_k\mathrm{E}_k$. Since $h(\mathrm{A}_n) = n+1$, $h(\mathrm{D}_n) = 2n-2$, $h(\mathrm{E}_6) = 12$, $h(\mathrm{E}_7) = 18$ and $h(\mathrm{E}_8) = 30$ it follows that R is of type $p\mathrm{A}_i + q\mathrm{D}_j + r\mathrm{E}_k$ with $pi+qj+rk=24$. We enumerate

1. $q=r=0$ gives $pi=24$ with $i \geq 1$
2. $p=r=0$ gives $qj=24$ with $j \geq 4$
3. $p=q=0$ gives $rk=24$ with $k=6,7,8$
4. $p,q \geq 1, r=0$ gives $i+1=2j-2$ with $j \geq 4$ and $p(2j-3)+qj=24$. Hence the possibilities are
   - $j=4 \Rightarrow 5p+4q=24 \Rightarrow (p,q)=(4,1)$
   - $j=5 \Rightarrow 7p+5q=24 \Rightarrow (p,q)=(2,2)$
   - $j=6 \Rightarrow 9p+6q=24 \Rightarrow (p,q)=(2,1)$
   - $j=9 \Rightarrow 15p+9q=24 \Rightarrow (p,q)=(1,1)$
5. $p+q, r \geq 1$ gives $pi+qj+rk=24$. Hence the possibilities are
   - $k=6 \Rightarrow 11p+\ \ 7q+6r=24 \Rightarrow (p,q,r)=(1,1,1)$
   - $k=7 \Rightarrow 17p+10q+7r=24 \Rightarrow (p,q,r)=(1,0,1),(0,1,2)$
   - $k=8 \Rightarrow 29p+16q+8r=24 \Rightarrow (p,q,r)=(0,1,1)$

which completes the proof. □

It turns out that for each of these 23 root systems R there exists a up to isomorphism unique Niemeier lattice N with $R(N)=R$. For $h \geq 25$ this is easy, and has been discussed in the examples of Section 6.1, but for small h (say $h=2,3$) this boils down to a problem in coding theory (existence and uniqueness of the binary and ternary Golay codes respectively). For the case by case details we refer to Chapter 18 of [18].

6.4 The Existence of the Leech Lattice

Let N be a Niemeier lattice with Coxeter number h and Weyl vector ρ. Let $L = \mathrm{U} \oplus N$ be the Lorentzian lattice of type $\mathrm{II}_{25,1}$, with coordinates $\lambda = (m,n,\nu)$ with $m,n \in \mathbb{Z}$, $\nu \in N$ and $\lambda^2 = -2mn + \nu^2$. If we denote

$$\kappa = -(0,1,0)\ ,\ \kappa' = -(h,h+1,\rho)$$

then $\kappa^2 = 0$ and $(\kappa')^2 = -2h(h+1) + \rho^2 = 0$ by Theorem 6.13. Since

$$(t\kappa + (1-t)\kappa')^2 = 2t(1-t)(\kappa, \kappa') = -2t(1-t)h < 0$$

for all $0 < t < 1$ the isotropic vectors κ and κ' lie in the same connected component of the set of nonzero isotrpic vectors. Let V_+ be the connected component of the set of negative norm vectors containing κ and κ' in its closure. Note that both κ and κ' are primitive norm zero vectors in L. Let us denote

$$\mathrm{ht}(\alpha) = (\alpha, \kappa)\ ,\ \mathrm{ht}'(\alpha) = (\alpha, \kappa')$$

for the height of $\alpha \in R = R(L)$ with respect to the controlling vectors κ and κ' respectively.

Lemma 6.19. *There is no root $\alpha \in R$ with $\mathrm{ht}(\alpha) = \mathrm{ht}'(\alpha) = 0$, in other words one can choose a closed fundamental chamber D for $W(L)$ containing both controlling vectors κ and κ'.*

Proof. If $\alpha \in R$ has $\mathrm{ht}(\alpha) = 0$ then $\alpha = (0, n, \nu)$ with $n \in \mathbb{Z}$ and $\nu \in R(N)$. Then $\mathrm{ht}'(\alpha) = nh - (\nu, \rho) \neq 0$ because $1 \leq |(\nu, \rho)| \leq h - 1$ for all $\nu \in R(N)$. $\square$

Lemma 6.20. *If $\alpha \in R$ with $\mathrm{ht}(\alpha) = 1$ then $\mathrm{ht}'(\alpha) \geq 1$.*

Proof. If $\alpha \in R$ with $\mathrm{ht}(\alpha) = 1$ then $\alpha = (1, \frac{1}{2}\nu^2 - 1, \nu)$ for some $\nu \in N$. Then we have

$$\begin{aligned} \mathrm{ht}'(\alpha) &= (h+1) + (\tfrac{1}{2}\nu^2 - 1)h - (\nu, \rho) = \tfrac{1}{2}h\nu^2 - (\nu, \rho) + 1 \\ &= \tfrac{1}{2}h[\nu^2 - 2(\nu, \rho/h)] + 1 = \tfrac{1}{2}h[(\nu - \rho/h)^2 - (\rho/h)^2] + 1 \end{aligned}$$

and hence $\mathrm{ht}'(\alpha) \geq 1$ by Corollary 6.16. $\square$

Combination of the two lemmas shows that the assumptions $\mathrm{ht}(\alpha) \geq 0$ and $\mathrm{ht}'(\alpha) = 0$ for $\alpha \in R$ necessarily imply that $\mathrm{ht}(\alpha) \geq 2$.

If $\lambda \in L$ is an isotropic vector, then the unimodularity of L implies the existence of $\mu \in L$ with $(\lambda, \mu) = 1$. Because L is even, we have $\mu^2 = 2m$ for some $m \in \mathbb{Z}$, and replacing μ by $\mu - m\lambda$ shows that we may assume $\mu^2 = 0$. Hence the sublattice $\mathbb{Z}\lambda + \mathbb{Z}\mu$ is isomorphic to the hyperbolic plane lattice U, and $L = \mathrm{U} \oplus \mathrm{U}^\perp$ with $\mathrm{U}^\perp$ an even unimodular Euclidean sublattice of L. The conclusion is that the classification of orbits under $\mathrm{O}(L)$ of primitive isotropic vectors in L is equivalent to the classification of Niemeier lattices up to isomorphism.

Lemma 6.21. *Let N' be the Niemeier lattice associated to the primitive isotropic vector κ'. Then the Coxeter number h' of N' satisfies $h' \leq \frac{1}{2}h$.*

Proof. If $h' = 0$ then there is nothing to prove. So assume $h' \geq 2$, and therefore $R(N')$ has rank 24 by Theorem 6.13. In turn this implies that $R' = \{\alpha \in R; \mathrm{ht}'(\alpha) = 0\}$ is the affine root system of rank 25 associated with $R(N')$. We can choose simple roots $\alpha_0, \alpha_1, \cdots, \alpha_r$ for R' with $\mathrm{ht}(\alpha_i) \geq 0$ and $\sum_0^r k_i\alpha_i = -\kappa'$. Here r is the rank of an irreducible component of $R(N')$ with highest root $\theta = \sum_1^r k_i\alpha_i$, $k_0 = 1$ and $h' = \sum_0^r k_i$. By the above lemmas we find $\mathrm{ht}(\alpha_i) \geq 2$ and therefore

$$h = -(\kappa, \kappa') = (\kappa, \sum_0^r k_i\alpha_i = \sum_0^r k_i \mathrm{ht}(\alpha_i) \geq 2\sum_0^r k_i = 2h'$$

and the lemma follows. □

Theorem 6.22. *There exists a Niemeier lattice Λ without roots.*

Proof. Start with a familiar Niemeier lattice N with Coxeter number h (for example $3\mathrm{E}_8$ with Coxeter number 30) and iterate the above procedure (at most 4 times). □

Remark 6.23. *The clean proof of the above theorem, as a consequence of Corollary 6.16, was found by Richard Borcherds in his thesis [5]. In fact we shall see in a later section that for each Niemeier lattice N with Coxeter number $h > 0$ after one step one has $h' = 0$ and so the lattice $N' = \Lambda$ has no roots.*

6.5 A Theorem of Conway

Let Λ be a Niemeier lattice without roots. A vector $\lambda \in \Lambda$ is called short if $\lambda^2 \leq 8$. Two vectors $\lambda, \mu \in \Lambda$ are called equivalent if $\lambda - \mu \in 2\Lambda$. Clearly the number of equivalence classes is equal to $|\Lambda/2\Lambda| = 2^{24}$. Since λ and $-\lambda$ are equivalent the short vectors different from 0 do occur in opposite pairs. The next result is due to Conway [17].

Theorem 6.24. *Each equivalence class in Λ contains a short vector. The equivalence classes that contain more than one single opposite pair of short vectors are precisely those that contain vectors of norm 8, and these classes contain exactly 24 mutually orthogonal opposite pairs of vectors of that length.*

Proof. Suppose $\lambda, \mu \in \Lambda$ are equivalent short vectors with $\lambda \neq \pm\mu$. Replacing μ by $-\mu$ if necessary we may assume that $(\lambda, \mu) \geq 0$. Since $\lambda - \mu = 2\nu$ for some $\nu \in \Lambda$ we have $(\lambda - \mu)^2 \geq 16$. Together with $\lambda^2 \leq 8$, $\mu^2 \leq 8$ and $(\lambda, \mu) \geq 0$ this implies $\lambda^2 = 8$, $\mu^2 = 8$ and $(\lambda, \mu) = 0$. So the number of equivalence classes in Λ that contain a short vector is at least equal to

$$N_0 + N_4/2 + N_6/2 + N_8/48$$

with $N_m = |\{\lambda \in \Lambda; \lambda^2 = m\}|$. Using Hecke's formula $\theta_\Lambda = E_2^3 - 720\Delta = \sum N_{2m} q^m$ of Corollary 6.3 we compute using the table for N_m from the book by Conway and Sloane [18]

$$\begin{aligned} N_0 &= 1 \\ N_4/2 = 196560/2 &= 98\,280 \\ N_6/2 = 16773120/2 &= 8\,386\,560 \\ N_8/48 = 398034000/48 &= 8\,292\,375 \end{aligned}$$

which add up to $16\,777\,216 = (4096)^2 = 2^{24}$. Therefore each equivalence class contains a short vector: of norm 0 a unique one, of norm 4 or 6 a unique opposite pair, and of norm 8 a unique collection of 24 mutually orthogonal opposite pairs. $\square$

Let $V = \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ be the ambient Euclidean space of Λ. The Voronoi cell $V(0)$ of Λ around 0 is defined by

$$V(0) = \{\xi \in V; \xi^2 \leq (\xi - \lambda)^2 \ \forall \lambda \in \Lambda\} .$$

It is a compact convex polytope. A vertex ξ of $V(0)$ and a translate of ξ over $\lambda \in \Lambda$ is called a hole of Λ. Since the bounding hyperplanes of $V(0)$ are rational it is clear that holes lie in $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$. The number

$$R(\xi) = d(\xi, \Lambda) = \inf\{|\xi - \lambda|; \lambda \in \Lambda\}$$

is called the radius of the hole ξ of Λ. The covering radius R of Λ can be defined by

$$R = \sup\{R(\xi); \xi \text{ a hole of } \Lambda\} .$$

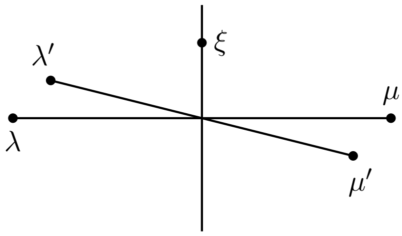
Deep holes are holes with radius equal to the covering radius. The other holes are called the shallow holes. In other words, the holes of Λ are those points $\xi \in V$ for which the distance $d(\xi, \Lambda)$ from ξ to Λ has a local maximum, and deep holes are those holes for $d(\xi, \Lambda)$ has a global maximum. For ξ a hole of Λ the nearby lattice points

$$\Lambda(\xi) = \{\lambda \in \Lambda; (\lambda - \xi)^2 = R(\xi)^2\}$$

are called the vertices of the hole ξ.

Corollary 6.25. *The distance between any two vertices of a hole is at most equal to* $2\sqrt{2}$*.*

Proof. Suppose ξ is a hole of Λ, and let $\lambda, \mu \in \Lambda(\xi)$ be two vertices with $(\lambda - \mu)^2 \geq 10$. By Theorem 6.24 there exists $\nu \in \Lambda$ with $(\lambda - \mu - 2\nu) \leq 8$. Hence $\lambda' = \lambda - \nu$ and $\mu' = \mu + \nu$ are both in Λ with distance at most $2\sqrt{2}$ apart, and with the same midpoint $(\lambda' + \mu')/2 = (\lambda + \mu)/2$ as λ and μ. Hence either λ' or μ' is closer to ξ than λ and μ as is clear from the picture



Indeed, if we assume

$$(\lambda' - \mu', \xi - (\lambda' + \mu')/2) \geq 0$$

then we get

$$\begin{gathered}(\lambda - \xi)^2 = ((\lambda - \mu)/2)^2 + (\xi - (\lambda + \mu)/2)^2 > \\ ((\lambda' - \mu')/2)^2 + (\xi - (\lambda' + \mu')/2)^2 \geq (\lambda' - \xi)^2\end{gathered}$$

and we arrive at a contradiction with $\lambda \in \Lambda(\xi)$. Hence $(\lambda - \mu)^2 \leq 8$. $\square$

This corollary will be used in the next section to show that the covering radius of Λ is equal to $\sqrt{2}$.

6.6 The Covering Radius of Λ

Let Λ be a Niemeier lattice without roots. Let $L = \mathrm{U} \oplus \Lambda$ be the Lorentzian lattice of type $\mathrm{II}_{25,1}$ with coordinates (m, n, λ) for $m, n \in \mathbb{Z}$, $\lambda \in \Lambda$ and norm $(m, n, \lambda)^2 = -2mn + \lambda^2$. Let $V = \mathbb{R} \otimes_{\mathbb{Z}} L$ be the ambient Lorentzian space. The controlling $\rho = (0, -1, 0)$ is called a Weyl vector for $R(L)$, and put $\mathrm{ht}(\alpha) = (\alpha, \rho)$ for $\alpha \in R(L)$. The roots in $R(L)$ of height 0 are the vectors $(0, n, \lambda)$ with $n \in \mathbb{Z}$ and $\lambda \in R(\Lambda)$, so there are none. Hence the root system $R = R(L)$ is decomposed as $R = R_+ \sqcup R_-$ with $R_+ = \{\alpha \in R; \mathrm{ht}(\alpha) \geq 1\}$ and $R_- = R_+$. Let $C = \{\xi \in V; (\xi, \alpha) > 0 \ \forall \alpha > 0\}$ the corresponding positive chamber.

The roots in R of height 1 are the vectors in L of the form

$$\alpha_\lambda = (1, \tfrac{1}{2}\lambda^2 - 1, \lambda)$$

with $\lambda \in \Lambda$, and these are all simple by the Vinberg algorithm as given in Theorem 5.32. In the next section we will show that there are no simple roots of height ≥ 2. The Gram matrix of the height 1 simple roots becomes

$$g_{\lambda\mu} = (\alpha_\lambda, \alpha_\mu) = 2 - \tfrac{1}{2}\lambda^2 - \tfrac{1}{2}\mu^2 + (\lambda, \mu) = 2 - \tfrac{1}{2}(\lambda - \mu)^2$$

for all $\lambda, \mu \in \Lambda$. The associated Coxeter has nodes indexed by $\lambda \in \Lambda$. The nodes with indices $\lambda, \mu \in \Lambda$ are connected by branches

if $(\lambda - \mu)^2 = 4, 6, 8$ or ≥ 10 respectively.

Proposition 6.26. *Any connected parabolic subdiagram of the Coxeter diagram Λ is contained in a subdiagram of Λ which is a disjoint union of m connected parabolic subdiagrams with $24 + m$ nodes.*

Proof. Suppose $\lambda_0, \lambda_1, \cdots, \lambda_r$ are the nodes of a connected parabolic subdiagram of Λ, and write $\alpha_i = (1, \frac{1}{2}\lambda_i^2 - 1, \lambda_i)$ for the corresponding simple roots of R_+. The vector $\kappa = -\sum_0^r k_i \alpha_i$ with k_i the weights on the nodes of the simply laced affine Dynkin diagrams as in Theorem 4.34 is a primitive norm 0 vector in the closure D of the fundamental chamber C. As such κ corresponds to a Niemeier lattice N with roots, and by Theorem 6.13 the root system $R(N)$ has rank 24 and all irreducible components of $R(N)$ have the same Coxeter number $h = \sum_0^r k_i = -(\kappa, \rho)$. Hence $R \cap \kappa^\perp$ is an affine root system of rank 25.

Let $\alpha'_0, \alpha'_1, \cdots, \alpha'_s \in R_+$ be the simple roots of some other connected component of the Coxeter diagram of $R \cap \kappa^\perp$, with weights $k'_j \geq 1$, the same Coxeter number $h = \sum_0^s k'_j$, and $\kappa = -\sum_0^s k'_j \alpha'_j$. Hence we get

$$h = -(\kappa, \rho) = \sum_0^s k'_j \mathrm{ht}(\alpha'_j)$$

and since $\mathrm{ht}(\alpha'_j) \geq 1$ for all j we conclude that $\mathrm{ht}(\alpha'_j) = 1$ for all j. Hence all simple roots of $R_+ \cap \kappa^\perp$ have height 1. $\square$

Proposition 6.27. *The rational lines of norm* 0 *vectors in* $\mathbb{Q} \otimes_{\mathbb{Z}} L$ *are in natural bijection with the points of* $(\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda) \sqcup \{\infty\}$ *by* $(x, \xi^2/(2x), \xi) \mapsto \xi/x$ *for* $x \in \mathbb{Q}^\times$ *and* $(0 : y : 0) \mapsto \infty$ *for* $y \in \mathbb{Q}^\times$. *The reflection* s_λ *in the simple root* $(1, \frac{1}{2}\lambda^2 - 1, \lambda)$ *acts on* $(\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda) \sqcup \{\infty\}$ *as inversion in the sphere with center* λ *and radius* $\sqrt{2}$.

Proof. The norm of $(x, y, \xi) \in \mathbb{Q} \otimes_{\mathbb{Z}} L$ is equal to $-2xy + \xi^2$. Hence either $x \in \mathbb{Q}^\times, y = \xi^2/(2x), \xi \in \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$ or $x = 0, y \in \mathbb{Q}^\times, \xi = 0$, and the first statement is just the change from the one sheeted hyperboloid model of hyperbolic geometry to the Poincaré upper half space model.

If $\nu = (1, \frac{1}{2}\xi^2, \xi)$ is a norm 0 vector and $\alpha_\lambda = (1, \frac{1}{2}\lambda^2 - 1, \lambda)$ a simple root then

$$(\nu, \alpha_\lambda) = 1 - \tfrac{1}{2}\lambda^2 - \tfrac{1}{2}\xi^2 + (\xi, \lambda) = 1 - \tfrac{1}{2}(\xi - \lambda)^2$$

and so $(\nu, \alpha_\lambda) = 0$ if and only if $(\xi - \lambda)^2 = 2$. □

We can now finish Conway's calculation of the covering radius of Λ [17].

Theorem 6.28. *The covering radius of a Niemeier lattice* Λ *without roots is at most equal to* $\sqrt{2}$.

Proof. Let $\xi \in (\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda)$ be a hole with radius $R(\xi) = \inf\{|\lambda - \xi|; \lambda \in \Lambda\}$ and with vertex set $\Lambda(\xi) = \{\lambda \in \Lambda; |\lambda - \xi| = R(\xi)\}$. Clearly the vectors $\{\lambda - \xi; \lambda \in \Lambda(\xi)\}$ form a linearly dependent set, and so its Gram matrix $(\lambda - \xi, \mu - \xi)$ with $\lambda, \mu \in \Lambda(\xi)$ is nonnegative definite with a nonzero kernel. On the other hand, we have

$$\begin{gathered} g_{\lambda\mu} = (\alpha_\lambda, \alpha_\mu) = 2 - \tfrac{1}{2}(\lambda - \mu)^2 = \\ 2 - \tfrac{1}{2}(\lambda - \xi)^2 - \tfrac{1}{2}(\mu - \xi)^2 + (\lambda - \xi, \mu - \xi) = \\ 2 - R(\xi)^2 + (\lambda - \xi, \mu - \xi) \end{gathered}$$

for $\lambda, \mu \in \Lambda(\xi)$.

Now suppose that $R(\xi) > \sqrt{2}$. Then the Coxeter subdiagram of Λ with nodes from $\Lambda(\xi)$ is hyperbolic. By Corollary 6.25 these numbers $g_{\lambda\mu}$ with $\lambda, \mu \in \Lambda(\xi)$ and $\lambda \neq \mu$ are from the set $\{0, -1, -2\}$. Any such hyperbolic Coxeter diagram contains a connected parabolic subdiagram with $r + 1 \leq 25$ nodes. Indeed, just keep on deleting nodes until in the next step one arrives at a connected elliptic Coxeter diagram. By Proposition 6.26 any such connected parabolic subdiagram of Λ is contained in a subdiagram of Λ, which is a disjoint union of m connected parabolic subdiagrams with $24 + m$ nodes altogether. By Proposition 6.27 this implies that $R(\xi) \leq \sqrt{2}$, which gives a contradiction. Hence $R(\xi) \leq \sqrt{2}$ for all holes ξ of Λ. □

6.7 Uniqueness of the Leech Lattice

We shall keep the notation of the previous section.

Theorem 6.29. *The simple roots in R_+ are just the height* 1 *simple roots* $\alpha_\lambda = (1, \frac{1}{2}\lambda^2 - 1, \lambda)$ *with* $\lambda \in \Lambda$*, with the height* $\mathrm{ht}(\alpha) = (\alpha, \rho)$ *taken with respect to the controlling vector* $\rho = (0, -1, 0) \in \mathrm{U} \oplus \Lambda$*.*

Proof. We just run the Vinberg algorithm with controlling vector ρ, for which we already found no height 0 roots and α_λ for $\lambda \in \Lambda$ as the height 1 roots. Suppose $\alpha = (m, (\mu^2 - 2)/(2m), \mu)$ with $\mu \in \Lambda$ is a simple root of next shortest height $m \geq 2$ with $(\alpha, \alpha_\lambda) \leq 0$ for all $\lambda \in \Lambda$. Since Λ has covering radius $\leq \sqrt{2}$ by Theorem 6.28, there is a vector $\lambda \in \Lambda$ with $(\lambda - \mu/m)^2 \leq 2$. But then

$$(\alpha, \alpha_\lambda) = -m(\tfrac{1}{2}\lambda^2 - 1) - (\mu^2 - 2)/(2m) + (\lambda, \mu) =$$
$$m + 1/m - \tfrac{1}{2}m(\lambda - \mu/m)^2 \geq m + 1/m - m = 1/m > 0$$

gives a contradiction with $(\alpha, \alpha_\lambda) \leq 0$. Hence there are no simple roots of height ≥ 2. $\square$

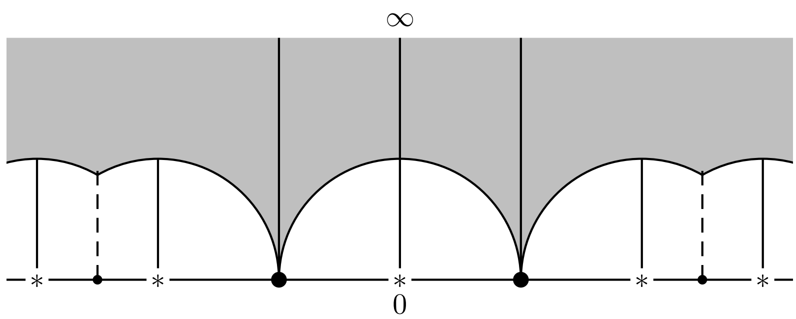
In the upper half space model $\mathbb{H}_+ = \{(\xi, x); \xi \in \mathbb{R} \otimes_{\mathbb{Z}} \Lambda, x > 0\}$ together with its rational boundary points

$$\overline{\mathbb{H}}_+ = \mathbb{H}_+ \sqcup (\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda) \sqcup \{\infty\}$$

the fundamental domain for the action of the reflection group $W < \mathrm{O}^+(L)$ becomes

$$D = \{(\xi, x) \in \mathbb{H}_+; (\xi - \lambda)^2 + x^2 \geq 2, x > 0\ \forall \lambda \in \Lambda\} \sqcup \mathbb{H} \sqcup \{\infty\}$$

with $\mathbb{H} \subset \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$ the set of deep holes with radius $\sqrt{2}$.



In the above picture the horizontal axis stands for the Euclidean space $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ of dimension 24 and the vertical axis is just the positive real line. The gray region is the fundamental domain D which is the complement in $\overline{\mathbb{H}}_+$ of the open spheres with centers $(\lambda, 0)$ for $\lambda \in \Lambda$ (indicated by a $*$ in the picture) and radius $\sqrt{2}$. The automorphism group $\mathrm{Aut}(D)$ is isomorphic to the crystallographic group $\Lambda \rtimes \mathrm{O}(\Lambda)$, and so

$$\mathrm{O}^+(L) = W(L) \rtimes (\Lambda \rtimes \mathrm{O}(\Lambda))$$

with $L = \mathrm{U} \oplus \Lambda$.

The orbits under $\Lambda \rtimes \mathrm{O}(\Lambda)$ of the ideal vertices of D correspond to the isomorphism classes of Niemeier lattices. The Niemeier lattices with roots correspond to the deep holes of the Leech lattice (indicated by a thick dot in the above picture), which necessesarily have hole radius equal to $\sqrt{2}$. The shallow holes (indicated by a thin dot in the above picture) correspond to the vertices of D in the upper half plane $\mathbb{H}_+$ itself. The next result is therefore clear from Theorem 6.28.

Corollary 6.30. *The covering radius of* Λ *is equal to* $\sqrt{2}$.

The vertex ∞ of D corresponds to the unique Niemeier lattice without roots.

Corollary 6.31. *The Leech lattice* Λ *is the (up to isomorphism) unique Niemeier lattice without roots.*

Since in the construction of Section 6.4 the vector ρ/h is the center of the inscribed sphere for the corresponding fundamental alcove, it is clear that the vertical geodesic departing from a deep hole lands after one step at the rational boundary point ∞ corresponding to Λ.

Corollary 6.32. *The construction of the Leech lattice* Λ *given in Section 6.4 in fact terminates after one step, and altogether we have* 23 *distinct "holy constructions" for* Λ*, namely one for each Niemeier lattice* N *with roots .*

Altogether there are 24 different Niemeier lattices. For the Niemeier lattices N with roots the groups $\mathrm{O}(N) = W(N) \rtimes G(N)$ can be computed in a case by case manner from the symmetry group $G(N)$ of the code $N/Q(N)$ with $Q(N) = \mathbb{Z}R(N)$ the root lattice of N. For example, for A_1^{24} one finds $\mathcal{S}_2^{24} \rtimes \mathrm{M}_{24}$ and for A_2^{12} one finds $\mathcal{S}_3^{12} \rtimes \mathrm{M}_{12}$ with $\mathrm{M}_{12} < \mathcal{S}_{12}$ and $\mathrm{M}_{24} < \mathcal{S}_{24}$ two sporadic simple Mathieu groups. They are among the list of 5 sporadic finite simple groups found by Mathieu in 1861.

Using the Minkowski–Siegel mass formula one can compute the order of $\mathrm{Co}_0 = \mathrm{O}(\Lambda)$. It turns out that the quotient of $\mathrm{O}(\Lambda)$ modulo its center of order 2 is the sporadic simple Conway group Co_1 of order $2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 12 \cdot 23$ found by Conway in 1969 [17].

The Mathieu groups $\mathrm{M}_{11} < \mathrm{M}_{12}$ and $\mathrm{M}_{22} < \mathrm{M}_{23} < \mathrm{M}_{24}$ as stabilizers of one or two elements are also sporadic simple groups. Likewise the subgroups $\mathrm{Co}_2 < \mathrm{Co}_0$ (of order $2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$) and $\mathrm{Co}_3 < \mathrm{Co}_0$ (or order $2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$) as stabilizers of a norm 4 and a norm 6 vector in Λ respectively are again sporadic simple groups.

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