DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN The Netherlands

THE MODULI SPACE OF RATIONAL ELLIPTIC SURFACES

Gert Heckman, Eduard Looijenga

Report No. 0111 (June 2001)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

The moduli space of rational elliptic surfaces

Gert Heckman and Eduard Looijenga

To Tonny Springer for his 75th birthday

Abstract. We show that the moduli space of rational elliptic surfaces admitting a section is locally a complex hyperbolic variety of dimension 8. We compare its Satake-Baily-Borel compactification with a compactification obtained by means of geometric invariant theory, considered by Miranda.

Introduction

By a *rational elliptic surface* we mean a smooth complete complex surface that can be obtained from a pencil of cubic curves in $\mathbb{P}^2$ with smooth members by successive blowing up (9 times) its base points. A more intrinsic characterization is to say that the surface is rational and admits a relatively minimal elliptic fibration possessing a section. Better yet: it is a smooth complete complex surface whose anticanonical system is base point free and defines a fibration. The description as a blown-up $\mathbb{P}^2$ is not canonical (in general the possible choices are in bijective correspondence with a weight lattice of an affine root system of type $\hat{E}_8$), but the last characterization makes it plain that the fibration is. The main goal of this paper is to investigate and describe the moduli space of these surfaces and certain compactifications thereof. By assigning to a fiber of a rational elliptic surface its Euler characteristic we find a divisor on its base curve, called the *discriminant divisor*. This discriminant divisor is effective and of degree 12. In general it is reduced, meaning that we have 12 singular fibers, each of which is a rational curve with a node. It is not difficult to show that in that case the discriminant divisor is a complete invariant: the projective equivalence class of the discriminant (as a 12-element subset of $\mathbb{P}^1$) determines the surface up to isomorphism. Let us denote by $\mathcal{M}$ the moduli space of rational elliptic surfaces with reduced discriminant. One compactification of $\mathcal{M}$ was obtained by Miranda [**24**] by

1991 *Mathematics Subject Classification.* 14J15, 14J27, 32N10.

Key words and phrases. rational elliptic fibration, moduli, ball quotient.

applying geometric invariant theory to the Weierstrass models of these surfaces. It gives a projective compactification of $\mathcal{M}$, denoted here by $\mathcal{M}^M$, with an interpretation of every boundary point as corresponding to an isomorphism class of rational elliptic surfaces.

Regarding the discriminant of a rational elliptic surface as its fundamental invariant leads to an altogether different compactification of $\mathcal{M}$. Let $\mathcal{D}_{12}$ denote the space of SL(2)-orbits in the configuration space of 12-element subsets of $\mathbb{P}^1$. A projective compactification $\mathcal{D}_{12}^*$ of $\mathcal{D}_{12}$ is obtained by means of Geometric Invariant Theory: take the closed SL(2)-orbits in the space of effective degree 12 divisors on $\mathbb{P}^1$ that are semistable in the sense that all multiplicities are $\leq \frac{1}{2} \cdot 12 = 6$. There is only one such orbit which is not stable: it is represented by a divisor which is 6 times a 2-element subset. The variety $\mathcal{D}_{12}^*$ appears in the work of Deligne and Mostow [**10**] as the Satake-Baily-Borel compactification of a 9-dimensional complex ball $\mathbb{B}^9$ with just one cusp (which corresponds to the closed strictly semistable orbit). It arises from a period mapping: for a 12-element subset D of $\mathbb{P}^1$, take the cyclic cover $C \to \mathbb{P}^1$ of degree 6 which totally ramifies in D and then assign to D the abelian variety (of dimension 10) obtained from the Jacobian of C by dividing out the Jacobian of intermediate covers (so that the Galois group acts on this quotient with primitive sixth roots of unity). The discriminant gives rise to a closed embedding of $\mathcal{M}$ in $\mathcal{D}_{12}$. Rational elliptic surfaces have 8 moduli, whereas $\dim \mathcal{D}_{12} = 9$ and so they define a SL(2)-invariant hypersurface in the 12th symmetric power of $\mathbb{P}^1$. This hypersurface can be characterized as defining the 12-element subsets admitting an equation that is the sum of a cube and a square. The compactification of $\mathcal{M}$ we alluded to is the normalization $\mathcal{M}^*$ of $\mathcal{M}$ in $\mathcal{D}_{12}^*$. A central result of this paper is a characterization of the morphism $\mathcal{M}^* \to \mathcal{D}_{12}^*$ in the spirit of Deligne and Mostow, namely as a morphism of Satake-Baily-Borel compactifications defined by an 'arithmetically defined' hyperball in $\mathbb{B}^9$. The origin of this description is explained by the fact that the degree 6 cover $C \to \mathbb{P}^1$ naturally comes with a morphism from C to the elliptic curve of J-invariant 0 when its ramification divisor is the discriminant of a rational elliptic surface.

We find that the boundary of $\mathcal{M}$ in $\mathcal{M}^*$ is of codimension one and has four irreducible components, each of which is the closure of a totally geodesic subvariety. Only two of these irreducible components also appear in Miranda's compactification and have there the interpretation as parametrizing rational elliptic surfaces with a special fiber (of bicyclic type I_2 and of cuspidal type II respectively).

Apart from that, the two compactifications are very much different. The natural birational map between $\mathcal{M}^M$ and $\mathcal{M}^*$ is not a morphism in either direction and many points of $\mathcal{M}^*$ fail to have an interpretation as describing an isomorphism class of a rational elliptic surface. We therefore consider the closure $\mathcal{M}^{M*}$ of the diagonally embedded $\mathcal{M}$ in the product of these two compactifications. A substantial part of this paper can be understood as a study of $\mathcal{M}^{M*}$ with its projections on $\mathcal{M}^M$ and $\mathcal{M}^*$. In the end it turns out that this diagram can be obtained in completely arithmetic terms (involving a hyperbolic Hermitian lattice over the Eisenstein ring); Examples 10.2 and 10.3 make this most explicit. The situation is quite similar to the relation one of us found between the Baily-Borel compactification of the moduli space of K3 surfaces of degree 2 (resp. 4) and Shah's GIT compactification of the sextic plane curves (resp. quartic surfaces) [**21**] and the one that Sterk [**32**] found between the moduli space of Enriques surfaces and Shah's GIT compactification of curves on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(4,4)$ invariant under a certain involution.

Vakil [**34**] recently showed that some interesting moduli spaces define finite monodromy covers of $\mathcal{M}$: the moduli spaces of (1) nonhyperelliptic genus 3 curves endowed with a canonical pencil, of (2) genus 4 curves with an effective even theta characteristic, and of (3) hyperelliptic genus 4 curves endowed with a noncanonical pencil all have this form. He observes that it then follows from our theorem that these moduli spaces are locally complex hyperbolic.

As is well-known, $\mathcal{M}$ has also the interpretation as the moduli space of Del Pezzo surfaces of degree 1. From this point of view, the above result neatly fits in a series of similar characterizations of the moduli spaces of Del Pezzo surfaces of given degree: this started with the work of Allcock, Carlson and Toledo [**3**] who associated to a cubic surface X in $\mathbb{P}^3$ the intermediate Jacobian of the cyclic degree 3 cover of $\mathbb{P}^3$ ramified along that surface. They found that in this way the moduli space of cubic surfaces has the structure of a ball quotient. In this case one is so fortunate as to have a GIT interpretation of the Baily-Borel compactification so that the boundary parametrizes (mildly) degenerate cubic surfaces. Van Geemen (unpublished) and Kondo [**16**] independently found a similar ball quotient description for the moduli space of Del Pezzo surfaces of degree two (or equivalently, of quartic plane curves). It seems that here the GIT compactication and the Baily-Borel compactification are related in a way that is quite similar to the case studied in the present paper. In particular, neither is a blowup of the other.

Let us take the occasion to point out that this is also the picture for Del Pezzo surfaces of degree four (in higher degree they are rigid, so this is the remaining case of interest). The anticanonical embedding of such a surface realizes that surface in $\mathbb{P}^4$ as the fixed point set of a pencil P of quadrics in $\mathbb{P}^4$. The singular quadrics in this pencil define a 5-element subset D of P and the isomorphism type of the pair (P, D) is a complete invariant of the surface. The work of Deligne-Mostow identifies the set of such of isomorphism types with an open subset of a ball quotient, essentially by passing to the Jacobian of the cyclic cover of P of degree 5 with total ramification in D.

We also mention here that Abramovich and Vistoli [**1**] defined (as a special case of a more general theory of theirs) a complete Deligne-Mumford stack of moduli of rational elliptic surfaces, which is modeled on what we call the *Kontsevich compactification.* Here the boundary points label no longer ordinary rational elliptic surfaces, but rather surfaces with an 'orbispace structure'

Let us now describe the contents of the separate sections. We begin the paper with a general discussion of the Picard group of a rational elliptic surface (Section 1). Although this section is brief, we do not use all the material expounded here and we therefore advise the reader to consult it as needed. Section 2 introduces the main character of this paper, the moduli space $\mathcal{M}$. We review Miranda's compactification $\mathcal{M}^M$ of $\mathcal{M}$, which parametrizes elliptic surfaces and we define another one, $\mathcal{M}^*$, which does not. In Section 3 we define yet another compactification that dominates these two and is based on Kontsevich's notion of a stable map. This compactification is useful by itself, but plays in the present paper only an auxiliary role: we use it to understand the birational map between $\mathcal{M}^M$ and $\mathcal{M}^*$ in geometric terms. In the next two Sections 4 and 5 we make a careful study of the homology of cyclic degree 6 covers of $\mathbb{P}^1$ totally ramified in 12 distinct points and the action of a corresponding central extension of the braid group of $\mathbb{P}^1$ with 12 strands. This discussion belongs to algebraic topology rather than to algebraic geometry and is independent of the preceding. Section 6 recalls the basics of the Satake-Baily-Borel compactification of a ball quotient and the next section discusses the work of Deligne-Mostow for the case that is relevant here. Since this result is a bit hidden in their general theory, we outline its proof. In passing we obtain a simple description of the monodromy group (a unitary group of a rank 10 lattice over the Eisenstein ring) as a quotient of the corresponding mapping class group (a centrally extended braid group). Section 8 leads up to the main Theorems 9.2 and 9.3 in the next section. The final Section 10 is for the most part descriptive. It provides what we feel is a natural general context for our results. It also suggests an extension of the theory of automorphic forms for ball quotients whose geometric counterpart is a theory of compactifications of ball quotients with a locally symmetric divisor removed. The appendix is devoted to unitary lattices over an Eisenstein ring. Part of this is a general discussion, but we have also put here the more specific results that we use.

Some of the initial steps of this work by one of us (GH) were carried out when he was a visitor of the École Normale Superieure at Paris in May 1998, and he is grateful for the hospitality. He also wants to thank Richard Borcherds for an inspiring lecture and discussion. We thank Rick Miranda for some helpful correspondence. Finally we thank the referee for his careful job.

Most of the results described here were obtained in the summer of 1999.

We happily dedicate this paper to our colleague Tonny Springer on the occasion of his 75th birthday.

Contents

1. Rational elliptic surfaces: basic properties

In this section we collect some facts (known and perhaps less known) concerning rational elliptic surfaces and Del Pezzo surfaces of degree one. General references are [**19**], [**12**], [**20**], [**26**] and [**14**].

By a *rational elliptic surface* we shall mean a smooth complete rational surface X that admits an elliptic fibration that is relatively minimal (in the sense that no exceptional curve is contained in a fiber) and has a section. Then this fibration is unique since its fibers are the anticanonical curves on X; in particular, its base P is canonically the projective line of lines of the plane $H^0(X, \omega_X^{-1})$. (In fact, any smooth complete surface whose anticanonical system is a pencil and defines a fibration is of this form.) The sections of this fibration are precisely the exceptional curves of the first kind of X. We can always obtain such a surface—though in general in more than one way—as follows: take a pencil of plane cubic curves having at least one smooth member. Its base locus will consist of nine points (possibly infinitely near) and blowing these up yields a rational elliptic surface in our sense (the last blowup giving a section).

It follows from this last description that the Picard lattice of X is isomorphic to the rank 10 lattice $I_{1,9}$ that has a basis $\ell, e_1, \dots, e_9$ on which the inner product takes the form $\ell.\ell = 1$, $\ell.e_i = 0$, $e_i.e_j = -\delta_{i,j}$. An isomorphism $I_{1,9} \cong \mathrm{Pic}(X)$ can be chosen such that ℓ is the class of a line in $\mathbb{P}^2$ and e_i the class of the exceptional curve of the ith blowup. The class of a fiber of $X \to P$ is the class of ω_X^{-1} and is therefore mapped to $f := 3\ell - e_1 - \cdots - e_9$.

We first investigate $I_{1,9}$ as an abstract lattice with distinguished isotropic vector f. A *root* of $I_{1,9}$ is a vector $\alpha \in I_{1,9}$ with $\alpha.f = 0$ and $\alpha.\alpha = -2$. The orthogonal reflection with respect to α,

$$s_\alpha : c \mapsto c + (\alpha.c)\alpha$$

preserves the lattice $I_{1,9}$ and fixes f. The set of roots (denoted here by $\mathcal{R}$) is an infinite root system; a root basis is $\alpha_0 := \ell - e_1 - e_2 - e_3, \alpha_1 := e_1 - e_2, \dots, \alpha_8 := e_8 - e_9$, which shows that it is of type $\hat{E}_8$. The associated Weyl group $W(\mathcal{R})$ of isometries of $I_{1,9}$ generated by the reflections with respect to roots is precisely the stabilizer of f in the orthogonal group of $I_{1,9}$ (see for instance [**35**]). We realize $\mathcal{R}$ as an affine root system (and $W(\mathcal{R})$ as an affine transformation group) as follows. The set of vectors $c \in I_{1,9}$ with $c.f = 0$ resp. $c.f = 1$ project in $I_{1,9}/\mathbb{Z}f$ onto a sublattice Q resp. an affine lattice A over Q. Given a root α, then taking the inner product with that root, makes α appear as an affine-linear form on A. If denote by $\check{\alpha}$ the image of $-\alpha$ in Q, then the action of s_α in A is given by $c \mapsto c - (\alpha.c)\check{\alpha}$ and thus $\mathcal{R}$ becomes an affine root system on A in the sense of [**22**]. The group $W(\mathcal{R})$ acts faithfully on A and the underlying real affine space $A_\mathbb{R}$ receives its standard affine reflection action. The image R of $\mathcal{R}$ in Q is a finite root system of type E_8 and spans Q. The full translation lattice Q is so realized as the translation subgroup of $W(\mathcal{R})$. More concretely, the transformation in $I_{1,9}$ associated to $u \in Q$ is the *Eichler-Siegel* transformation

$$T_u : c \mapsto c + (c.f)\hat{u} - (c.\hat{u})f - \frac{1}{2}(\hat{u}.\hat{u})(c.f)f,$$

where $\hat{u} \in \hat{Q}$ lifts $u \in Q$. The transformation T_u indeed only depends on u and we have thus defined an injective homomorphism $T : Q \to \mathrm{SO}(I_{1,9})$ of groups.

Let us denote by $\mathcal{E} \subset I_{1,9}$ the set of e with $e.f = 1$ and $e.e = -1$. The natural map $\mathcal{E} \to A$ is a bijection: if $c \in I_{1,9}$ is such that $c.f = 1$, then $(c.c)$ is odd (this follows from the fact that Q is even and that this is true for one such c, e.g., $c = e_1$) and so $e := c - \frac{1}{2}(1 + (c.c))f$ is the unique element of $c + \mathbb{Z}f$ with self-product -1. So the translation subgroup $T(Q)$ of $W(\hat{\mathcal{R}})$ acts simply transitively on $\mathcal{E}$.

It is clear that this discussion makes sense in $\mathrm{Pic}(X)$ without any reference to an isomorphism of $(I_{1,9}, f)$ onto $(\mathrm{Pic}(X), [\omega_X^{-1}])$. We adapt our notation to this situation in an obvious way and write $f_X, \mathcal{R}_X, \mathcal{E}_X, Q_X, A_X, \dots$.

An element of $\mathcal{R}_X$ resp. $\mathcal{E}_X$ that is the class of an irreducible curve is called a *nodal* resp. *exceptional* class and we denote by $\mathcal{R}_X^{\mathrm{irr}} \subset \mathcal{R}_X$ resp. $\mathcal{E}_X^{\mathrm{irr}} \subset \mathcal{E}$ the corresponding subset. The following is well-known.

Proposition 1.1. *Any irreducible component of a reducible fiber has a nodal class and this establishes a bijection between the set of irreducible components of reducible fibers and $\mathcal{R}_X^{\mathrm{irr}}$.*

The set $\mathcal{R}_X^{\mathrm{irr}}$ decomposes according to the set of reducible fibers $(X_p)_{p \in S}$:

$$\mathcal{R}_X^{\mathrm{irr}} = \sqcup_{p \in S} \mathcal{R}_{X_s}^{\mathrm{irr}}.$$

It is convenient to introduce the *closed nodal chamber* as the set of $c \in A_{\mathbb{R}}$ satisfying $\alpha.c \geq 0$ for all $\alpha \in \mathcal{R}_X^{\rm irr}$. This is a product of closed simplices (a factor for every reducible fiber) times an affine space. It is a strict fundamental domain for the action of the Weyl subgroup $W(\mathcal{R}_X^{\rm irr}) \subset W(\mathcal{R}_X)$ in $A_{\mathbb{R}}$. Let us denote by $Q_X^{\rm irr} \subset Q_X$ the image of the integral span of $\mathcal{R}_X^{\rm irr}$ in $\mathrm{Pic}(X)$.

PROPOSITION 1.2. *A section of $X \to P$ is an exceptional curve of the first kind and this identifies the set of sections with $\mathcal{E}_X^{\rm irr}$. Given $e \in \mathcal{E}_X$, let $e_0 \in \mathcal{E}_X$ be the unique element of its $W(\mathcal{R}_X^{\rm irr})$-orbit mapping to the closed nodal chamber. Then e_0 is the class of a section and $e - e_0$ is a nonnegative linear combination of nodal classes. The composite map $\mathcal{E}_X^{\rm irr} \subset \mathcal{E}_X \cong A_X \to A_X/Q_X^{\rm irr}$ is a bijection.*

All of this is known, though perhaps stated somewhat differently in the literature (see for example [**26**]). So $\mathcal{E}_X^{\rm irr}$ gets smaller when $\mathcal{R}_X^{\rm irr}$ gets bigger. The generic situation is when $\mathcal{R}_X^{\rm irr} = \emptyset$: then $\mathcal{E}_X^{\rm irr} = \mathcal{E}_X$. The other extreme, $\mathcal{E}_X^{\rm irr}$ finite, happens precisely when $Q_X/Q_X^{\rm irr}$ is finite. The following proposition identifies the rational points of the Picard group of the generic fiber.

PROPOSITION 1.3. *The group of automorphisms* $\mathrm{Aut}^0(X/P)$ *of X that induce a translation in every smooth fiber is faithfully represented in* $\mathrm{Pic}(X)$. *It acts simply transitively on $\mathcal{E}_X^{\rm irr}$ and via the identification of $\mathcal{E}_X^{\rm irr}$ with $A_X/Q_X^{\rm irr}$, this group is identified with the abelian group $Q_X/Q_X^{\rm irr}$. It is also the group of automorphisms of* $\mathrm{Pic}(X)$ *that lie in $T(Q) \cdot W(\mathcal{R}_X^{\rm irr})$ and preserve $\mathcal{R}_{X_s}^{\rm irr}$ for every reducible fiber X_s. (This group contains the image of $(Q_X^{\rm irr})^{\perp} \subset Q$ under T as a subgroup of finite index.)*

This proposition should be known, but since we did not find it stated this way, we give a proof. For this we need a property of affine Coxeter groups that we recall from [**6**], Ch. VI, § 2. Let $(W, (s_i)_{i \in I})$ be an irreducible Coxeter system of affine type (with the s_i's distinct) and identify W with its canonical representation as an affine transformation group. Denote by $D(I)$ the Dynkin diagram on I. The normalizer $N(W)$ of W in the affine transformation group acts on $D(I)$ and identifies $N(W)/W$ with $\mathrm{Aut}(D(I))$. If $I_0 \subset I$ is the set of *special* vertices of $D(I)$ (an $i \in I$ is special precisely when every element of W is the composite of a translation and an element of $W_{I-\{i\}}$), then $N(W)/W$ acts faithfully on I_0 and the subgroup of $N(W)/W$ induced by translations acts simply transitively on I_0. In particular, if a translation in $N(W)$ fixes a special vertex of $D(I)$, then it lies in W. On the other hand, any element of $N(W)/W$ not coming from a translation fixes a special vertex.

We see this illustrated by a Kodaira fibration over a smooth curve germ $\mathcal{X} \to \mathbb{D}$ with special fiber X_o (the general fiber is a smooth curve of genus one, the special fiber is of Kodaira type). If $(C_i)_{i \in I}$ are the distinct irreducible components of X_o, then we have $C_i.C_i = -2$ for all i and if $\sum_i n_i C_i$ is the class of the general fiber, then the reflections $s_i : c \mapsto c + (c, C_i)[C_i]$ in $H_2(X_0)$ generate an irreducible Coxeter system $(W, (s_i)_{i \in I})$ of affine type acting naturally in the affine hyperplane in

$\mathrm{Hom}(H_2(X_o),\mathbb{R}) = H^2(X_o;\mathbb{R})$ of forms that take the value 1 on $\sum_i n_i C_i$. Its Dynkin diagram is just the intersection graph of the C_i's. We have $n_i > 0$ for all i and $i \in I$ is special precisely when $n_i = 1$. Any automorphism of the general fiber which induces a translation on that fiber extends to the whole fibration. If it preserves a special component, then it preserves every component. So it follows from the preceding that that its action on $A_{\mathbb{R}}$ is the composite of an element of W and a translation.

Proof of 1.3. That $\mathrm{Aut}(X)$ acts faithfully on $\mathrm{Pic}(X)$ is well-known and easy to prove. If $e, e' \in \mathcal{E}_X^{\mathrm{irr}}$ are represented by sections E, E', then there is a fiberwise translation in the part of X that is smooth over P which sends E to E'. As recalled above, this translation extends as an automorphism h of X. Then h fixes the difference of any two sections, so it certainly acts as the identity in Q_X/Q_X^{irr}. If E and E' meet a reducible fiber X_s in the same component, then this component is special. So h fixes every irreducible component of X_s. The rest of the argument is now straightforward or follows from the above mentioned property of Kodaira fibrations. □

Lemma 1.4. *We have* $T(Q_X) \subset \mathrm{Aut}^0(X/P) \cdot W(\mathcal{R}_X^{\mathrm{irr}})$.

Proof. Let $u \in Q_X$. So for every $\alpha \in \mathcal{R}_X$ we have $T_u(\alpha) = \alpha - (\alpha.u)f$. It follows that for every reducible fiber X_s, T_u preserves the root subsystem $\mathcal{R}_{X_s}$ of $\mathcal{R}_X$ generated by $\mathcal{R}_{X_s}^{\mathrm{irr}}$. So T_u normalizes the associated affine Weyl group $W(\mathcal{R}_{X_s}^{\mathrm{irr}})$. Choose a section E. Then T_u sends its class $e \in \mathcal{E}_X^{\mathrm{irr}}$ to an element of the form $w(e')$ with $w \in W(\mathcal{R}_X^{\mathrm{irr}})$, where $e' \in \mathcal{E}_X^{\mathrm{irr}}$ is the class of a section E'. There is a unique $h \in \mathrm{Aut}^0(X/P)$ that sends E to E'. We show that $g := h_*^{-1}w^{-1}T_u$ is in $W(\mathcal{R}_X^{\mathrm{irr}})$. It is clear that g is the identity on the orthogonal complement of $\mathcal{R}_X^{\mathrm{irr}}$ and fixes e. Also, for every reducible fiber X_s, g normalizes $W(\mathcal{R}_{X_s}^{\mathrm{irr}})$ and its image in $N(W(\mathcal{R}_{X_s}^{\mathrm{irr}}))/W(\mathcal{R}_{X_s}^{\mathrm{irr}})$ is induced by a translation. Since $g(e) = e$, it follows that this image is trivial: g acts in the span of $\mathcal{R}_{X_s}^{\mathrm{irr}}$ as an element of $W(\mathcal{R}_{X_s}^{\mathrm{irr}})$. This is true for all reducible fibers and hence $g \in W(\mathcal{R}_X^{\mathrm{irr}})$. □

Remark 1.5. Contraction of an exceptional curve of the first kind with class $e \in \mathcal{E}_X^{\mathrm{irr}}$ produces a smooth rational surface surface X_e with $\omega_{X_e}.\omega_{X_e} = 1$. It follows from Proposition 1.3 that its isomorphism type is independent of the choice of e. If all fibers of $X \to P$ are irreducible (in other words, $\mathcal{R}_X^{\mathrm{irr}} = \emptyset$), then $\omega_{X_e}^{-1}$ is ample, in other words, X_e is a Del Pezzo surface of degree one. Conversely, if we are given a Del Pezzo surface of degree one, then its anticanonical system consists of irreducible curves and has a unique fixed point. Blowing up that point yields an elliptic surface with all its fibers irreducible. So the coarse moduli space of Del Pezzo surfaces of degree one can be identified with the coarse moduli space of smooth rational elliptic surfaces with all its fibers irreducible. Notice that we have a natural identification of Q_X with the orthogonal complement of $[\omega_{X_e}]$ in $\mathrm{Pic}(X_e)$.

2. Moduli of rational elliptic surfaces I

2.1. The Weierstrass model. Let $f : X \to P$ be a rational elliptic surface. The *discriminant divisor* of f is the divisor on P for which the multiplicity of $p \in P$ is the Euler characteristic of the fiber X_p. This is an effective divisor whose degree must be the Euler characteristic of X, which is 12. Assigning to each fiber its modular invariant defines a morphism $J : P \to \mathbb{P}^1$. Let us assume that all the singular fibers are of type I_k. Then $D_\infty := J^*(\infty)$ is the discriminant divisor of f. In order to understand J over the special points 0 and 1, let us recall that the affine J-line is obtained as the analytic orbifold $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$ with 0 resp. 1 corresponding to the singular orbits of $\omega := e^{2\pi\sqrt{-1}/6}$ resp. $\sqrt{-1}$. The order of ramification of the quotient map over such a point is the order of its $\mathrm{PSL}(2,\mathbb{Z})$-stabilizer, that is 3 resp. 2. Since the fibers of f over $P - D_\infty$ are smooth, the morphism J is at every point of $P - D_\infty$ locally liftable to a morphism to $\mathbb{H}$. This implies that $J^*(0) = 3D_0$ and $J^*(1) = 2D_1$ with D_0 resp. D_1 a divisor of degree 4 resp. 6. So D_∞ is in the pencil generated by $3D_0$ and $2D_1$. This imposes a nontrivial condition on D_∞.

To see this, we fix a projective line P and denote by H the space of sections of $\mathcal{O}_P(1)$. For a nonnegative integer k, $H_k := \mathrm{Sym}^k H$ is then the space of sections of $\mathcal{O}(k)$ and the associated (k-dimensional) projective space P_k is the linear system of effective degree k divisors on P. The set of triples $(D_0, D_1, D_\infty) \in P_4 \times P_6 \times P_{12}$ with D_0 and D_1 not a common multiple of an element of P_2 (to ensure that they generate a pencil) and D_∞ in the pencil generated by $3D_0$ and $2D_1$ is an irreducible subvariety of dimension $6 + 4 + 1 = 11$. Denote by $\tilde{\Sigma}$ its closure in $P_4 \times P_6 \times P_{12}$ and by Σ the projection of $\tilde{\Sigma}$ in P_{12}. It is clear that Σ is irreducible of dimension ≤ 11. In fact:

PROPOSITION 2.1. *The projection $\tilde{\Sigma} \to \Sigma$ is birational so that Σ is a rational ruled hypersurface in P_{12}. A point $(D_0, D_1, D_\infty) \in \tilde{\Sigma}$ for which $3D_0 \neq 2D_1$ comes from a rational elliptic surface. It is unique up to P-automorphism if D_0 and D_1 have disjoint support (a condition fulfilled if D_∞ is reduced).*

PROOF. Let $W \subset \tilde{\Sigma}^2$ be the locus of pairs of distinct points of $\tilde{\Sigma}$ with the same image in P_{12}. For the first assertion it is enough to show that W is of dimension ≤ 10. A point of $(D_0, D_1, D_\infty) \in \tilde{\Sigma}$ for which $3D_0, 2D_1, D_{12}$ are mutually distinct can be represented by a triple $(f_0, f_1, f_\infty) \in H_4 \oplus H_6 \oplus H_{12}$ with $f_\infty = f_0^3 + f_1^2$ so that D_i is the divisor defined by f_i. Notice that the vector $(f_0^3, f_1^2) \in H_{12}^{\oplus 2}$ is unique up to a scalar factor. An element of W is representable by a quadruple $(f_0^3, f_1^2, g_0^3, g_1^2)$ in H_{12} with $f_0^3 + f_1^2 = g_0^3 + g_1^2$. This identity can also be written as $(f_1 - g_1)(f_1 + g_1) = g_0^3 - f_0^3$. If the righthand side is nonzero, then it is factored by the lefthand side into two forms of degree six. The family of such factorizations (with fixed nonzero righthand side) is of dimension one. Since $[f_0 : g_0]$ lies in a projective space of dimension 9, it follows that $\dim W \leq 9 + 1 = 10$.

To prove the second assertion we consider the vector bundle $\mathcal{E} := \mathcal{O}_P(2) \oplus \mathcal{O}_P(3) \oplus \mathcal{O}_P$ over P. Denote the projections on its summands by X, Y, Z repectively. So for (f_0, f_1, f_∞) as above, the expression $-Y^2Z + X^3 + 3f_0XZ^2 + 2f_1Z^3$ defines a homomorphism $\mathcal{E} \to \mathcal{O}_P(6)$. Its zero set in the associated projectivized bundle $\mathbb{P}(\mathcal{E})$ is a Weierstrass curve over P with modular function $J = f_0^3/f_\infty$. If f_0^3 and f_1^2 are linearly independent, then minimal resolution of its singularities gives an elliptic surface for which the first summand of $\mathcal{E}$ defines a section. This surface is rational.

If f_0 and f_1 have no nontrivial common zero, then J has degree 12 and $J^*(i) = D_i$ for $i = 0, 1, \infty$. Kodaira's theory (see for example [**4**], Thm. 11.1 and Subsection 3.1 below) implies that this elliptic surface is unique up to P-isomorphism. □

Remark 2.2. Vakil [**34**] recently proved that the degree of Σ is equal to 3762. In the same paper he also gives several remarkable characterizations of this hypersurface.

Example 2.3. This is an example to which we will later return. Take for D_∞ the 12th roots of unity in $\mathbb{C}$, viewed as a reduced divisor on $\mathbb{P}^1$. If we take $D_0 = 4(0)$ and $D_1 = 6(\infty)$, then clearly $(D_0, D_1, D_\infty) \in \tilde{\Sigma}$. By the preceding argument there is a rational elliptic surface with D_∞ as discriminant divisor.

We thus recover a result of Dolgachev.

Corollary 2.4 (Dolgachev, [**13**]). *The coarse moduli space of rational elliptic surfaces (and hence also the coarse moduli space of Del Pezzo surfaces of degree one) is rational.*

Proof. In view of 2.1 we must show that the $\mathrm{Aut}(P)$-orbit space of $\tilde{\Sigma}$ is rational. Generically $\tilde{\Sigma}$ is fibered in lines over the product of projective spaces $P_4 \times P_6$. Let $P_4' \subset P_4$ be the locus where $\mathrm{Aut}(P)$ acts freely. Then P_4' is open-dense in P_4, and the orbit space $B := \mathrm{Aut}(P)\backslash P_4'$ is a rational curve. So if $\tilde{\Sigma}'$ denotes the preimage of P_4' in $\tilde{\Sigma}$, then $\mathrm{Aut}(P)\backslash\tilde{\Sigma}' \to B$ is a morphism to a rational curve whose generic fiber has the structure of a fibration of lines over a projective space. This implies that $\mathrm{Aut}(P)\backslash\tilde{\Sigma}'$ is rational. □

2.2. Miranda's compactification. R. Miranda gave in his thesis [**23**] a geometric invariant theory compactification of the space of pencils of cubic plane curves. Since pencils with a smooth member define rational elliptic surfaces, this leads a compactification of the moduli space of (generic) rational elliptic surfaces. Later he found that the geometric invariant theory of Weierstrass fibrations did that job more directly [**24**] and so it is this approach that we shall follow.

Let $U \subset H_4 \oplus H_6$ be the open subset of (f_0, f_1) such that $f_0^3 + f_1^2$ is square free. As was noted in the proof of 2.1, the locus $Y^2Z = X^3 + 3f_0XZ^2 + 2f_1Z^3$ defines in $\mathbb{P}(\mathcal{O}_P(2) \oplus \mathcal{O}_P(3) \oplus \mathcal{O}_P) \times U$ a rational elliptic surface $\mathcal{X}_U \to P \times U$ over U with section over $P \times U$. The group $\mathrm{GL}(H)$ acts on this fibration. Two points of U define isomorphic elliptic surfaces with section if and only if they are in the same $\mathrm{GL}(H)$-orbit. Since the automorphism group of a rational elliptic surface acts transitively on its sections, it follows that $\mathrm{GL}(H)\backslash U$ is the coarse moduli space of rational elliptic surfaces with reduced discriminant. We denote that orbit space by $\mathcal{M}$. A natural projective completion $\mathcal{M}^M$ of $\mathcal{M}$ is obtained by means of geometric invariant theory applied to the $\mathrm{SL}(H)$-action on $H_4 \oplus H_6$. With Miranda one easily finds that (f_0, f_1) is semistable (resp. stable) relative to this action if and only if f_0^3 and f_1^2 have no nontrivial common zero of order > 6 (resp. ≥ 6). The proj of the algebra of $\mathrm{SL}(H)$-invariants of the algebra of regular functions on $H_4 \oplus H_6$,

$$\mathbb{C}[H_4 \oplus H_6]^{\mathrm{SL}(H)} = \big(\mathbb{C}[H_4] \otimes \mathbb{C}[H_6]\big)^{\mathrm{SL}(H)}$$

is a projective completion of $\mathcal{M}$. In more geometric terms: if $\mathbb{P}(H_4 \oplus H_6)$ stands for the weighted projective space gotten by dividing $H_4 \oplus H_6 - \{(0,0)\}$ out by the action of the central subgroup $\mathbb{G}_m \subset \mathrm{GL}(H)$, then

$$\mathcal{M}^M = \mathrm{SL}(H)\backslash\backslash\mathbb{P}(H_4 \oplus H_6)^{\mathrm{ss}}.$$

Here the double backslash indicates that we are forming a categorical orbit space. In this case, its closed points are in bijective correspondence with the closed $\mathrm{SL}(H)$-orbits in $\mathbb{P}(H_4 \oplus H_6)^{\mathrm{ss}}$. We shall refer to $\mathcal{M}^M$ as the *Miranda compactification* of $\mathcal{M}$. The geometric counterpart of the graded algebra of invariants $\mathbb{C}[H_4 \oplus H_6]^{\mathrm{SL}(H)}$ is an orbifold line bundle $\mathcal{L}_{\mathcal{M}^M}$ over $\mathcal{M}^M$ such that $\mathbb{C}[H_4 \oplus H_6]^{\mathrm{SL}(H)}$ is the graded algebra of sections of its tensor powers with twice the degree. For instance, $H^0(\mathcal{M}^M, \mathcal{L}_{\mathcal{M}^M}^{\otimes 2}) = H_4^* \otimes 1$ and $H^0(\mathcal{M}^M, \mathcal{L}_{\mathcal{M}^M}^{\otimes 3}) = 1 \otimes H_6^*$.

The minimal strictly semistable orbits in $H_4 \oplus H_6$ are represented by the pairs of the form $(\lambda f^2, \mu f^3)$ with f a product of two distinct linear forms and λ, μ constants that are not both zero. In that case the modular function is constant equal to $[\lambda^3 : \lambda^3 + \mu^2] \in \mathbb{P}^1$ and is a complete invariant of the orbit.

A stable orbit can be given more of a geometric content by associating to a stable pair $(f_0, f_1) \in H_4 \oplus H_6$ the divisor triple $(D_0, D_1, D_\infty) \in \tilde{\Sigma}$ of $(f_0, f_1, f_0^3 + f_1^2)$: this triple determines the pair (f_0, f_1) up to the action of the central subgroup $\mathbb{G}_m \subset \mathrm{GL}(H)$. We thus have defined an invariant open subset $\tilde{\Sigma}^{\mathrm{st}}$ of $\tilde{\Sigma}$ characterized by the condition that $3D_0$ and $2D_1$ have no point in common of multiplicity ≥ 6.

Proposition 2.5 (Miranda [**24**]). *A stable orbit defines a rational elliptic surface all of whose fibers are reduced that is, are of Kodaira type I_k (k-gon), II (cuspidal curve), III (two rational curves with a common tangent), or IV (three confluent smooth rational curves). Conversely, any such rational elliptic surface determines a stable orbit.*

A semistable orbit defines a rational elliptic surface such that the irreducible components of its fibers have multiplicity ≤ 2, that is, in addition to the fibers above, we also allow those of type I_k^*. Conversely, such a rational elliptic surface determines a semistable orbit in $\tilde{\Sigma}$. The minimal strictly semistable orbits correspond to rational elliptic surfaces with a I_4^*-fiber (such a surface is unique) or with two distinct I_0^*-fibers (such a surface has constant modular function—see below—and this constant is a complete invariant of the surface).

An elliptic surface with two I_0^* fibers is always of the following form: start out with a smooth elliptic curve E and consider the involution in $E \times \mathbb{P}^1$ defined by $(p, [z:1]) \mapsto (-p, [-z:1])$. This involution has 8 fixed points that give ordinary doubly points on the quotient surface. A single blowup resolves these and the resulting smooth surface X is rational and fibers over the rational curve that is the quotient of $\mathbb{P}^1$ by the involution $[z:1]) \mapsto [-z:1])$. So to a strictly semistable orbit of this type is associated a J-invariant.

2.3. Discriminant compactification. We think of $P_{12} = \mathrm{Sym}^{12} P$ as the projective space of effective divisors of degree 12 on H. Write $\mathcal{D} := \mathrm{SL}(H)\backslash P_{12}^r$ for the moduli space of reduced effective degreee 12 divisors on P. Let us recall that a $\mathrm{SL}(H)$-orbit in P_{12} is stable (resp. semistable) if and only if it has no point of multiplicity ≥ 6 (resp. 7). The minimal strictly semistable elements are of the form $6(a) + 6(b)$ with a and b distinct, hence lie in a single $\mathrm{SL}(H)$-orbit. Let us write $\mathcal{D}^{\mathrm{st}}$ for the ordinary orbit space $\mathrm{SL}(H)\backslash P_{12}^{\mathrm{st}}$ and put

$$\mathcal{D}^* := \mathrm{SL}(H)\backslash\backslash P_{12}^{\mathrm{ss}} = \mathrm{Proj}(\mathbb{C}[H_{12}]^{SL(H)}).$$

So $\mathcal{D}^*$ is a projective one point compactification of $\mathcal{D}^{\mathrm{st}}$; the added singleton will be denote d_∞. The hypersurface $\mathcal{D}^* - \mathcal{D}$ in $\mathcal{D}^*$ parametrizes the nonreduced divisors and is classically called the *discriminant*. There is an orbifold line bundle $\mathcal{L}_{\mathcal{D}^*}$ on $\mathcal{D}^*$ such that the degree n part of $\mathbb{C}[H_{12}]^{SL(H)}$ is the space of sections of its nth tensor power. The discriminant is given by the equation $\prod_{1\leq i<j\leq 12}(z_i - z_j)^2$ and hence the divisor of a section of $\mathcal{L}_{\mathcal{D}^*}^{11\cdot 12}$.

Consider the open part $\mathcal{M}' \subset \mathcal{M}^M$ that parametrizes rational elliptic surfaces whose discriminant divisor has no point of multiplicity ≥ 6. This means that we discard the surfaces with a nonreduced fiber or a fiber of type I_6 or worse. So $\mathcal{M}^M - \mathcal{M}'$ is of dimension ≤ 3 and hence everywhere of codimension ≥ 5 in $\mathcal{M}^M$. There is an obvious *discriminant morphism* $F : \mathcal{M}' \to \mathcal{D}^*$. Assigning to $(f_0, f_1) \in H_4 \oplus H_6$ the *discriminant form* $f_0^3 + f_1^2$ defines an isomorphism

$$F^*\mathcal{L}_{\mathcal{D}^*} \cong \mathcal{L}_{\mathcal{M}^M}^{\otimes 6}|_{\mathcal{M}'}.$$

Hence we find:

Corollary 2.6. *The algebra of sections $\oplus_{k\in\mathbb{Z}} H^0(\mathcal{M}', F^*\mathcal{L}_{\mathcal{D}^*}^{\otimes k})$ is zero in negative degrees and of finite type. Its proj defines the projective compactification $\mathcal{M}' \subset \mathcal{M}^M$.*

By Proposition 2.1, $\mathcal{M}$ embeds in $\mathcal{D}$ as a closed hypersurface. We denote the normalization of $\mathcal{M}$ in $\mathcal{D}^*$ by $\mathcal{M}^*$ and in $\mathcal{M}^M \times \mathcal{D}^*$ (via the diagonal embedding) by $\mathcal{M}^{M*}$. The projection $\mathcal{M}^{M*} \to \mathcal{M}^M$ will be special over the singleton corresponding to the case where $3D_0 = 2D_1$ (in other words, $(D_0, D_1) = (2(a) + 2(b), 3(a) + 3(b))$ with $a, b \in P$ distinct) and over the locus where the linear span of $3D_0, 2D_1$ has a member with a point of multiplicity ≥ 7. A major goal of this paper is to describe the diagram

$$\mathcal{M}^M \leftarrow \mathcal{M}^{M*} \rightarrow \mathcal{M}^*$$

in terms of complex hyperbolic geometry. In particular, we will show that $\mathcal{M}^*$ is naturally the Baily-Borel compactification of a ball quotient such that $\mathcal{M}^* - \mathcal{M}$ is the closure of a union of locally symmetric divisors. This requires a better geometric understanding of the above diagram and that is the topic of the next section.

3. A geometrically meaningful compactification

We found two compactifications of $\mathcal{M}$ obtained from Geometric Invariant Theory: one ($\mathcal{M}^M$) based on the Weierstrass description of a rational elliptic surface, the other ($\mathcal{M}^*$) based on the fact that a generic elliptic surface is defined by its discriminant. It is our goal to define a rather explicit compactification of $\mathcal{M}$ which dominates both. We also want it to be geometrically meaningful in the sense that the newly added points define degenerate elliptic surfaces of some sort. Together these desiderata imply that the modular function of these elliptic surfaces must always be of degree 12. Since there exist rational elliptic surfaces whose modular function has lower degree, there is a price to pay: we must allow the base to have ordinary double points.

3.1. Kodaira's theorem. We begin with restating a fundumental result of Kodaira in more geometric form. If P is a smooth complete curve, then a nonconstant morphism $J : P \to \mathbb{P}^1$ defines over $P - J^{-1}\{0, 1, \infty\}$ a fibration by elliptic curves given up to involution. Associated to such a 'Kummer fibration' is a μ_6-covering of P which will play a central role in this paper. It is defined as follows. We recall that the abelianization of $\mathrm{PSL}(2, \mathbb{Z})$ is the cyclic group of order 6 with $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ mapping to a generator. We denote that group by C_6 and its generator by τ. So the $\mathrm{PSL}(2, \mathbb{Z})$ principal bundle over $P - J^{-1}\{0, 1, \infty\}$ defined by J determines an unramified C_6-covering of $P - J^{-1}\{0, 1, \infty\}$. We extend that covering to a possibly ramified one over P, $C \to P$, by normalizing over P. In the case of the universal example—J is then the identity—this corresponds to the modular covering $E_o \to \mathbb{P}^1$ defined by the commutator subgroup of $\mathrm{PSL}(2, \mathbb{Z})$. The curve E_o is of genus one and has only one cusp (in other words, it is totally ramified over ∞). If we choose that cusp to be the origin, E_o becomes an elliptic curve and the fact that it comes with a faithful action of μ_6 implies that E_o has J-invariant 0. In the general case, $C \to P$ is simply the normalized pull-back of $E_o \to \mathbb{P}^1$. Here is the list of Kodaira fibers expressed in terms of the behavior of J at p:

$J(p)$	$\deg_p J$	type
∞	$k(\geq 1)$	I_k or I_k^*
0	$0 \pmod 3$	I_0 or I_0^*
	$1 \pmod 3$	II or IV^*
	$2 \pmod 3$	II^* or IV
1	$0 \pmod 2$	I_0 or I_0^*
	$1 \pmod 2$	III or III^*
$\notin \{0,1,\infty\}$		I_0 or I_0^*

The abelianization of $\mathrm{SL}(2,\mathbb{Z})$ is cyclic of order 12 with $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ mapping to a generator. We denote group and generator by C_{12} and η. So the nontrivial element -1 of the kernel of $\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z})$ maps to η^6. A relatively minimal elliptic fibration $X \to P$ with J as modular function determines a C_{12}-covering $\tilde{C} \to P$ which factorizes over $C \to P$. Thus we associated to every Kodaira fiber an integer modulo 12, which together with the local behaviour of J at the corresponding base point determines that fiber. Kodaira's basic result says that the lift of the C_6-covering to a C_{12}-covering determines $X \to P$ up to P-isomorphism and that any such lift so arises. This residue class is in fact the reduction modulo 12 of the Euler characteristic of the fiber. So the Euler characteristics of the fibers define a further lift to the integers. (This implies that the Euler characteristic of X is always divisible by 12.) For a fiber with finite J-value, its Euler characteristic is the unique representative of $\mathbb{Z}/(12)$ in $\{0,1,\dots,11\}$ (though 1, 5, 7 and 11 will not occur), whereas for a fiber X_p with $J(p)=\infty$ it is $\deg_p J$ (type $I_{\deg_p J}$) or $\deg_p J + 6$ (type $I^*_{\deg_p J}$).

A cyclic covering over a smooth rational curve is already given by the orders of the stabilizers. So if P is rational, then an elliptic fibration associated to J is already specified by a lift of the map $P \to \mathbb{Z}/(6)$ defined by J (whose support will be in $J^{-1}\{0,1,\infty\}$) to a finitely supported map with values in $\mathbb{Z}/(12)$. The above receipe defines a lift to the nonnegative integers and the 'integral' of the latter is the Euler characteristic of the total space. The total space is rational precisely when the sum of its fiber Euler characteristics is equal to 12. This describes a procedure to obtain all rational elliptic fibrations and it is the one employed by Miranda in [**25**] to recover Persson's classification [**29**] of rational elliptic fibrations up to homeomorphism.

3.2. Kontsevich compactification. Let be given a pair $(J : P \to \mathbb{P}^1, D)$, where

(a) P is a complete connected normal crossing curve of arithmetic genus zero,
(b) $J : P \to \mathbb{P}^1$ a morphism of degree 12,
(c) D is a 12-element subset of the regular part of P contained in $J^{-1}(\infty)$.

For later purposes it will be useful to observe that there then exists a μ_6-covering $C \to P$ such that

(i) C is a connected normal crossing curve,

(ii) $C \to P$ is unramified over $P_{\text{reg}} - D$ and
(iii) $C \to P$ is totally ramified over D and the action of μ_6 in the tangent space of such a ramification point is the tautological one (i.e., given by scalar multiplication).

and that this covering is unique up to isomorphism. (The arithmetic genus of C is easily calculated to be 25.) So to give the pair $(J : P \to \mathbb{P}^1, D)$ is equivalent to giving a complete normal crossing curve C with C_6-action as above and a morphism $C \to \mathbb{P}^1$ constant on orbits of degree $6 \cdot 12$. The cover $C \to P$ need not be the pull-back of the modular elliptic curve $E_o \to \mathbb{P}^1$ for there may be irreducible components of P in a fiber of J (on which $C \to P$ is necessarily nontrivial). But if we contract all such components then this is true. In other words, J is covered by a C_6-equivariant morphism $\tilde{J} : C \to E_o$.

We say that $(J : P \to \mathbb{P}^1, D)$ is *Kontsevich stable* if the group of its automorphisms that induce the identity of $\mathbb{P}^1$ is finite. In other words, we require that every connected component of $P_{\text{reg}} - D$ on which J is constant has negative Euler characteristic. There is an obvious extension of this notion to families of such pairs which leads to a well-defined moduli problem. Following Kontsevich ([**18**] 1.3.2) such pairs have a moduli stack that is complete, smooth. He also shows that the locus parametrizing pairs $(J : P \to \mathbb{P}^1, D)$ with P singular defines a normal crossing divisor. His argument shows at the same time that the singular points of P are *fully smoothable* in the sense that they are independently smoothable, already at first order. The underlying variety can be regarded as a coarse moduli space of pairs $(C, C \to \mathbb{P}^1)$ obtained as above: here C is a complete connected normal crossing curve of arithmetic genus 25 endowed with C_6-action having in C_{reg} exactly 12 fixed points, each with tangent character χ such that the morphism $C \to \mathbb{P}^1$ is constant on orbits and has degree $6 \cdot 12$, and the group of $\mathbb{P}^1$-automorphisms of C is finite. But the corresponding stack is slightly different.

Remark 3.1. If $(J : P \to \mathbb{P}^1, D)$ is a Kontsevich stable pair, then (P, D) need not be (Deligne-Knudsen-Mumford) stable as a 12-punctured curve, but successive contraction of its unstable components yields such a curve $(\bar{P}, \bar{D})$ and this curve is unique. There results a morphism from the Kontsevich moduli space to the Knudsen-Deligne-Mumford space $\mathcal{S}_{12}\backslash\overline{\mathcal{M}}_{0,12}$ of stable 12-punctured rational curves.

We embed $\mathcal{M}$ in this moduli space by assigning to a generic rational elliptic fibration $X \to P$ the pair consisting of its modular function $J : P \to \mathbb{P}^1$ and the fiber $J^{-1}(\infty)$. The normalization of $\mathcal{M}$ in this moduli space will be called the *Kontsevich compactification* and denoted by $\mathcal{M}^K$.

If $(J : P \to \mathbb{P}^1, D)$ represents a closed point of $\mathcal{M}^K$, then clearly D will be contained in $J^{-1}(\infty)$. Specifically, a connected component of $J^{-1}(\infty)$ contains as many points of D as the degree of J on a deleted neighborhood of that component in P. Moreover, every connected component of $J^{-1}(0)$ resp. $J^{-1}(1)$ has a basis of deleted neighborhoods in P on which J has degree divisible by 3 resp. 2. The interest of this construction is that such a J is still the modular function of an elliptic fibration defined over the union of the connected components of $P_{\mathrm{reg}} - D$ on which J is nonconstant: if P' is an irreducible component of P on which J is nonconstant, then J determines an elliptic fibration up to canonical involution. If p is a smooth point of P, then the fiber over p will be smooth or of type I_1, depending on whether $J(p)$ is finite. In case p is singular, then we have a singular Kodaira fiber not of type I_1. So if P' has exactly one singular point p, then the Euler characteristic of the fiber over p is determined by the fact that the Euler characteristics of the singular fibers sum up to 12. This gives also the answer in the general case since we can smooth all the singular points of P different from p and do the calculation for this new situation. We thus conclude that the Euler characteristic of the fiber over p in P' must be equal to 12 minus the degree of J on the connected component of $P - \{p\}$ containing $P' - \{p\}$ plus the multiplicity of p in $(J|P')^*(\infty)$. But beware that in general a singular fiber over a crossing point will depend on the choice of a branch through it. For instance, if P has two connected components P_2, P_{10} of degree 2 and 10 meeting in a point p with $J(p) = 0$, then the fiber over p in P_2 is of type II^* whereas the fiber over p in P_{10} is of type II. This issue is addressed and resolved by Abramovich and Vistoli in [**1**] by consistently working in a setting of Deligne-Mumford stacks. We shall not go into this here as it is not needed for what follows. We content ourselves with observing that $\mathcal{M}^K$ comes as a stack with a universal morphisms $\mathcal{P}^K \to \mathcal{M}^K \times \mathbb{P}^1$ of degree 12 such that the part of $\mathcal{P}^K$ where this morphism is smooth supports an elliptic fibration for which J is the modular function. Moreover, $\mathcal{P}^K$ comes (as a stack) with a C_6-covering $\mathcal{C}^K \to \mathcal{P}^K$.

Proposition 3.2. *The identity map of $\mathcal{M}$ extends to a morphism from the Kontsevich compactification $\mathcal{M}^K$ to the Miranda compactification $\mathcal{M}^M$. Precisely, if $J : P \to \mathbb{P}^1$ represents a closed point of $\mathcal{M}^K$ and*

(i) *if the fibration has a component P' of P on which J has degree > 6, then we assign to J the fibration over this component (since a nonreduced Kodaira fiber takes off at least 6 from the degree of modular function, this fibration will have only reduced Kodaira fibers);*

(ii) *if P has a singular point p with finite J-value such that each component of $P - \{p\}$ has degree 6 over $\mathbb{P}^1$, then we assign to J the elliptic fibration with constant modular function $J(p)$ and with two fibers of type I_0^* and*

(iii) *if P has a singular point p over ∞, such that each component of $P - \{p\}$ has degree 6 over $\mathbb{P}^1$, then we assign to J the point $m_\infty \in \mathcal{M}^M$ (the unique point representing a rational elliptic surface with a I_4^*-fiber).*

Proof. We begin with proving the first part of (iii). Suppose that P has a singular point p over ∞. Denote the closures of the connected components of $P - \{p\}$ by P_1 and P_2. Then on (P_i, p) we have a Kodaira fiber of type $I_{k_i}^*$ for some $k_i \geq 1$. The Euler characteristic of such a fiber is $6 + k_i$ and hence the degree of J on P_i is 6.

To see that the birational map from $\mathcal{M}^K$ to $\mathcal{M}^M$ is in fact a morphism, we consider the closure $\overline{\mathcal{M}}$ of the diagonal embedding of $\mathcal{M}$ in $\mathcal{M}^K \times \mathcal{M}^M$. Since $\mathcal{M}^K$ is normal it suffices to prove that the projection $\overline{\mathcal{M}} \to \mathcal{M}^K$ is a bijection. Or equivalently, that any curve germ in $\mathcal{M}^M$ is the image of one in $\mathcal{M}^K$. Moreover, we want this lift to be as prescribed by the proposition. This can be checked in a straightforward manner. □

REMARK 3.3. It can be shown that the natural morphism $\mathcal{M}^K \to \mathcal{S}_{12}\backslash\overline{\mathcal{M}}_{0,12}$ is finite. This implies that is also possible to define $\mathcal{M}^K$ as the normalization of $\mathcal{M}$ in $\mathcal{S}_{12}\backslash\overline{\mathcal{M}}_{0,12}$. Though this avoids appeal to the Kontsevich moduli space, we shall need the more powerful interpretation that comes with the latter.

The identity also extends as a morphism $\mathcal{M}^K \to \mathcal{M}^*$ as follows. Let be given an allowable pair $(J : P \to \mathbb{P}^1, D)$ representing a closed point of $\mathcal{M}^K$. If there exists an irreducible component P_c of P such that the direct image of D under the natural retraction $P \to P_c$ is a stable divisor (all multiplicities < 6), then this irreducible component is unique—we shall call it the *central component* of (P, D)—and we assign to $(J : P \to \mathbb{P}^1, D)$ the corresponding point of $\mathcal{D}^*$. If no such component exists, then there is a unique singular point p_c—the *central point* of (P, D)—such that D has 6 points in each connected component of $P - \{p_c\}$, and we then assign to $(J : P \to \mathbb{P}^1, D)$ the point of $\mathcal{D}^*$ that corresponds to the unique minimal semistable orbit (the orbit of divisors that have two distinct points, each with multiplicity 6). In either case we allow ourselves a mild abuse of language by referring to this point of $\mathcal{D}^*$ as the *discriminant* of $(J : P \to \mathbb{P}^1, D)$. It is not difficult to verify that this defines a morphism $\mathcal{M}^K \to \mathcal{D}^*$. Since $\mathcal{M}^K$ is normal this morphism will factorize over $\mathcal{M}^*$.

So $\mathcal{M}^K$ dominates $\mathcal{M}^{M*}$. Understanding of $\mathcal{M}^K$ will help us in understanding $\mathcal{M}^{M*}$.

3.3. A partial list of strata. Let us describe the generic points of $\mathcal{M}^K - \mathcal{M}$ (these turn out to be all hypersurfaces). If $X \to P$ is a generic rational elliptic fibration (so with smooth base P and reduced discriminant), then the modular function $J : P \to \mathbb{P}^1$ is a degree 12 covering with the property that the local degree of J at a point over 0 resp. 1 is always equal to 3 resp. 2. Following Riemann-Hurwitz, the discriminant of J must then have the form $8(0) + 6(1) + R$, with R of degree 8. This divisor gives us the 8 moduli parameters. Degeneracies will occur when $\mathrm{supp}(R)$ meets 0, 1 or ∞. The computation of (co)dimension is based on the full smoothability property.

In the list below we make use of a small part of Persson's classification [**29**]. For instance, we use the fact that the rational elliptic fibrations with a fiber of Kodaira type I_k ($8 \neq k \leq 9$), II, III, IV, I_k^* ($k \leq 4$) respectively are parametrized by an irreducible variety. We excluded the I_8-case since there are two types of fibrations with an I_8 fiber: in one case (I_8') the classes of the irreducible components in the Picard group generate a primitive sublattice and in the other case (I_8'') the sublattice is of index two in a primitive sublattice and either case is parametrized by an irreducible variety. The two cases can be distinguished by the fact that in case (I_8') the fiber can degenerate in a I_9-fiber, whereas this is not possible for the (I_8'') case. But either can degenerate into a I_4^*-fiber and is a degeneration of a I_7-fiber.

$(I_{k\geq 2})$ Then P has an irreducible component P_{12} of degree 12 over $\mathbb{P}^1$ and there is a $z \in P_{12}$ where $J|P_{12}$ has local degree k. We have an extra component P_0 in $J^{-1}(\infty)$ which meets P_{12} in the ramification point. This component contains k points of D and so it is central if and only if $k \geq 7$. Hence the discriminant has a point of multiplicity $\min\{k, 12-k\}$. The image of this hypersurface of $\mathcal{M}^K$ in $\mathcal{M}^M$ is of dimension $9-k$, whereas its image in $\mathcal{M}^*$ is of dimension $9-k$ for $k = 2, 3, 4, 5$, of dimension 0 for $k = 6$, and of dimension $k-2$ for $k = 7, 8, 9$. The hypersurface in question is irreducible unless $k = 8$, in which case there are two irreducible components.

(II) Then P has two irreducible components P_{10}, P_2 of degrees resp. 10 and 2 over $\mathbb{P}^1$ meeting in a point p with J-value 0. The component P_2 ramifies simply over 0 and 1; the component P_{10} has a fiber over 0 resp. 1 of type $(3^3, 1)$ resp. (2^5). Over (P_{10}, p) we have a fiber of type II (a cuspidal fiber) and over (P_2, p) a fiber of type II^* (an $\hat{E}_8$-fiber). The central component is P_{10} and the discriminant has a point of multiplicity 2. The images of this subvariety in $\mathcal{M}^*$ and $\mathcal{M}^M$ are hypersurfaces.

(III) This case and the next are similar to the preceding case. Here P has two irreducible components P_9, P_3 of degrees 9 resp. 3 over $\mathbb{P}^1$ meeting in a point p with with J-value 1. The component P_3 ramifies totally over 0 and has a point of simple ramification over 1; the component P_9 has fiber over 0 resp. 1 of type (3^3) resp. $(2^4, 1)$. Over (P_9, p) we have a Kodaira fiber of type III, and over (P_3, p) one of type III^* (an $\hat{E}_7$-fiber). The central component is P_9 and the discriminant has a point of multiplicity 3. The images of this subvariety in $\mathcal{M}^*$ and $\mathcal{M}^M$ are of codimension two (since we forget P_3).

(IV) Now P has two irreducible components P_8, P_4 of degrees 8 resp. 4 over $\mathbb{P}^1$ meeting in a point p with J-value 0. The component P_4 has fiber over 0 resp. 1 of type $(3, 1)$ resp. (2^2), whereas for P_8 these data are $(3^2, 2)$ resp. (2^4). They meet in their points of smallest ramification. Over (P_8, p) we have a fiber of type IV and over (P_4, p) a fiber of type IV^* (an $\hat{E}_6$-fiber). The central component is P_8 and the discriminant has a point of multiplicity 4. The images of this subvariety in $\mathcal{M}^*$ and $\mathcal{M}^M$ are of codimension three.

The following cases involve Kodaira fibers of type I_k^*. In all these cases, P has two irreducible components P_6, P_6' that are both of degree 6 over $\mathbb{P}^1$.

(I_0^*) P_6 and P_6' meet in a point p with $J(p)$ finite. Over (P_6, p) and (P_6', p) we have fibers of type I_0^*. The point p is central and so the discriminant is the orbit of the divisor with two points of multiplicity 6. The image of this subvariety in $\mathcal{M}^M$ is a curve and its image in $\mathcal{M}^*$ is a singleton.

($I^*_{k,k'}$) Here P_6 and P'_6 are separated by a central component P_c contained in $J^{-1}(\infty)$. If P_c meets P_6 in p, then we have a Kodaira fiber of type I^*_k at (P,p), where $k = \deg_p(J|P) \in \{1,2,3,4\}$. Similarly we find a Kodaira fiber of type $I^*_{k'}$ for P'_6. So P_c meets D in $k+k'$ points. Hence the discriminant has a point of multiplicity $6-k$ and one of multiplicity $6-k'$. This defines a hypersurface in $\mathcal{M}^K$, whose image in $\mathcal{M}^*$ has dimension $k+k'-1$ (so we get a hypersurface in $\mathcal{M}^*$ precisely when $k=k'=4$). Its image in $\mathcal{M}^M$ is a singleton.

So the boundary of $\mathcal{M}$ in $\mathcal{M}^K$ is a union of irreducible hypersurfaces $\mathcal{M}^K(F)$, where F runs over the Kodaira symbols I_k, $k=2,\dots,7,9$, I'_8, I''_8, II, III, IV, I^*_0, $I^*_{k,k'}$ with $k,k'=1,2,3,4$. Let us write $\mathcal{M}^{M*}(F)$ for the image of $\mathcal{M}^K(F)$ in $\mathcal{M}^{M*}$ and let $\mathcal{M}^M(F)$ and $\mathcal{M}^{M*}(F)$ have a similar meaning. The dimensions of these subvarieties are listed in the table below.

It is not hard to check that $\mathcal{M}^{M*}(I_2)$ contains $\mathcal{M}^{M*}(I_k)$ when $k\le 5$, that $\mathcal{M}^{M*}(II)\supset\mathcal{M}^{M*}(III)\supset\mathcal{M}^{M*}(IV)$ and that $\mathcal{M}^{M*}(I^*_{4,4})$ contains $\mathcal{M}^{M*}(I^*_{k,k'})$. From these and similar incidence relations we deduce:

(i) The irreducible components of the boundary of $\mathcal{M}$ in $\mathcal{M}^{M*}$ are the hypersurfaces $\mathcal{M}^{M*}(I_2)$, $\mathcal{M}^{M*}(II)$, $\mathcal{M}^{M*}(I_7)$, $\mathcal{M}^{M*}(I'_8)$, $\mathcal{M}^{M*}(I''_8)$, $\mathcal{M}^{M*}(I_9)$, $\mathcal{M}^{M*}(I^*_{4,4})$, the curve $\mathcal{M}^{M*}(I^*_0)$ and the threefold $\mathcal{M}^{M*}(I_6)$.

(ii) For $k=7,9$ we have $\mathcal{M}^{M*}(I_k)=\mathcal{M}^*(I_k)\times\mathcal{M}^M(I_k)$ and $\mathcal{M}^{M*}(I_8)^{(i)} = \mathcal{M}^*(I_8^{(i)})\times\mathcal{M}^M(I_8^{(i)})$ for $i=1,2$.

(iii) We have inclusions

$$\mathcal{M}^*(I_9)\supset\mathcal{M}^*(I'_8)\supset\mathcal{M}^*(I_7),$$
$$\mathcal{M}^M(I_9)\subset\mathcal{M}^M(I'_8)\subset\mathcal{M}^M(I_7),$$
$$\mathcal{M}^*(I^*_{4,4})\supset\mathcal{M}^*(I''_8)\supset\mathcal{M}^*(I_7),$$
$$\mathcal{M}^M(I^*_{4,4})\subset\mathcal{M}^M(I''_8)\supset\mathcal{M}^M(I_7).$$

(iv) The projection of $\mathcal{M}^{M*}(F)\to\mathcal{M}^*(F)$ is birational for $F=I_2$, I_9, II, $I^*_{4,4}$ and a collapse onto a point for $F=I_6$, I^*_0.

(v) The projection of $\mathcal{M}^{M*}(F)\to\mathcal{M}^M(F)$ is birational for $F=I_2$, II, I_6, I^*_0 and and a collapse onto a point for $F=I_9$, $I^*_{4,4}$.

The following statements then follow in a straightforward manner:

Corollary 3.4. *The boundary of $\mathcal{M}$ in $\mathcal{M}^*$ is the union of the irreducible hypersurfaces $\mathcal{M}^*(I_2)$, $\mathcal{M}^*(II)$, $\mathcal{M}^*(I_9)$ and $\mathcal{M}^*(I^*_{4,4})$. Moreover,*

(i) $\mathcal{M}^*(I_9)\cap\mathcal{M}^*(I^*_{4,4})=\mathcal{M}^*(I'_8)$,

(ii) $\mathcal{M}^*(I''_8)\subset\mathcal{M}^*(I^*_{4,4})$,

(iii) $\mathcal{M}^*(I'_8)\cap\mathcal{M}^*(I''_8)=\mathcal{M}^*(I_7)$,

(iv) *$\mathcal{M}^*(I_6)$ is a singleton contained in $\mathcal{M}^*(I_7)$ and $\mathcal{M}^*(I_0^*)$ is a singleton contained in $\mathcal{M}^*(I_{4,4}^*)$. These two make up the preimage of $d_\infty \in \mathcal{D}^*$ in $\mathcal{M}^*$.*

F	$\dim \mathcal{M}^*(F)$	$\dim \mathcal{M}^M(F)$	$\dim \mathcal{M}^{M*}(F)$
I_2	7	7	7
$I_{k\leq 5}$	$9-k$	$9-k$	$9-k$
I_6	0	3	3
I_7	5	2	7
I_8', I_8''	6	1	7
I_9	7	0	7
II	7	7	7
III	6	6	6
IV	5	5	5
I_0^*	0	1	1
$I_{k,k'}^*$	$k+k'-1$	0	$k+k'-1$

4. Homology of a cyclic covering

4.1. Symplectic lattices with symmetries. Let be given a finite abelian group G that acts (morphically) on a symplectic lattice L. We then extend the symplectic form as a sesquilinear form over $\mathbb{Z}[G]$ by

$$\langle\ ,\ \rangle : L \times L \to \mathbb{Z}[G], \quad (a,b) \mapsto \sum_{g\in G}(a\cdot gb)g = \sum_{g\in G}(g^{-1}a\cdot b)g.$$

Indeed, this form is $\mathbb{Z}[G]$-linear in the first argument and $\langle b,a\rangle = -\overline{\langle a,b\rangle}$ (where the overline is the involution which sends each element of G to its inverse). So if we multiply the form by any anti-invariant element of $\mathbb{Z}[G]$ (such as $g - g^{-1}$ for some $g \in G$), then we get a Hermitian form over $\mathbb{Z}[G]$.

We take G to be a cyclic group of order 6, C_6, with a given generator $\tau \in C_6$. Let $\chi : \mathbb{Z}[C_6] \to \mathbb{C}$ be the character that sends τ to $\omega := e^{2\pi\sqrt{-1}/6}$. The image of this character is the ring of integers $\mathbb{Z} + \mathbb{Z}\omega$. We call this ring the *Eisenstein ring* and denote it by $\mathcal{O}$. For the lattice L as above, $L_{\mathcal{O}} := \mathcal{O} \otimes_{\mathbb{Z}[C_6]} L/(\text{torsion})$ is the biggest torsion free quotient of L on which C_6 acts through $\mathcal{O}$. This quotient of L is realized as the image of L under the natural 'eigenprojection' $\mathbb{C} \times_{\mathbb{Z}} L \to (\mathbb{C} \times_{\mathbb{Z}} L)_\chi$. The composition of the sesquilinear form above with χ factorizes over a skew-hermitian ($\mathcal{O}$-valued) form:

$$\phi : L_{\mathcal{O}} \times L_{\mathcal{O}} \to \mathcal{O}.$$

We make this a Hermitian form by multiplying with a square root of -3: we put

$$\theta := \omega - \omega^{-1},$$

and let our Hermitian form be

$$\psi(a,b) := -\theta\phi(a,b).$$

As we will show in the Appendix, such Hermitian lattices can also be gotten from quadratic forms with C_6-symmetry.

EXAMPLE 4.1. Let E_o be the elliptic curve of J-invariant 0. It admits a faithful action of C_6 with τ acting on the tangent space at the origin as multiplication by ω. Note that $H^1(E_o)$ is a free $\mathcal{O}$-module of rank one. The generators make up a C_6-orbit and if c is any one of them, then $\phi(c,c) = \sum_{i=0}^5 (c,(\tau^*)^{-i}c)\omega^i = 2\theta$ and so $\psi(c,c) = -\theta.2\theta = 6$. For later reference we also note that $(\tau^*)^{-1}$ acts on $H^{1,0}(E_o)$ as multiplication by ω^{-1}.

EXAMPLE 4.2. Here is another example. Take $L := \mathbb{Z}[C_6]/(\sum_{i=0}^5 \tau^i)$ (which, as a $\mathbb{Z}[C_6]$-module, is isomorphic to the augmentation ideal of $\mathbb{Z}[C_6]$). We equip it with the symplectic form

$$\tau^i \cdot \tau^j = \begin{cases} \pm 1 & \text{if } j = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\langle 1,1\rangle = \tau - \tau^{-1}$ and so for the image e of 1 in $L_{\mathcal{O}}$ we have $\psi(e,e) = -\theta.\theta = 3$.

4.2. Cyclic covers. Let $\pi : C_o \to \mathbb{P}^1$ be the smooth C_6-covering of the projective line that has total ramification over the 12th roots of unity in the unit circle and with the generator τ of C_6 acting as multiplication by $e^{2\pi\sqrt{-1}/6}$ on the tangent space of the ramification points. An affine equation for this curve is $w^6 + z^{12} = 1$ with τ acting as $\tau(z,w) = (z,\omega w)$ and $\pi(z,w) = z$. There is also C_{12}-symmetry, with a generator η of C_{12} acting as $\eta(z,w) = (e^{2\pi\sqrt{-1}/12}z, w)$. So we have an action $C_6 \times C_{12}$ on C_o. Our first goal is to describe $H_1(C_o)$ as a module over

$$R := \mathbb{Z}[C_6 \times C_{12}] = \mathbb{Z}[\tau,\eta]/(\tau^6 - 1, \eta^{12} - 1).$$

We make use of F. Pham's description [**30**] of the homology (with its intersection form) of the affine piece $C_o' := C_o - \pi^{-1}(\infty)$. Consider the real part of C_o' defined by $x^{12} + u^6 = 1$ with x and u in the unit interval. We orient it as going from $(0,1)$ to $(1,0)$ and denote the singular 1-simplex thus defined by $\mathbf{e}$. Since $\mathbf{e}$ is not fixed by any non-trivial element of $C_6 \times C_{12}$, $\mathbf{e}$ generates a free R-submodule of the module of singular 1-chains on C_o'. Pham observes that

$$e := (1-\tau)(1-\eta)\mathbf{e}$$

is a 1-cycle with the property that it generates $H_1(C_o)$ as an R-module. Since $R\mathbf{e}$ does not contain nonzero boundaries, $H_1(C_o')$ gets identified (as an R-module) with the ideal $(1-\tau)(1-\eta)R$. The annihilator of $(1-\tau)(1-\eta)$ in R is the ideal $(\sum_{i=0}^5 \tau^i, \sum_{i=0}^{11} \eta^i)R$ and so the dual module $H^1(C_o')$ appears naturally as a quotient:

$$\mathbb{Z}[\tau,\eta]/(\sum_{i=0}^{5}\tau^i, \sum_{i=0}^{11}\eta^i) \cong H^1(C_o'), \quad 1 \mapsto e^*.$$

Pham also describes the intersection pairing: the adjoint homomorphism $H_1(C_o) \to H^1(C_o)$ is the antihomomorphism of R-modules given by

$$e = (1-\tau)(1-\eta)\mathbf{e} \mapsto -(1-\tau)(1-\eta)(1-\tau\eta)e^*.$$

Notice that the kernel of this map is $(1-\tau)(1-\eta)(\sum_{i=0}^{11}(\tau\eta)^i)R\mathbf{e}$. The inclusion $C_o' \subset C_o$ induces a surjection on H_1; in fact, $H_1(C_o)$ can be identified with the image of $H_1(C_o) \to H^1(C_o)$ (compatibly with the intersection pairing). So we find an isomorphism

$$(1-\tau)(1-\eta)R/(1-\tau)(1-\eta)(\sum_{i=0}^{11}(\tau\eta)^i) \cong H_1(C_o), \quad (1-\tau)(1-\eta) \mapsto e.$$

We will identify the lefthand side with the quotient ring

$$A := \mathbb{Z}[\tau,\eta]/(\sum_{i=0}^{5}\tau^i, \sum_{i=0}^{11}\eta^i, \sum_{i=0}^{11}(\tau\eta)^i),$$

so that 1 corresponds to e. (So as a $\mathbb{Z}[C_6]$-module, A is generated by $\{\eta^i\}_{i=1}^{10}$.) The sesquilinear extension of the intersection pairing is given by

$$\langle a,b\rangle_R = (1-\tau)(1-\eta)a.\overline{b(\tau\eta-1)} \in R, \quad a,b\in A.$$

If we merely regard $H_1(C_o)$ as a $\mathbb{Z}[C_6]$-module, then the intersection form defines a sesquilinear pairing

$$\langle\ ,\ \rangle_{\mathbb{Z}[C_6]} : H_1(C_o)\times H_1(C_o) \to \mathbb{Z}[C_6]$$

that is $\mathbb{Z}[C_{12}]$-invariant. The two are of course related by

$$\langle a,b\rangle_R = \sum_{i=0}^{11}\langle ae, \eta^i be\rangle_{\mathbb{Z}[C_6]}\eta^i.$$

Reducing modulo the ideal generated by $\tau^2-\tau+1$ yields sesquilinear pairings

$$\langle\ ,\ \rangle_{\mathcal{O}} : H_1(C_o)_{\mathcal{O}}\times H_1(C_o)_{\mathcal{O}} \to \mathcal{O} \quad \text{and} \quad \langle\ ,\ \rangle_{\mathcal{O}[C_{12}]} : A_{\mathcal{O}}\times A_{\mathcal{O}} \to \mathcal{O}[C_{12}]$$

that are related in the same way. The associated Hermitian forms are defined by multiplying these by $-\theta = -\omega(1+\omega)$:

$$\psi(ae,be) := -\omega(1+\omega)\langle ae,be\rangle_{\mathcal{O}} \quad \text{and} \quad \Psi(a,b) := -\omega(1+\omega)\langle a,b\rangle_{\mathcal{O}[C_{12}]},$$

so that

$$\begin{aligned}\sum_{i=0}^{11}\psi(e,\eta^i e)\eta^i &= \Psi(1,1)\\ &= -\omega(1+\omega)(1-\omega)(1-\eta)(\omega^{-1}\eta^{-1}-1)\\ &= -(1+\omega)((-1-\omega^{-1})+\eta+\omega^{-1}\eta^{-1})\\ &= 3-(1+\omega)\eta-(1+\omega^{-1})\eta^{-1}.\end{aligned}$$

In other words,

$$\psi(e, \eta^i e) = \begin{cases} 3 & \text{if } i = 0, \\ -1-\omega & \text{if } i = 1, \\ -1-\omega^{-1} & \text{if } i = -1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Since ψ is C_{12}-invariant, these formulae completely describe ψ on the generators $\eta^i e$.

Let us denote by Λ the Hermitian $\mathcal{O}$-module underlying $A_{\mathcal{O}}$. So if $r_i \in \Lambda$ denotes the image of $\omega^{2i}\eta^i$, then $(r_1, \dots, r_{10})$ is a $\mathcal{O}$-basis of $A_{\mathcal{O}}$ on which ψ is given by

$$\psi(r_i, r_j) = \begin{cases} 3 & \text{if } j = i, \\ \theta & \text{if } j = i+1, \\ 0 & \text{if } j > i+1. \end{cases} \tag{2}$$

Notice that for $k \leq 10$, the annihilator of the span of $r_1, \dots, r_{k-1}$ contains the span of $r_{k+1}, \dots, r_{11}$. It is not hard to see that it is in fact equal to it.

Remark 4.3. The homology class of e can be represented more simply as follows. The closed sector of the (closed) unit disk in the z-line with $\arg(z)$ between 0 and $2\pi/12$ has a unique lift to C passing through $(0,1)$. If we give this lift its complex orientation, then it becomes a singular 2-simplex whose boundary of is of the form $\mathbf{e} + \epsilon - \eta\mathbf{e}$, where ϵ is a lift of the arc on the unit circle. So $(1-\eta)\mathbf{e}$ is homologous to $-\epsilon$. Hence $e = (1-\tau)(1-\eta)\mathbf{e}$ is homologous to $(\tau - 1)\epsilon$.

Remark 4.4. It is easy to check that the $\mathcal{O}$-sublattice of $A_{\mathcal{O}}$ spanned by η^i, $i = 0, \dots, k$ is of rank $\min\{k+1, 10\}$ and positive definite for $k \leq 3$, positive indefinite for $k = 4$, and hyperbolic for $k \geq 5$. Since multiplication by η is a lattice automorphism it follows that the $\mathcal{O}$-sublattice spanned by all η^i with $i \not\equiv 5 \pmod 6$ is a positive (indefinite) sublattice of rank at least 9. This is clearly also the maximal rank of a positive sublattice, so it is of the form $l_0^\perp$ for some 0-vector l_0. A small calculation shows that we can take $l_0 = (1 + (1+\omega)\eta + 2\omega\eta^2 + (2\omega - 1)\eta^3 + (\omega - 1)\eta^4)e$.

5. A central extension of a braid class group

5.1. Braid and braid class groups. This section reviews some facts concerning the braid groups of $\mathbb{C}^\times$ and $\mathbb{P}^1$. We adhere to the categorical convention for the composition law in fundamental groupoids: $\alpha\beta$ means that the path α comes after the path β.

We first establish the terminology. Fix a positive integer d. For any topological surface X we denote by $X(d)$ the configuration space of d-element subsets of X. The *braid group of X with d strands* $\mathrm{Br}_d(X)$ of X is by definition the fundamental group of $X(d)$. The latter requires a choice of base point and so strictly speaking this group is only defined up to conjugacy. The group $\mathrm{Homeo}(X)$ of self-homeomorphisms of X

acts on $X(d)$. The image of the the fundamental group of the identity component, $\pi_1(\mathrm{Homeo}(X)^0, 1)$ in the fundamental group of $X(d)$ is normal and we shall refer to the quotient group as the d-pointed *braid class group* of X, $\mathrm{BCl}_d(X)$. For $X = \mathbb{P}^1$ we will often omit X and simply write Br_d and BCl_d.

An alternative characterization of $\mathrm{BCl}_d(X)$ is as a mapping class group: if we fix a d-element subset S of X, then $\mathrm{BCl}_d(X)$ is the group of isotopy classes of self-homeomorphisms of the pair (X, S) that are trivial as an absolute isotopy class of self-homeomorphisms of X. This also gives BCl_d the interpretation as the orbifold fundamental group of the moduli space $\mathcal{S}_d\backslash\mathcal{M}_{0,d}$ of smooth rational curves with d punctures.

We first consider the case $X = \mathbb{C}^\times$. We take as a base point $*$ for $\mathbb{C}^\times(d)$ the set μ_d of dth roots of 1. We have two special elements R and T of $\mathrm{Br}_d(\mathbb{C}^\times)$: R is defined in $\mathrm{Br}_d(\mathbb{C}^\times)$ by $t \in [0,1] \mapsto e^{2\pi\sqrt{-1}t/d} \cdot \mu_d$, and T is represented by the loop that leaves all elements of μ_d in place except 1 and $e^{2\pi\sqrt{-1}/d}$: these traverse (in counterclockwise direction) half of the circle that has the segment $[1, e^{2\pi\sqrt{-1}/d}]$ as a diameter. These two elements generate $\mathrm{Br}_d(\mathbb{C}^\times)$, but in order to get a useful presentation of $\mathrm{Br}_d(\mathbb{C}^\times)$ it is better to enlarge the number of generators. Let $T_k := R^k T R^{-k}$ $(k \in \mathbb{Z}/d)$. Clearly, T_k relates to the pair $(e^{2\pi\sqrt{-1}k/d}, e^{2\pi\sqrt{-1}(k+1)/d})$ in the same way as T to $(1, e^{2\pi\sqrt{-1}/d})$. These elements satisfy:

$$\begin{aligned} T_k T_{k+1} T_k &= T_{k+1} T_k T_{k+1}, \quad k \in \mathbb{Z}/d, \\ T_k T_l &= T_l T_k, \quad k, l \in \mathbb{Z}/d \text{ and } k - l \neq \pm 1. \end{aligned} \tag{3}$$

Together with the obvious relations

$$R T_k R^{-1} = T_{k+1}, \quad k \in \mathbb{Z}/d, \tag{4}$$

these present $\mathrm{Br}_d(\mathbb{C}^\times)$ in terms of the generators $R, T_0, \dots, T_{d-1}$. It is clear that in the braid class group R^d comes from a loop in $\mathbb{C}^\times \subset \mathrm{Homeo}^0(\mathbb{C}^\times)$ (the image of R corresponds to multiplication by $e^{2\pi\sqrt{-1}/d}$). So it dies in $\mathrm{BCl}_d(\mathbb{C}^\times)$, and indeed, $\mathrm{BCl}_d(\mathbb{C}^\times)$ is gotten from $\mathrm{Br}_d(\mathbb{C}^\times)$ by imposing this extra relation.

The loop defined by R^d gives the nontrivial element of $\pi_1(\mathrm{PSL}(2,\mathbb{C})) \cong \mathbb{Z}/2$. So R^{2d} dies in Br_d. The reader may check that in Br_d we also have the relations

$$R \equiv T_1 T_2 \cdots T_{d-1}, \quad R^{-1} \equiv T_{d-1} T_{d-2} \cdots T_1. \tag{5}$$

One can verify that the relations (3) imply that $T_1 T_2 \cdots T_{d-1}$ and $T_{d-1} T_{d-2} \cdots T_1$ have the same dth power in $\mathrm{Br}_d(\mathbb{C}^\times)$. So the relations (5) already imply that R^{2d} maps to 1 in Br_d. Conjugating them with R shows that the images of $T_1 T_2 \cdots T_{d-1}$ and $T_{d-1} T_{d-2} \cdots T_1$ in Br_d are invariant under the cyclic permutation $(0, 1, \dots, d-1)$. (By suppressing R and adding the cyclic invariance we get a presentation of Br_d in terms of the T_i's. The cyclic invariance also allows us to eliminate another generator and this then leads to a presentation due to Fadell-Van Buskirk in [**15**].) Finally, the braid class group BCl_d is gotten by putting $R^d \equiv 1$.

5.2. Action of a centrally extended braid class group. We continue with the situation of Section 4. We use the presentation of the braid class group BCl_{12} with generators $R = \eta, T_0, \dots, T_{11}$ subject to the relations (3), (4), (5) (and $\eta^{12} = 1$). The loop defining $T = T_0$ can be represented by a homeomorphism of the pair $(\mathbb{P}^1, \mu_{12})$ with support in a neighborhood U of the arc from 1 to $e^{2\pi\sqrt{-1}/12}$, the loop defining R is represented by η. This homeomorphism lifts uniquely over $\pi : C_o \to \mathbb{P}^1$ to a homeomorphism with support in $\pi^{-1}U$. Let $\hat{T}$ denote its isotopy class in the group of homeomorphisms of C_o that commute with the C_6-action. (Perhaps we should remark that $\hat{T}$ is also the monodromy that we get from a Milnor fibration: if we let the two points of ramification $1, e^{2\pi\sqrt{-1}/12}$ coalesce along the segment that connects them, then the C_6-covers acquire a singularity with local equation $w^6 + \zeta^2$ (an A_5-singularity) and $\hat{T}$ is the monodromy of this degeneration.)

The action of $\hat{T}$ on $H_1(C_o)$ will be a $\mathbb{Z}[C_6]$-linear automorphism that preserves the intersection pairing. Hence $\hat{T}$ will also act on the $\mathcal{O}$-module $H_1(C_o)_{\mathcal{O}}$ and preserve the Hermitian form ψ defined in Section 4.1. Let us make these actions explicit in terms of Pham's basis. A suitable representative $\hat{T}$ (in the given isotopy class) will act on 1-chains on C_o with boundary supported by the π-preimage of 0 and the 12th roots of unity. Clearly, $\hat{T}$ will not affect the class of $\eta^i\mathbf{e}$ if $i \neq 0, 1 \pmod{12}$. It is also easily seen that $\hat{T}$ maps the class of $\eta\mathbf{e}$ to that of $\mathbf{e}$. On the other hand $\hat{T}\mathbf{e}$ will be represented by the path which first follows $\mathbf{e}$, stops just before 1, makes then a full counterclockwise loop around the ramification point over 1, then returns to a point over 0, and finally follows a lift over the segment $[0, e^{2\pi\sqrt{-1}/12}]$. From this description it follows that this path is as a 1-chain homologous to $(1 - \tau + \tau\eta)\mathbf{e}$.

Corollary 5.1. *The monodromy operator $\hat{T}$ acts on $H_1(C_o)$ as follows:*

$$\hat{T}(\eta^i e) - \eta^i e = \begin{cases} -(1+\tau)e & \text{if } i = 0, \\ e & \text{if } i = 1, \\ \tau e & \text{if } i = -1, \\ 0 & \text{otherwise.} \end{cases}$$

It is in particular of order 6. *Its action on $H_1(C_o)_{\mathcal{O}}$ is the given by the complex reflection*

$$\hat{T}_{\mathcal{O}}(x) = x - \tfrac{1}{3}(1+\omega)\psi(x,e)e = x + \omega^{-1}\theta^{-1}\psi(x,e)e$$

of order 3.

Proof. The first statement follows in a straightforward manner from the fact that $e = (1-\tau)(1-\eta)\mathbf{e}$, our computation of $\hat{T}(\eta^i\mathbf{e})$, and the $\mathbb{Z}[C_6]$-linearity of $\hat{T}$. The second follows from the first if we bear in mind the Formulae 1 for $\psi(\eta^i e, e) = \psi(e, \eta^{-i} e)$. ☐

Consider the mapping class group $\widehat{\mathrm{BCl}}_{12}$ of C_6-equivariant isotopy classes generated by $\hat{T}$ and $C_6 \times C_{12}$. So $\widehat{\mathrm{BCl}}_{12}$ is a central extension of BCl_{12} by C_6. Let

$\hat{T}_k := \eta^k \hat{T} \eta^{-k} \in \widehat{\mathrm{BCl}}_{12}$, $k \in \mathbb{Z}/12$. These elements also obey the braid relations

$$\begin{aligned} \hat{T}_k \hat{T}_{k+1} \hat{T}_k = \hat{T}_{k+1} \hat{T}_k \hat{T}_{k+1}, \quad k \in \mathbb{Z}/12, \\ \hat{T}_k \hat{T}_l = \hat{T}_l \hat{T}_k, \quad k, l \in \mathbb{Z}/12, k - l \neq \pm 1. \end{aligned} \tag{6}$$

In view of the relations (5) it is natural to put

$$\hat{R} := \hat{T}_1 \hat{T}_2 \cdots \hat{T}_{11}, \quad \hat{R}^* := \hat{T}_{11} \hat{T}_{10} \cdots \hat{T}_1.$$

Lemma 5.2. *We have $\hat{R} = \tau\eta$ and $\hat{R}^* = \eta^{-1}$.*

Proof. From the definitions we find that $\hat{R} = (\eta\hat{T})^{11}\eta$ and $\hat{R}^* = \eta^{-1}(\hat{T}\eta^{-1})^{11}$. We know a priori that $(\eta\hat{T})^{11}$ and $(\hat{T}\eta^{-1})^{11}$ are covering transformations, hence it is enough to show that these elements act on $H_1(C_o)$ as resp. τ and 1. This is verified in a straightforward manner using Corollary 5.1. □

So the $\hat{T}_i$'s generate all of $\widehat{\mathrm{BCl}}_{12}$. It also follows that $\widehat{\mathrm{BCl}}_{12}$ is a nontrivial central extension of BCl_{12}.

Recall from Section 4 that we identified $H_1(C_o)_{\mathcal{O}}$ with the hermitian rank 10 $\mathcal{O}$-module Λ. We noted in Remark 4.4 (see also the more precise identification in the Appendix) one finds that the form ψ on Λ has hyperbolic signature $(9,1)$. Since the action of $\widehat{\mathrm{BCl}}_{12}$ in $H_1(C_o)$ preserves the $\mathbb{Z}[C_6]$-module structure and the sesquilinear form, we have an induced monodromy representation $\widehat{\mathrm{BCl}}_{12} \to \mathrm{U}(\Lambda)$ with $\hat{R}\hat{R}^*$ mapping to ω. This drops to a projective representation $\mathrm{BCl}_{12} \to \mathrm{PU}(\Lambda)$.

Theorem 5.3 (Allcock, [**2**]). *The monodromies*

$$\rho : \mathrm{BCl}_{12} \to \mathrm{PU}(\Lambda) \quad \textit{and } \hat{\rho} : \widehat{\mathrm{BCl}}_{12} \to \mathrm{U}(\Lambda)$$

are surjective.

Corollary 5.4. *Every unitary automorphism of Λ comes from a symplectic automorphism of $H_1(C_o)$ that commutes with the C_6-action.*

It follows from 5.1 that in either case the image of $\hat{T}_i$ has order three. So if we define $\widehat{\mathrm{BCl}}_{12}[3]$ as the quotient of $\widehat{\mathrm{BCl}}_{12}$ by the relations $\hat{T}_i^3 \equiv 1$ and define $\mathrm{BCl}_{12}[3]$ similarly, then the monodromy representations factorize over homomorphisms $\widehat{\mathrm{BCl}}_{12}[3] \to \mathrm{U}(\Lambda)$ and $\mathrm{BCl}_{12}[3] \to \mathrm{PU}(\Lambda)$. We shall see that these are isomorphisms.

6. Satake-Baily-Borel compactification

Let V be a complex vector space equipped with a Hermitian form $\psi : V \times V \to \mathbb{C}$ of hyperbolic signature $(n,1)$, with $n \geq 2$. Denote by $\mathbb{L} = \mathbb{L}(V) \subset V$ the set of $v \in V$ with $\psi(v,v) < 0$. Then its projectivization $\mathbb{B} = \mathbb{B}(V) \subset \mathbb{P}(V)$ (a complex ball) is a symmetric space for the projective unitary $\mathrm{PU}(V)$. We regard $\mathbb{L}$ as an equivariant $\mathbb{C}^\times$-bundle over $\mathbb{B}$. For any integer k we denote by $\mathbb{L}(k)$ the line bundle defined by the representation of $\mathbb{C}^\times$ on $\mathbb{C}$ given by $z \in \mathbb{C}^\times \mapsto z^k$. Then $\mathbb{L}(n+1)$

is equivariantly isomorphic to the canonical bundle of $\mathbb{B}$. (To see this, observe that if $p \in \mathbb{B}$ is given by the negative definite line $L \subset \Lambda_{\mathbb{C}}$, then the tangent space of $\mathbb{B}$ at p is canonically isomorphic to $\mathrm{Hom}(L, \Lambda_{\mathbb{C}}/L)$ and hence the determinant line of the cotangent space with $L^{n+1} \otimes \det(\Lambda_{\mathbb{C}})^{-1}$.) So the canonical bundle of $\mathbb{B}$ is $\mathrm{SU}(V)$-equivariantly isomorphic to $\mathbb{L}(n+1)$.

6.1. Suppose V has also the structure of a vector space over an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{-d})$ in $\mathbb{C}$ (d a positive square free integer), such that ψ is defined over K and let be given an arithmetic subgroup Γ of $\mathrm{U}(V_K)$. Then Γ acts properly on the $\mathbb{C}^\times$-bundle $\mathbb{L}$ and the analytic orbifold

$$\mathbb{L}_\Gamma := \Gamma \backslash \mathbb{L}.$$

retains a $\mathbb{C}^\times$ action.

The space of Γ-*automorphic forms* of weight k is by definition

$$A^k := H^0(\mathbb{B}, \mathbb{L}(k))^\Gamma.$$

Its elements may be thought of as Γ-invariant functions on $\mathbb{L}$ that are homogeneous of degree $-k$ on every fiber. The space A^k is known to be finite dimensional for all $k \in \mathbb{Z}$ and trivial for $k < 0$. Observe that $A^k = 0$ when k is not divisible by the order of $\Gamma \cap K^\times$. (In the case that interests us this order will be 6.) Examples of such forms are the Poincaré series: if $v_0 \in \mathbb{L}$, then

$$F(a) := \sum_{\gamma \in \Gamma} \psi(a, \gamma v_0)^{-k}$$

converges uniformly on compact subsets of $\mathbb{L}$, provided that $k \geq 2\dim(\Lambda_{\mathbb{C}}) = 2n+2$. Hence F defines an element of A^k. The direct sum

$$A^\bullet := \oplus_{k \geq 0} A^k$$

is a $\mathbb{C}$-algebra of regular functions on $\mathbb{L}_\Gamma$. It is an algebra of finite type whose spectrum we denote by $\mathbb{L}_\Gamma^*$. This is a normal affine variety which contains $\mathbb{L}_\Gamma$ as an open-dense subvariety; we therefore call it the *automorphic hull* of $\mathbb{L}_\Gamma$. The group $\mathbb{C}^\times$ acts on $\mathbb{L}_\Gamma^*$ with a unique fixed point. The corresponding projective variety at infinity, $\mathrm{Proj}(A^\bullet)$, will be denoted by $\mathbb{B}_\Gamma^*$. As the notation suggests, the underlying spaces are in fact orbit spaces of a Γ-space extensions $\mathbb{L}^* \supset \mathbb{L}$ and $\mathbb{B}^* \supset \mathbb{B}$. The Satake-Baily-Borel theory constructs these spaces and we briefly recount how this is done.

A point of $\partial\mathbb{B}$ defined over K is called a *cusp* (of the form ψ_K). Then the union $\mathbb{B}^*$ of $\mathbb{B}$ and the set of cusps is just the convex hull of the K-points of the closure of $\mathbb{B}$ in $\mathbb{P}(V)$. A nonzero isotropic vector n defined over K defines a cusp $[n] \in \mathbb{B}^*$ and conversely, a cusp defines an isotropic line $I \subset V$ defined over K. For such a line I, let

$$\pi_{I^\perp} : V \to V/I^\perp$$

denote the obvious projection. If $n \in I$ is a generator, then $\psi(\ , n)$ defines a coordinate for $V/I^\perp$, so that $\pi_{I^\perp}$ is basically given by the inner product with n. The image of

$\mathbb{L} \subset V$ is the set of generators $(V - I^\perp)/I^\perp$. Let $\mathbb{L}^*$ be the disjoint union of $\mathbb{L}$, the punctured lines $\pi_{I^\perp}(\mathbb{L})$ and a singleton $V/V = \{*\}$. Notice that $U(V_K)$ acts naturally on this union (with $*$ as fixed point). We give $\mathbb{L}^*$ the topology generated by

(i) the open subsets of $\mathbb{L}$,
(ii) unions $\Omega_n \cup \pi_{n^\perp}(\Omega_n)$ with $n \in V_K - \{0\}$ isotropic and $\Omega_n \subset \mathbb{L}$ is the subset defined by $-\psi(z,z) > |\psi(z,n)|^2$.
(iii) unions $\Omega_N \cup \cup_{n\in N}\pi_{n^\perp}(\Omega_N) \cup \{*\}$ with $N \subset V_K - \{0\}$ a finite union of Γ-orbits of isotropic vectors and Ω_N the subset of $\mathbb{L}$ defined by the inequalities $-\psi(z,z) > |\psi(z,n)|^2$, for all $n \in N$.

The group $\mathrm{U}(V_K)$ acts on $\mathbb{L}^*$ as a group of homeomorphisms. The action of the central subgroup $K^\times$ extends in an obvious way to $\mathbb{C}^\times$ so that in fact $\mathrm{U}(V_K).\mathbb{C}^\times$ acts. The orbit space $\Gamma\backslash\mathbb{L}^*$ is the $\mathbb{C}^\times$-space underlying the automorphic hull (it is not difficult to verify that the Poncaré series defined above extends continously to $\mathbb{L}^*$). The cuspidal lines define finitely many (regular) $\mathbb{C}^\times$-orbits in $\mathbb{L}^*_\Gamma$, because Γ acts with finitely many orbits in the set of cusps.

Similarly, the space underlying $\mathbb{B}^*_\Gamma$ is the the $\mathbb{C}^\times$-orbit space of the Γ-orbit space of $\mathbb{B}^*$ endowed with the *horoball* (or *Satake*) topology: this is the topology of $\mathbb{B}^*$ generated by

(i) the open subsets of $\mathbb{B}$,
(ii) unions $\mathbb{P}(\Omega_n)\cup\{[n]\}$, where $\mathbb{P}(\Omega_n) \subset \mathbb{B}$ is of the form $-\psi(z,z)/|\psi(z,n)|^2 > 1$, with $n \in V_K$ nonzero isotropic.

6.2. The automorphic hull possesses plenty of totally geodesic hypersurfaces: Suppose that $\mathcal{H}$ is a Γ-invariant collection of K-hyperplanes in V of hyperbolic signature (that is, orthogonal to a positive vector). We assume that Γ has finitely many orbits in this collection. An example is the case when $\mathcal{H}$ is the set of hyperplanes that are perpendicular to a vector $v \in V_K$ with $\psi(v,v) = k$ (k a fixed positive integer). For every $H \in \mathcal{H}$, $\mathbb{B}(H)$, is totally geodesic subball of $\mathbb{B}$ and the collection of these is locally finite on $\mathbb{B}$. So

$$\mathbb{L}(\mathcal{H}) := \cup_{H\in\mathcal{H}}\mathbb{L}(H)$$

is closed in $\mathbb{L}$ and defines a closed analytic subset $\mathbb{L}(\mathcal{H})_\Gamma$ of $\mathbb{L}_\Gamma$. If $n > 2$, then $\mathbb{L}^*_\Gamma - \mathbb{L}_\Gamma$ is of codimension > 2 in $\mathbb{L}^*_\Gamma$ and an extension theorem implies that the closure $\mathbb{L}(\mathcal{H})^*_\Gamma$ of $\mathbb{L}(\mathcal{H})_\Gamma$ is analytic in $\mathbb{L}^*_\Gamma$. (This is also true when $n = 2$, but that needs an additonal argument.)

This will be a $\mathbb{C}^\times$-invariant hypersurface, hence algebraic. Notice that $\mathbb{L}(\mathcal{H})^*_\Gamma$ supports an effective Cartier divisor if and only if $\mathbb{L}(\mathcal{H})$ is defined by a single automorphic form. (That form then will admit a product expansion.)

7. The moduli space of rational curves with 12 punctures

By a *smooth C_6-curve* we will mean a complete nonsingular complex-projective curve C endowed with an action of the cyclic group C_6 that is isomorphic to a curve

C_D with affine equation $w^6 = \prod_{p\in D}(z-p)$, where D is a 12-element subset of $\mathbb{C}$, with $\tau(w,z) = (\omega w, z)$ (recall that τ is a fixed generator of C_6 and $\omega = e^{2\pi\sqrt{-1}/6}$). A more intrinsic characterization is to say that C has genus 25 and that the C_6-action has 12 distinct fixed points, each with (tangent space) character χ, and is free elsewhere. (The Riemann-Hurwitz formula shows that its orbit space is then a rational curve.)

Given such a smooth C_6-curve C, let $H^{1,0}(C)_\chi$ denote the space of regular differentials α on C on which C_6 acts with character χ, that is, which satisfy $\tau^*\alpha = \omega^{-1}.\alpha$. We claim that $H^{1,0}(C)_\chi$ has dimension one. To see this, represent C by an affine equation $w^6 = \prod_{p\in D}(z-p)$ as above. Then $w^{-1}dz$ is a regular differential on C and $\tau^*(w^{-1}dz) = \omega^{-1}.w^{-1}dz$. Notice that the only zeroes of $w^{-1}dz$ are the ramification points and that each such point appears with multiplicity 4. This implies that it is the only such form up to scalar: any other must be of the form $f(z)w^{-1}dz$ with f a rational function. In order that it be regular f should have no poles, so f must be constant. (If we let the ramification points move in $\mathbb{P}^1$, then a period of such a form is a Lauricella function, see [**10**].)

The coarse moduli space of the C_6-curves under consideration is the same as the one of 12 element subsets of a projective line (given up to a projective transformation), and so can be identified with $\mathcal{D}$. This suggests to allow as singular objects the C_6-coverings of a projective line $C_D \to P$ with D a semistable divisor on P such that over a point of multiplicity k of D we have a (plane curve) singularity with local equation $z^k = w^6$ $(k = 1, \ldots, 6)$. A good substitute for the sheaf of regular differentials is then the dualizing sheaf ω_C.

LEMMA 7.1. *For a C_6-covering $C = C_D \to P$ with D semistable, the χ-eigenspace in $H^0(C, \omega_C)$ is one-dimensional. The pull-back of a generator to a normalization of C is a logarithmic differential whose polar set is the preimage of multiplicity* 6 *locus of the discriminant.*

PROOF. Choose an affine equation for C as before. First note that $w^{-1}dz$ lies in $H^0(C, \omega_C)_\chi$. At a point of multiplicity k, a local equation of C is $z^k = w^6$. A straightforward calculation shows that the pull-back of $w^{-1}dz$ under normalization has in each the preimage of this singularity a zero of order $4, 1, 0, 0, 0, -1$ for $k = 1, 2, 3, 4, 5, 6$. Any other element of $H^0(C, \omega_C)_\chi$ is of the form $f(z)w^{-1}dz$ and as in the smooth case we find that f cannot have any poles, hence must be constant. $\square$

LEMMA 7.2. *The orbifold line bundle $\mathcal{L}_{\mathcal{D}^*}$ over $\mathcal{D}^*$ is naturally isomorphic to the coarse moduli space of pairs $(C, \alpha^{\otimes 6})$ with C a C_6-curve with semistable discriminant divisor and $\alpha \in H^0(C, \omega_C)_\chi$.*

PROOF. We use our fixed two dimensional vector space Π equipped with a generator ζ of $\wedge^2\Pi$. Given a semistable $F \in \Pi_{12}$, regard F as a homogeneous function on Π^*. Then $w^6 = F$ defines a degree 6 covering of Π^*. It is an affine surface with good $\mathbb{C}^\times$-action (so that w has weight 2) whose curve at infinity is a C_6-curve C as above.

Then $w^{-1}\zeta$ is a $\mathbb{C}^\times$-invariant rational form whose residue at infinity, α, is a nonzero element of $H^0(C,\omega_C)_\chi$. So $\alpha^{\otimes 6}$ is the residue of $w^{-6}\zeta^{\otimes 6} = F^{-1}\zeta^{\otimes 6}$. Think of F^{-1} as the linear form on the line $\mathbb{C}F$ in Π_{12} spanned by F which takes the value 1 on F. The $\mathrm{SL}(\Pi)$-orbit of such a linear form defines an element of the complement of the zero section of $\mathcal{L}_{\mathcal{D}^*}$ and vice versa. Since the constructions are $\mathrm{SL}(\Pi)$-equivariant, we thus get a map from the complement of the zero section of $\mathcal{L}_{\mathcal{D}^*}$ to the moduli space in question. It is easy to see this this extends to an isomorphism of $\mathcal{L}_{\mathcal{D}^*}$ to the moduli space. □

Let C be a smooth C_6-curve as above. The intersection pairing identifies $H^1(C)$ with $H_1(C)$ as $\mathbb{Z}[C_6]$-modules with symplectic form. Since $H_1(C)$ is isomorphic (as a $\mathbb{Z}[C_6]$-module with symplectic form) to A, the choice of such an isomorphism induces an isomorphism of Hermitian $\mathcal{O}$-modules $\Lambda = A_{\mathcal{O}} \to H^1(C)_{\mathcal{O}}$. We shall refer to a Hermitian isomorphism $\Phi : \Lambda \to H^1(C)_{\mathcal{O}}$ as a Λ-*marking* of the C_6-curve C. By Corollary 5.4 such a marking always comes from a sesquilinear isomorphism $A \to H^1(C)$.

Lemma 7.3. *The automorphism group of the C_6-curve C acts faithfully on the quotient $H^1(C)_{\mathcal{O}}$.*

Proof. This is clear for the group of covering transformations. Any such automorphism that is not a covering transformation must permute the ramification points nontrivially. It is easy to see that such an automorphism acts nontrivially on $H^1(C)_{\mathcal{O}}$. □

This implies that a Λ-marked C_6-curve has no automorphisms. Hence there is fine moduli space $\tilde{\mathcal{D}}$ in the analytic category of these objects. It is an analytic manifold of dimension 9 (use three of the ramification points as coordinates for the projective line $C_6\backslash C$; the other nine then run over an open subset of $\mathbb{C}^9$) and comes with an evident action of the unitary group $\mathrm{U}(\Lambda)$ of Λ: $u \in \mathrm{U}(\Lambda)$ sends (C,Φ) to $(C,\Phi u^{-1})$. This action is proper and the orbit space can be identified with $\mathcal{D}$. Lemma 7.2 suggests we also consider the moduli space $\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}}$ of triples (C,Φ,α) consisting of a Λ-marked genus C_6-curve (C,Φ) and an element $\alpha \in H^0(C,\omega_C)_\chi$. It is clear that the projection $\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}} \to \tilde{\mathcal{D}}$ is a $\mathrm{U}(\Lambda)$-equivariant line bundle.

Lemma 7.4. *The morphism $\tilde{\mathcal{D}} \to \mathcal{D} \subset \mathcal{D}^{\mathrm{st}}$ extends naturally to a branched $\mathrm{U}(\Lambda)$-covering $\tilde{\mathcal{D}}^{\mathrm{st}} \to \mathcal{D}^{\mathrm{st}}$. Moreover, the $\mathrm{U}(\Lambda)$-equivariant line bundle $\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}} \to \tilde{\mathcal{D}}$ extends naturally to a $\mathrm{U}(\Lambda)$-equivariant line bundle $\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}^{\mathrm{st}}} \to \tilde{\mathcal{D}}^{\mathrm{st}}$.*

Proof. Let D be a stable effective degree 12 divisor in $\mathbb{C}$ (so all multiplicities ≤ 5). Given a neighborhood U of D in the space of effective degree 12 divisors, denote by $U' \subset U$ the divisors that are reduced. Then $D' \in U' \mapsto H^1(C_{D'})_{\mathcal{O}}$ defines a locally constant sheaf of $\mathcal{O}$-modules. If D has multiplicities $5 \geq n_1 \geq n_2 \geq \cdots n_r \geq 1$ (so that $\sum n_i = 12$), and U is sufficiently small, then the local monodromy group is isomorphic to subgroup of $\prod_i U(\Lambda^{n_i-1})$. Since the ranks $n_i - 1$ are all ≤ 4, the latter is finite by Subsection A.1, and hence so is the monodromy group. The assertions of the lemma are a formal consequence of this fact. $\square$

Remark 7.5. Closer inspection shows that there is in fact a moduli interpretation of the added points: an element of $\tilde{\mathcal{D}}^{\mathrm{st}}$ is represented by a pair (C, Φ), where C is a C_6-curve with stable ramification divisor and $\Phi : \Lambda \to H^1(C)_{\mathcal{O}}$ is a certain epimorphism of $\mathcal{O}$-modules. The kernel of Φ is isomorphic to an orthogonal direct sum of sublattices $\Lambda^{n_1-1} \perp \Lambda^{n_2-1} \perp \cdots$ and Φ is given up to composition with an element of the local monodromy group $\prod_i U(\Lambda^{n_i-1})$. A point of $\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}^{\mathrm{st}}}$ is obtained by also giving an element of $H^0(C, \omega_C)_\chi$.

Remark 7.6. If D is stable, then we have a square norm on $H^0(C, \omega_C)_\chi$ defined by

$$\alpha \in H^0(C, \omega_C)_\chi \mapsto \theta \int_C \alpha \wedge \bar{\alpha}.$$

In case D is reduced, then this is just the restriction of our Hermitian form $-\psi$ via the embedding

$$H^0(C, \omega_C)_\chi \subset H^1(C; \mathbb{C})_\chi = \mathbb{C} \otimes_{\mathcal{O}} H^1(C)_{\mathcal{O}}.$$

This norm blows up over the point d_∞. To see this, use the fact that if D becomes strictly semistable, then $w^{-1}dz$ becomes a differential on the normalization of C_D with poles of order one. So the integral of the generating section defined by $|w^{-1}dz|^2$ blows up over d_∞.

We now define a period mapping. Let $(\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}})^\times$ be the complement of the zero section of $\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}}$. Let $\mathbb{L}$ be as defined in Section 6 with $V = \Lambda_{\mathbb{C}}$, $K = \mathbb{Q}(\omega) = \mathbb{Q}(\sqrt{-3})$ and $\Gamma = \mathrm{U}(\Lambda)$. If (C, Φ, α) represents a point of $(\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}})^\times$, then assign to this triple the vector $\Phi^{-1}(\alpha)$. This defines the *period mapping*:

$$\widetilde{\mathrm{Per}} : (\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}})^\times \to \mathbb{L}.$$

This mapping is clearly equivariant with respect to the actions of $\mathbb{C}^\times$ and $\mathrm{U}(\Lambda)$ and both its domain and range are analytic manifolds of dimension 10. This period mapping extends across the locus with finite monodromy: we have an extension

$$\widetilde{\mathrm{Per}} : (\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}^{\mathrm{st}}})^\times \to \mathbb{L}.$$

Indeed, if a point of the domain is represented as in Remark 7.5 by a triple (C, Φ, α), then Lemma 7.1 implies that α defines a nonzero element of $H^1(C)_\chi$ and the image of (C, Φ, α) is the point of $\mathbb{L} \cap \ker(\Phi)^\perp$ that is mapped by Φ to α. For the details we refer to [**10**]. The period mapping drops to a morphism

$$\mathrm{Per} : (\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}^{\mathrm{st}}}) \to \mathbb{L}_{\mathrm{U}(\Lambda)},$$

and if we pass to $\mathbb{C}^\times$-orbit spaces, we also get

$$\mathbb{P}(\widetilde{\mathrm{Per}}) : \tilde{\mathcal{D}}^{\mathrm{st}} \to \mathbb{B} \text{ and } \mathbb{P}(\mathrm{Per}) : \mathcal{D}^{\mathrm{st}} \to \mathbb{B}_{\mathrm{U}(\Lambda)}.$$

The following theorem is a special case of a theorem of Deligne-Mostow [**10**].

THEOREM 7.7 (Deligne-Mostow [**10**], see also [**8**]). *The period map* $\widetilde{\mathrm{Per}}$ *establishes a* $\mathrm{U}(\Lambda)$*-equivariant isomorphism between the* $\mathbb{C}^\times$*-bundle* $(\mathcal{L}^{1/6}_{\tilde{\mathcal{D}}^{\mathrm{st}}})^\times$ *and* $\mathbb{L}$. *The induced isomorphism* $\mathcal{D}^{\mathrm{st}} \to \mathbb{B}_{\mathrm{U}(\Lambda)}$ *extends to an isomorphism between the GIT compactification* $\mathcal{D}^* \supset \mathcal{D}^{\mathrm{st}}$ *and the Baily-Borel compactification* $\mathbb{B}^*_{\mathrm{U}(\Lambda)} \supset \mathbb{B}_{\mathrm{U}(\Lambda)}$.

Statement and proof are somewhat hidden in the paper and so we give an

OUTLINE OF PROOF. Since $\widetilde{\mathrm{Per}}$ is $\mathbb{C}^\times$-equivariant, it is enough to prove that $\mathbb{P}(\widetilde{\mathrm{Per}}) : \tilde{\mathcal{D}}^{\mathrm{st}} \to \mathbb{B}$ is an isomorphism. To this end, one first shows that $\mathbb{P}(\widetilde{\mathrm{Per}})$ is a local isomorphism in codimension one (this is based on simple type of local Torelli theorem) and has discrete fibers. This implies that $\mathbb{P}(\widetilde{\mathrm{Per}})$ has no ramification. So $\mathbb{P}(\widetilde{\mathrm{Per}})$ is a local isomorphism every where. We wish to show that $\mathbb{P}(\widetilde{\mathrm{Per}})$ is proper; the simple connectivity of $\mathbb{B}$ will then imply that $\mathbb{P}(\widetilde{\mathrm{Per}})$ is an isomorphism. This will follow if we prove that $\mathbb{P}(\mathrm{Per}) : \mathcal{D}^{\mathrm{st}} \to \mathbb{B}_{\mathrm{U}(\Lambda)}$ is proper. In other words, we want to show that $\mathbb{P}(\mathrm{Per})$ extends continuously to the one-point compactifications of its domain and range.

Let D be a strictly semistable divisor of degree 12 on $P = \mathbb{P}^1$. So D has a point of multiplicity 6. Let γ be a small oriented circle around this point. Then the preimage of γ in C_D consists of 6 disjoint circles. If $\tilde{\gamma}$ is one of these, then $\int_{\tilde{\gamma}} w^{-1}dz$ is by 7.1 the residue of a differential with a simple pole and hence nonzero. The cycle $\tilde{\gamma}$ subsists under small deformations of D and for D' in a neighborhood of D the corresponding integral $\int_{\tilde{\gamma}(D')} w^{-1}dz$ is then analytic in D' and nowhere zero. If D' is reduced, then $\tilde{\gamma}(D')$ defines an isotropic element of $H_1(C_{D'})_{\mathcal{O}}$. On the other hand, by Remark 7.6, $\int_{C_{D'}} |w^{-1}dz|^2$ tends to $+\infty$, as D' approaches D. So the same is true for the expression

$$\frac{\int_{C_{D'}} |w^{-1}dz|^2}{|\int_{\tilde{\gamma}(D')} w^{-1}dz|^2}.$$

It now follows from our explicit description of the Satake topology in Section 6 that the image of D' under $\mathbb{P}(\mathrm{Per})$ tends to the cusp of $\mathbb{B}^*_{\mathrm{U}(\Lambda)}$, as D' tends to D. This proves that $\mathbb{P}(\mathrm{Per}) : \mathcal{D}^{\mathrm{st}} \to \mathbb{B}_{\mathrm{U}(\Lambda)}$ is proper.

So both $\mathbb{P}(\widetilde{\mathrm{Per}}) : \tilde{\mathcal{D}}^{\mathrm{st}} \to \mathbb{B}$ and $\mathbb{P}(\mathrm{Per}) : \mathcal{D}^{\mathrm{st}} \to \mathbb{B}_{\mathrm{U}(\Lambda)}$ are isomorphisms. Since $\mathcal{D}^*$ and $\mathbb{B}^*_{\mathrm{U}(\Lambda)}$ are normal one point compactifications of $\mathcal{D}^{\mathrm{st}}$ and $\mathbb{B}_{\mathrm{U}(\Lambda)}$ respectively, the continuous extension $\mathcal{D}^* \to \mathbb{B}^*_{\mathrm{U}(\Lambda)}$ is in fact an isomorphism. □

We can also tell what the image of $\mathcal{D}$ is. Let us call a hyperplane in Λ a *mirror* if it is the orthogonal complement of a 3-vector. A mirror has hyperbolic signature and by Lemma A.6 any two mirrors are $\mathrm{U}(\Lambda)$-equivalent. So the collection $\mathcal{H}$ of mirrors defines an irreducible hypersurface $\mathbb{B}(\mathcal{H})^*_{\mathrm{U}(\Lambda)}$ in $\mathbb{B}^*_{\mathrm{U}(\Lambda)}$. If we let of 12 distinct points in $\mathbb{P}^1$ two coalesce, then we get a curve germ in $\mathcal{D}^*$ with generic point in $\mathcal{D}$ and closed point a generic point in $\mathcal{D}^* - \mathcal{D}$. Associated to this there is a 'vanishing 3-vector' which shows that $\mathcal{D}^* - \mathcal{D}$ is mapped to $\mathbb{B}(\mathcal{H})^*_{\mathrm{U}(\Lambda)}$. Since both $\mathcal{D}^* - \mathcal{D}$ and $\mathbb{B}(\mathcal{H})^*_{\mathrm{U}(\Lambda)}$ are irreducible we find:

THEOREM 7.8. *The period mapping defines an isomorphism*

$$(\mathcal{D}^*, \mathcal{D}^{st}, \mathcal{D}) \cong (\mathbb{B}^*_{\mathrm{U}(\Lambda)}, \mathbb{B}_{\mathrm{U}(\Lambda)}, \mathbb{B}_{\mathrm{U}(\Lambda)} - \mathbb{B}(\mathcal{H})_{\mathrm{U}(\Lambda)}).$$

REMARK 7.9. The geometric invariant theory quotient $\mathcal{D}^*$ is in a natural way stratified by the multiplicities of an effective degree 12 divisor on P. The strata in $\mathcal{D}^{st}$ are indexed by partitions of 12 with parts at most 5, whereas the point $\mathcal{D}^* - \mathcal{D}^{st}$ corresponds to the partition (6^2) of 12. The ball quotient $\mathbb{B}^*_{\mathrm{U}(\Lambda)}$ is also stratified in a natural way by mirror intersection pattern. The orthogonal complement in Λ of an intersection of mirrors, which is not empty inside $\mathbb{L}$, is isomorphic to a primitive sublattice of Λ generated by 3-vectors. As such it is isomorphic to a lattice $\Lambda^{k_1} \perp \Lambda^{k_2} \perp \cdots$ with $k_1 \geq k_2 \geq \cdots$ (see Appendix), and we assign to it the partition $(k_1 + 1, k_2 + 1, \cdots)$ of 12. With respect to this geometric stratification of $\mathcal{D}^*$ and this arithmetic stratification of $\mathbb{B}^*_{\mathrm{U}(\Lambda)}$ the period mapping becomes an isomorphism of stratified spaces.

REMARK 7.10. We observed in 2.3 that the discriminant hypersurface $\mathcal{D}^* - \mathcal{D}$ has degree $11 \cdot 12$ (with respect to the $\mathcal{L}_{\mathcal{D}^*}$). Hence the locally symmetric hypersurface $\mathbb{B}(\mathcal{H})^*_{\mathrm{U}(\Lambda)}$ is defined by a section of $\mathbb{L}(6 \cdot 11 \cdot 12)$. Since $\mathbb{L} \to \mathbb{L}_{\mathrm{U}(\Lambda)}$ ramifies with order three along $\mathbb{L}(\mathcal{H})$, it follows that the divisorial preimage of $\mathbb{L}(\mathcal{H})_{\mathrm{U}(\Lambda)}$ is $3\mathbb{L}(\mathcal{H})$. So $\mathbb{L}(\mathcal{H})$ is given by an automorphic form of weight $2 \cdot 11 \cdot 12$ with a character of order 3. Since Allcock finds this degree to be 44 [**2**], we assume that his weight is 1/6 of ours (the center of $\mathrm{U}(\Lambda)$ consists of the 6th roots of unity and so the degree of any nonzero $\mathrm{U}(\Lambda)$-automorphic form on $\mathbb{B}$ is divisible by 6).

COROLLARY 7.11. *The kernel of the monodromy representation* $\rho : \widehat{\mathrm{BCl}}_{12} \to \mathrm{U}(\Lambda)$ *is the normal subgroup generated by* $\hat{T}_0^3$ *so that* ρ *induces isomorphisms* $\widehat{\mathrm{BCl}}_{12}[3] \cong \Gamma$ *and* $\mathrm{BCl}_{12}[3] \cong \mathrm{PU}(\Lambda)$.

PROOF. The group BCl_{12} may be identified with the orbifold fundamental group of $\mathcal{D}$. Via the orbifold isomorphism $\mathcal{D} \cong \mathbb{B}_{\mathrm{U}(\Lambda)} - \mathbb{B}(\mathcal{H})_{\mathrm{U}(\Lambda)}$, we then get a $\mathrm{BCl}_{12}[3]$-covering. This covering factorizes over a covering of $\mathbb{B} - \mathbb{B}(\mathcal{H})$ with the kernel of $\mathrm{BCl}_{12}[3] \to \mathrm{PU}(\Lambda)$ as covering group.

Since $\hat{T}_0^3$ is trivial in $\widehat{\mathrm{BCl}}_{12}[3]$, a simple loop around a deleted hyperplane has monodromy of order three, and so the covering over $\mathbb{B} - \mathbb{B}(\mathcal{H})$ extends as an *unramified* covering over the smooth part of $\mathbb{B}(\mathcal{H})$: we now have a connected unramified covering over $\mathbb{B} - \mathbb{B}(\mathcal{H})_{\mathrm{sing}}$. Since $\mathbb{B} - \mathbb{B}(\mathcal{H})_{\mathrm{sing}}$ is simply connected, this covering must be trivial. We conclude that $\mathrm{BCl}_{12}[3] \to \mathrm{PU}(\Lambda)$ is injective. From this it follows that $\widehat{\mathrm{BCl}}_{12}[3] \cong \mathrm{U}(\Lambda)$ is injective as well. $\square$

8. Rational elliptic surfaces and the Eisenstein curve

Recall from our discussion of Kodaira's theorem 3.1 that the commutator subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ defines a modular curve E_o of genus one with a simple cusp. We regard it as an elliptic curve by taking the cusp as its origin. It comes with a faithful action the abelianization C_6 of $\mathrm{PSL}(2,\mathbb{Z})$, and so this elliptic curve has J-invariant 0. In other words, it can be analytically obtained as the quotient $\mathbb{C}/\mathcal{O}$ with the generator $\tau \in C_6$ acting as complex multiplication by ω. So $\mathbb{Z}[C_6]$ acts on $H^1(E_o)$ via $\mathcal{O}$. We will refer to E_o as the *Eisenstein curve*. Since τ acts on the tangent space of the origin with eigenvalue ω, the same is true for the action of τ^* on $H^{1,0}(E_0)$. It follows that $H^1(E_0,\mathbb{C})_\chi = H^{0,1}(E_0)$.

The natural map to the J-line, $E_o \to \mathbb{P}^1$, ramifies over 0 (two points of order three), 1 (three points of order two) and ∞ (total ramification).

Lemma 8.1. *Let $X \to P$ be a rational elliptic surface with reduced discriminant D_∞. Let $J : P \to \mathbb{P}^1$ be its modular function and let C be the normalization of $P \times_{\mathbb{P}^1} E_o$. Then the C_6-covering $C \to P$ is the one considered in Section 7: it is only ramified over D_∞, the ramification over D_∞ is total and τ acts in the tangent space of each ramification point as multiplication by ω.*

Proof. It is clear that the projection $C \to P$ is a C_6-covering. There is no ramification outside the discriminant divisor $J^*(\infty)$ since J is there locally liftable to a morphism to E_o. The remaining statements follow easily. □

A special feature of this situation is that C comes with a C_6-equivariant morphism $\tilde{J} : C \to E_o$. Its degree is clearly 12.

Theorem 8.2. *In the situation of Lemma 8.1 we have:*

(i) *The morphism $\tilde{J} : C \to E_o$ induces an embedding $\tilde{J}^*_{\mathcal{O}} : H^1(E_o)_{\mathcal{O}} \to H^1(C)_{\mathcal{O}}$ of $\mathcal{O}$-modules that multiplies the hermitian form by* 12*,*

(ii) *the line $H^{1,0}(C)_\chi$ is perpendicular to the image of $\tilde{J}^*_{\mathcal{O}}$ and*

(iii) *there exists a* 6*-vector $z \in H^1(C)_{\mathcal{O}}$ such that the image of $\tilde{J}^*_{\mathcal{O}}$ is the $\mathcal{O}$-submodule $H^1(C)_{\mathcal{O}}$ spanned by $2\theta z$.*

Proof. The first assertion follows from the fact that $\tilde{J}$ is C_6-equivariant and of degree 12 and the second from the observation that $H^1(E_o,\mathbb{C})_\chi = H^{0,1}(E_o)$.

The last clause requires more work. In view of the connectedness of $\mathcal{M}$, it is enough to prove that assertion for one particular rational elliptic surface. We take the case studied in Section 4, where $D_\infty \subset \mathbb{P}^1$ is the set of 12th roots of unity and $C_o \to \mathbb{P}^1$ is the curve with $C_6 \times C_{12}$-action. As noted in Example 2.3, D_∞ is the discriminant divisor of an elliptic surface, but we will exhibit such a fibration more directly. Consider the action of the (order 12) subgroup $G \subset C_6 \times C_{12}$ generated by $\tau^3\eta$. The orbit space $G\backslash C_o$ is a C_6-covering of $C_{12}\backslash\mathbb{P}^1$. If we identify the latter with $\mathbb{P}^1$ by means of the affine coordinate z^{12}, then we see that $G\backslash C_o \to \mathbb{P}^1$ has total ramification over 1, a fiber with two points over 0 and a fiber with three points over ∞. These properties imply that $G\backslash C_o$ has genus one and more than that, namely that $G\backslash C_o$ is C_6-equivariantly isomorphic to the Eisenstein curve E_o. The Eisenstein curve supports a C_6-equivariant elliptic fibration. This pulls back to a C_6-equivariant elliptic fibration over C_o and that in turn descends to an elliptic fibration on $\mathbb{P}^1$. We therefore denote the resulting C_6-morphism $\tilde{J} : C_o \to E_o$. The induced map on the first cohomology $\tilde{J}^* : H^1(E_o) \to H^1(C_o)$ is C_6-equivariant. We identify the $\mathbb{Z}[\tau,\eta]$-module $H^1(C_o)$ with the algebra A defined in Section 4. It is clear that the image of $\tilde{J}^*$ is the $\mathcal{O}$-submodule spanned by

$$\sum_{i=0}^{11}(\tau^3\eta)^i \in A.$$

The image u of this element in

$$H^1(C_o)_{\mathcal{O}} \cong A_{\mathcal{O}} = \mathcal{O}[\eta]/(\sum_{i=0}^{11}\eta^i, \sum_{i=0}^{11}(\omega\eta)^i)$$

is easily calculated to be of the form $2\theta z$, with

$$z = \omega^{-1}(\eta^2+\eta^8) + (\eta^3+\eta^4+\eta^9+\eta^{10}) + \omega(\eta^5+\eta^{11}).$$

We claim that u is a $12\cdot 6$-vector: this is a straightforward computation or one invokes Example 4.1 and the fact that the Hermitian form is multiplied by 12. So z is a 6-vector. $\square$

The last assertion of the above proposition implies that the condition for a 12 element subset of $\mathbb{P}^1$ to be the discriminant of a rational elliptic surface *imposes a linear constraint on the period map* defined in Section 7. We investigate this in more detail in the next section.

9. Moduli of rational elliptic surfaces II

From now on, we make free use of notions, notation and results of the theory of $\mathcal{O}$-lattices, as collected and proved in the Appendix.

In the Appendix we fix a sublattice Λ_o that is the orthogonal complement of a 6-vector $z_o \in \Lambda$. (It is proved in Proposition A.6 that all such sublattices are $\mathrm{U}(\Lambda)$-equivalent.) According to Proposition A.8 the stabilizer of Λ_o in $\mathrm{U}(\Lambda)$ restricts isomorphically to the unitary group $\mathrm{U}(\Lambda_o)$ of Λ_o. It follows from Proposition A.6 that $\mathrm{U}(\Lambda_o)$ has two orbits in the set of primitive 0-vectors in Λ_o: type (θ) and (0). So the Baily-Borel compactification $\mathbb{B}^*_{o,\mathrm{U}(\Lambda_o)}$ adds two points to $\mathbb{B}_{o,\mathrm{U}(\Lambda_o)}$. We denote them ∞_θ and ∞_0.

We call a hyperplane H of Λ_o a *mirror trace* if it is the intersection of a mirror of Λ with Λ_o and has hyperbolic signature. This amounts to requiring that the orthogonal complement $H^\perp$ of H in Λ is positive definite and contains the 6-vector z_o and a

3-vector. According to Lemma A.9 the discriminant of $H^\perp$ then takes the values 6, 9, 15 or 18; we denote that number by $d(H)$ and call it the *d-invariant* of H. A special role will be played by the mirror traces with d-invariant 6 or 9 as in these cases there exist 3-vectors r_1, r_2 in $H^\perp$ such that $r_1 + r_2$ spans $\Lambda_o^\perp$. Proposition A.12 can be restated as:

PROPOSITION 9.1. *Two mirror traces with the same d-invariant are equivalent under the* $\mathrm{U}(\Lambda_o)$*-action.*

We denote the collection of mirror traces by $\mathcal{H}_o$, and those with d-invariant in a subset $S \subset \{6, 9, 15, 18\}$ by $\mathcal{H}_o(S)$. So we get a hypersurface $\Delta := \mathbb{B}_o(\mathcal{H}_o)^*_{\mathrm{U}(\Lambda)_o}$ in $\mathbb{B}^*_{o,\mathrm{U}(\Lambda_o)}$ that has four irreducible components: $\Delta(d) := \mathbb{B}_o(\mathcal{H}_o(d))^*_{\mathrm{U}(\Lambda_o)}$, $d = 6, 9, 15, 18$.

The inclusion $\mathbb{L}_o \subset \mathbb{L}$ induces a natural map

$$\mathbb{L}^*_{o,\mathrm{U}(\Lambda_o)} \to \mathbb{L}^*_{\mathrm{U}(\Lambda)}$$

that is finite and birational onto a hypersurface of $\mathbb{L}^*_{\mathrm{U}(\Lambda)}$ (it need not be injective though) so that $\mathbb{L}^*_{o,\mathrm{U}(\Lambda_o)}$ can be identified with the normalization of this hypersurface. It is clear that $\mathbb{L}_o(\mathcal{H}_o)^*_{\mathrm{U}(\Lambda_o)}$ is the preimage of $\mathbb{L}(\mathcal{H})^*_{\mathrm{U}(\Lambda)}$ under the map displayed above.

Let $f : X \to P$ be a rational elliptic surface with reduced discriminant. We have an associated C_6-covering $C \to P$ together with an equivariant morphism $C \to E_o$. We say that a Λ-marking $\Phi : H_1(C)_{\mathcal{O}} \cong \Lambda$ is *adapted* if $\Phi\tilde{J}^*$ maps $H^1(E_o)$ to the orthogonal complement of Λ_o. Rational elliptic surfaces with adapted markings define analytic covers $\tilde{\mathcal{M}}$ and $\mathcal{E}_{\tilde{\mathcal{D}}}|\tilde{\mathcal{M}}$ of $\mathcal{M}$ and $\mathcal{E}_{\mathcal{D}}|\mathcal{M}$ respectively, the latter with Galois group $\mathrm{U}(\Lambda_o)$, the former with Galois group $\mathrm{U}(\Lambda_o)$ modulo its scalars. The period map induces an equivariant morphism $\mathcal{E}_{\mathcal{D}}|\tilde{\mathcal{M}} \to \mathbb{L}_o$. It follows from the preceding that this morphism is injective; in fact from Proposition 2.1, Theorem 7.8 and Theorem 8.2 we get:

THEOREM 9.2. *The period mapping induces an isomorphism of arrows:*

$$\begin{array}{ccc}
(\mathcal{M}^*, \mathcal{M}) & \cong & (\mathbb{B}^*_{o,\mathrm{U}(\Lambda_o)}, \mathbb{B}_{o,\mathrm{U}(\Lambda_o)} - \mathbb{B}_o(\mathcal{H}_o)_{\mathrm{U}(\Lambda)_o}) \\
\downarrow & & \downarrow \\
(\mathcal{D}^*, \mathcal{D}) & \cong & (\mathbb{B}^*_{\mathrm{U}(\Lambda)}, \mathbb{B}_{\mathrm{U}(\Lambda)} - \mathbb{B}(\mathcal{H})_{\mathrm{U}(\Lambda)}).
\end{array}$$

According to 3.4, the boundary of $\mathcal{M}$ in $\mathcal{M}^*$ consists of four irreducible hypersurfaces of $\mathcal{M}^*$: $\mathcal{M}^*(I_2)$, $\mathcal{M}^*(II)$, $\mathcal{M}^*(I_9)$ and $\mathcal{M}^*(I^*_{4,4})$, whereas the irreducible components of Δ are $\Delta(18)$, $\Delta(15)$, $\Delta(9)$, $\Delta(6)$. The period isomorphism 9.2 must set up a bijection between these two sets. Something similar should hold for the strata $\mathcal{M}^*(I_6)$ and $\mathcal{M}(I^*_0)$ lying over the two cusps ∞_0 and ∞_θ of $\mathbb{B}^*_{\mathrm{U}(\Lambda)}$. We complete the picture by determining which goes to which.

THEOREM 9.3. *The period isomorphism maps the irreducible components denoted* $\mathcal{M}^*(I_2)$, $\mathcal{M}^*(II)$, $\mathcal{M}^*(I^*_{4,4})$, $\mathcal{M}^*(I_9)$ *onto* $\Delta(18)$, $\Delta(15)$, $\Delta(9)$, $\Delta(6)$ *respectively. Moreover, the singletons* $\mathcal{M}^*(I^*_6)$ *and* $\mathcal{M}(I^*_0)$ *are mapped to* $\{\infty_\theta\}$ *and* $\{\infty_0\}$ *respectively.*

Before we begin the proof, we note that this theorem is equivalent to the corresponding statements for $\mathcal{M}^K$ (instead of $\mathcal{M}^*$), for by definition $\mathcal{M}^*(F)$ is the image of $\mathcal{M}^K(F)$ under the modification $\mathcal{M}^K \to \mathcal{M}^*$. We will prove the theorem in this form.

Let $(J : P \to \mathbb{P}^1, D)$ represent a closed point of $\mathcal{M}^K$ and let $C \to P$ be the corresponding μ_6-covering. Consider a deformation of $(J : P \to \mathbb{P}^1, D)$ over a smooth curve germ $(\mathbb{D}, o)$ with smooth generic fiber. After a finite base change this is covered by a smoothing of C:

$$\mathcal{C} \to \mathcal{P} \to \mathbb{P}^1 \times \mathbb{D},$$

where the first morphism is the quotient by an μ_6-action and the second is of degree 12. We observed in 3.2 that there is a natural μ_6-equivariant morphism $\mathcal{C} \to E_o$. In a situation like this there is a standard procedure for comparing the cohomology of the special fiber and the general fiber: the pull-back of $\mathcal{C} \to \mathbb{D}$ over the universal cover $\tilde{\mathbb{D}}^\times$ of $\mathbb{D}^\times := \mathbb{D} - \{o\}$, $\mathcal{C}_{\tilde{\mathbb{D}}^\times} \to \tilde{\mathbb{D}}^\times$, is homologically trivial and after a choice of an adapted Λ-marking we get an isomorphism of $\mathcal{O}$-modules $H^1(\mathcal{C}_{\tilde{\mathbb{D}}^\times})_{\mathcal{O}} \cong \Lambda$ such that the image of $H^1(E_o)_{\mathcal{O}}$ is a multiple of z_o. This gives rise to a period morphism $\tilde{\mathbb{D}}^\times \to \mathbb{B}_o$. The inclusion $C \subset \mathcal{C}$ is a homological isomorphism, and hence the diagram $C \subset \mathcal{C} \leftarrow \mathcal{C}_{\tilde{\mathbb{D}}^\times}$ induces a homomorphism of $\mathcal{O}$-modules $\Lambda \to H^1(C)_{\mathcal{O}}$ such the image of $\mathcal{O}2\theta z_o$ is mapped onto $H^1(E_o)_{\mathcal{O}}$.

PROOF OF 9.3. Consider the case when the closed fiber represents a general point of $\mathcal{M}^K(I_2)$, $\mathcal{M}^K(II)$, $\mathcal{M}^K(I_9)$ or $\mathcal{M}^K(I^*_{4,4})$. The image of such a point in $\mathcal{D}^*$ is a semistable orbit of a degree 12 divisor on $\mathbb{P}^1$ of type $(2, 1^{10})$, $(2, 1^{10})$, $(3, 1^9)$, $(2^2, 1^8)$ respectively, as explained in Section 3. So its image under the period isomorphism is going to be perpendicular to a (primitive) sublattice L of Λ of type Λ^1, Λ^1, Λ^2, $\Lambda^1 \times \Lambda^1$ respectively, by Section 7. In the last two cases, the central component of P_c is in $J^{-1}(\infty)$ and so the morphism $H^1(E_o) \to H^1(C) \to H^1(C_c)$ will be zero. This implies that in these cases L contains z_o. This shows that in terms of the notation of Lemma A.9 L is of type δ_6 in the $\mathcal{M}^K(I_9)$-case and of type δ_9 and in the $\mathcal{M}^K(I^*_{4,4})$-case. So we then find a point of $\Delta(6)$ and $\Delta(9)$ respectively.

We now show that for I_2 we cannot end up with a point of $\Delta(15)$. Since we have a period isomorphism, it then will follow that we must get a point of $\Delta(18)$ and that in the remaining case II we get a point of $\Delta(15)$. We note that in the I_2-case, the lattice $L \cong \Lambda^1$ is accounted for by $H^1(C')_{\mathcal{O}}$, where C' is the irreducible component of C that lies over ∞. Since the map $C \to E_o$ is constant on C', it follows that $L \subset \Lambda_o$. It follows that $L + \mathcal{O}z_0$ is of type δ_{18}. A priori this lattice might be imprimitive, but it certainly does not contain a lattice of type δ_{15}.

We know that both $\mathcal{M}^K(I_6)$ and $\mathcal{M}^K(I_0^*)$ map to $d_\infty \in \mathcal{D}^*$. So they will map to distinct cusps of $\mathbb{B}^*_{o,\mathrm{U}(\Lambda_o)}$. Hence it is enough to show that $\mathcal{M}^K(I_6)$ maps to ∞_θ: then $\mathcal{M}^K(I_0^*)$ must necessarily map to the other cusp ∞_0. A similar argument as used for $\mathcal{M}^K(I_2)$ shows that a generic point of $\mathcal{M}^K(I_6)$ is mapped to cusp of $\mathbb{B}_o^*$ that is perpendicular to a sublattice $L \subset \Lambda_o$ isomorphic to Λ^5. Then $I := L \cap L^\perp$ is a primitive isotropic line whose image in $\mathbb{B}_o^*$ is the cusp in question. A primitive isotropic line of type (0) is not perpendicular to a lattice of type Λ^5, whereas one of type (θ) is. So $\mathcal{M}^K(I_6)$ maps to a cusp of type θ. □

From Corollary 2.6 we deduce a description of the Miranda compactification in terms of automorphic forms:

Theorem 9.4. *The graded $\mathbb{C}$-algebra of automorphic forms on $\mathbb{B}_o$ with values in a tensor power $\mathbb{L}(k)$ with arbitrary poles along the hyperball arrangement $\mathbb{B}_o(\mathcal{H}_o(6,9))$ is zero in negative degrees and of finite type. Its proj reproduces the Miranda compactifiation of $\mathbb{B}_{o,\mathrm{U}(\Lambda_o)} - \mathbb{B}_o(\mathcal{H}_o(6,9))_{\mathrm{U}(\Lambda_o)}$.*

This means that the hypersurface $\Delta(6) \cup \Delta(9)$ in $\mathbb{B}_{o,\mathrm{U}(\Lambda_o)}$ can never be the zero set of an automorphic form, since the inverse of such a form would produce an element of the above algebra of negative degree. This is in contrast with Δ itself (see [**2**]).

Remark 9.5. An intersection of mirror traces in $\mathbb{B}_o$ of d-invariant 6 or 9 is by definition the orthogonal complement of a positive definite sublattice $L \subset \Lambda$ spanned by z_o and 3-vectors of d-invariant 6 or 9. According to Proposition A.11 there are, apart from the mirror traces themselves, three types: $(6,9)$, $(9,9)$ and $(6,9,9)$, in which cases L is spanned by z_o and 3-vectors of the indicated d-invariant. It also follows from Proposition A.11 that each of these three types represents a single $\mathrm{U}(\Lambda_o)$-orbit. So these define irreducible subvarieties $\Delta(6,9)$, $\Delta(9,9)$, and $\Delta(6,9,9)$ of $\mathbb{B}^*_{o,\mathrm{U}(\Lambda_o)}$ of codimension 2, 2 and 3 respectively. Using Corollary 3.4 one identifies these subvarieties in $\mathcal{M}^*$ as $\mathcal{M}^*(I_8')$, $\mathcal{M}^*(I_8'')$ and $\mathcal{M}^*(I_7)$ respectively.

10. Modification of the Baily-Borel compactification.

Although this section is mostly of a descriptive nature, it may help to put our results into perspective: we outline an extension of the Baily-Borel theory which produces the compactifications obtained here in an algebro-geometrical setting in a canonical fashion. This is closely related to the construction described in [**21**].

10.1. Modifications defined by arrangements. Suppose we are given a complex manifold X of dimension n and a collection $\mathcal{H}$ of smooth hypersurfaces of X that is locally finite on X and is *arrangementlike*, in the sense that at each point of X there exist local analytic coordinates such that each $H \in \mathcal{H}$ passing through that point is given by a linear equation. Denote by $D = \cup_{H\in\mathcal{H}} H$ their union. There is a simple and straightforward way to find a modification $\tilde{X} \to X$ of X such that strict transforms of the members of $\mathcal{H}$ get separated: if $D^{(k)}$ denotes the union of the codimension k intersections of members of the $\mathcal{H}$, then first blow up $D^{(n)}$, then the strict transform of $D^{(n-1)}$, and so on, finishing with blowing up a strict transform of $D^{(2)}$:

$$X = \tilde{X}_n \leftarrow \tilde{X}_{n-1} \leftarrow \cdots \leftarrow \tilde{X}_1 = \tilde{X}.$$

If we denote the strict transform of H in $\tilde{X}_k$ by $\tilde{H}_k$, then the collection $\{\tilde{H}_k\}_{H\in\mathcal{H}}$ is also arrangementlike and has no intersections of codimension $> k$. In particular, the $\{\tilde{H}_1\}_{H\in\mathcal{H}}$ are disjoint. It is clear that the blowup is an isomorphism over $\Omega := X - D$.

LEMMA 10.1. *The morphism $\tilde{X} \to X$ is obtained by blowing up the fractional ideal $\sum_{H\in\mathcal{H}} \mathcal{O}_X(H)$.*

PROOF. Let k be the maximal integer for which $D^{(k)}$ is nonempty. So $\tilde{X}_k \to X$ is an isomorphism, but $\tilde{X}_{k-1} \to \tilde{X}_k$ is not. So $D^{(k)}$ is locally the intersection of k members of $\mathcal{H}$ in general position. From this it follows that the blowup of $\mathcal{I}^D$ factorizes over $\tilde{X}_{k-1}$. The pull-back of $\sum_{H\in\mathcal{H}} \mathcal{O}_X(H)$ to $\tilde{X}_{k-1}$ is up to a twist with a principal ideal equal to $\sum_{H\in\mathcal{H}} \mathcal{O}_{\tilde{X}_{k-1}}(\tilde{H}_{k-1})$. The lemma now follows with induction. $\square$

A case of interest is when X is the projective space $\mathbb{P}(V)$ of a complex vector space V. If $H \in \mathcal{H}$ is given by the linear form ϕ_H on V, then the blowup above is simply obtained as follows: consider the morphism $\Omega \to \mathbb{P}(\mathbb{C}^{\mathcal{H}})$ defined by $[z] \mapsto [(\phi_H(z)^{-1})_{H\in\mathcal{H}}]$ and take the closure of its graph in $\mathbb{P}(V) \times \mathbb{P}(\mathbb{C}^{\mathcal{H}})$.

Assume now that in this situation the collection $\mathcal{H}$ is nonempty and that the $H \in \mathcal{H}$ have no point in common (in other words, $\mathcal{H}$ contains a set of coordinate hyperplanes). Then the projection of $\tilde{\mathbb{P}}(V) \to \mathbb{P}(\mathbb{C}^{\mathcal{H}})$ is birational onto its image. That image can be regarded as a projective completion of the hyperplane complement Ω and we therefore denote it by $\hat{\Omega}$. (In case $V = \mathbb{C}^{n+1}$ and $\mathcal{H}$ consists of the set of coordinate hyperplanes, then the resulting birational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is the natural n-dimensional generalization of the standard Cremona transformation.) The variety $\hat{\Omega}$ comes with a natural stratification $\{\Omega(W)\}_W$ into smooth subvarieties. Here the index set runs over all linear subspaces $W \subset V$ with the property that $\mathbb{P}(W)$ is an intersection of members of $\mathcal{H}$. To be precise: $\Omega(W)$ is the image of Ω under the projection $\Omega \to \mathbb{P}(V/W)$. So it is in fact the hyperplane complement in $\mathbb{P}(V/W)$ defined by the collection of $H \in \mathcal{H}$ that pass through $\mathbb{P}(W)$.

The variety $\hat{\Omega}$ defined in the above example always exists as a locally compact Hausdorff space. If X is projective, then conditions can be specified under which $\hat{\Omega}$ will exist as a projective variety. Let us explain briefly how.

The connected components of the indecomposables of the Boolean algebra generated by the members of $\mathcal{H}$ define a stratification of X. This stratification is analytically locally trivial. In a similar fashion, the collection of irreducible components of the preimages of the members of $\mathcal{H}$ determine a stratification of $\tilde{X}$. The preimage of a stratum of X is a union of strata of $\tilde{X} \to X$ and it is easy to see that this preimage is trivial over the given stratum as a stratified variety. We consider now a somewhat coarser partition of $\tilde{X}$ whose members are indexed by the irreducible components of intersections of members of $\mathcal{H}$, in which we include the empty set as index (this will no longer be a stratification in general: the closure of a member need not be union of parts): if S is an irreducible component of some $D^{(k)}$ with $k \geq 1$, then let P_S be the closure of the preimage of $S - (S \cap D^{(k+1)})$ in $\tilde{X}$ minus the points that lie in the closure of the preimage of $D - D^{(k)}$ and $P_\emptyset$ will be the preimage of Ω. So the open member of this partition can be identified with Ω, and the closed members of this partition are the strict transforms of the members of $\mathcal{H}$. For $S \neq \emptyset$, the morphism $P_S \to S$ is trivial: P_S is then canonically a product $S \times \Omega(S)$, where $\Omega(S)$ is the complement of a hyperplane configuration in a projective space. This structure defines an equivalence relation on $\tilde{X}$: declare two points of $\tilde{X}$ to be equivalent if they are in the same member P_S of the partition and have the same image in $\Omega(S)$ (when $S = \emptyset$, read this as: have the same image in Ω). This equivalence relation is closed and the quotient space $\hat{\Omega}$ is locally compact Hausdorff.

If X is projective, and we seek to put a projective structure on $\hat{\Omega}$, then the above example suggests we look for a line bundle $\mathcal{L}$ on X with the property that the restriction of $\mathcal{L}$ to H is isomorphic to the normal bundle of H. Its pull-back to $\tilde{X}$ will then be trivial on the equivalence classes and so we would like that $\sum_{H\in\mathcal{H}} \mathcal{L}(-H)$ is generated by its sections and that these sections separate the equivalence classes on $\tilde{X}$. In fact, it would be enough to know that $\mathcal{L}$ restricted to H is isomorphic to a positive power n_H of the normal bundle of H and then we would ask the corresponding property for $\sum_{H\in\mathcal{H}} \mathcal{L}(-n_H H)$.

10.2. Intermediate modification of a cusp. We will look at an analogue of this situation in the case where X is a locally symmetric variety (a quotient of a bounded symmetric domain by an arithmetic automorphism group) and the hypersurfaces H are totally geodesic. We then also wish to understand what happens if we take the closure D^* of D in the Baily-Borel compactification $X \subset X^*$ and how the blowup over X extends across that compactification. The irreducible bounded symmetric domains admitting totally geodesic complex hypersurfaces are the domains of type IV (associated to a real orthogonal group of type $\mathrm{SO}(2, n)$) and the complex balls. Only the complex balls are relevant here, and as they are easier to deal with than the type IV domains, we concentrate on them.

So let us take up the situation of Section 6. It is known [**5**] that Γ has a neat subgroup of finite index (this means that this subgroup has the property that the subgroup of $\mathbb{C}^\times$ generated by the eigen values of its elements has no torsion). For the purposes of this discussion, there is no loss in generality when passing to such a subgroup and therefore we assume that Γ is neat from the start.

Let us now agree on a bit of notation. If $W \subset V$ is a degenerate positive subspace defined over K with radical I, then $\mathbb{B}$ is disjoint with $\mathbb{P}(W)$ and so the projection $\mathbb{P}(V) - \mathbb{P}(W) \to \mathbb{P}(V/W)$ is defined on $\mathbb{B}$. We denote the image by $\mathbb{B}(W)$ and the projection $\pi_W : \mathbb{B} \to \mathbb{B}(W)$. It is easy to see that $\mathbb{B}(W) = \mathbb{P}(V/W) - \mathbb{P}(I^\perp/W)$. So this is an *affine space* over $I^\perp/W$.

There is an evident factorization

$$\pi_W : \mathbb{B} \xrightarrow{\pi_I} \mathbb{B}(I) \longrightarrow \mathbb{B}(W).$$

The second projection is one of affine spaces. Let us explicate π_I.

Suppose $v = (z_0, z_1, z_2, \dots, z_n)$ are K-coordinates for V such that $I^\perp$ is defined by $z_0 = 0$ and ψ assumes the form

$$\psi(z, w) = z_0\bar{w}_n + z_n\bar{w}_0 + \sum_{i=1}^{n-1} z_i\bar{w}_i.$$

The intersection of the affine hyperplane defined by $z_0 = \psi(z, e_n) = 1$ with $\mathbb{L}$ projects isomorphically onto $\mathbb{B}$. This intersection is given by $\Re(-z_n) > \|z'\|^2$, where $z' = (z_1, \dots, z_{n-1})$. In terms of these coordinates the projection π_I is simply $(z', z_n) \mapsto z'$, and hence a fibration into left half planes, indeed. The topology near the cusp defined by I is easily described in these terms also: a neighborhood basis of this cusp intersected with $\mathbb{B}$ is the family of shifted fibrations defined by $\Re(-z_n) > \|z'\|^2 + a$ with a a positive constant. The boundary of such subset, in other words a fiber of the function $\Re(z_n) + \|z'\|^2$, is an orbit of the unipotent radical of the $\mathrm{U}(V)$-stabilizer of I. This unipotent radical is a Heisenberg group and is described in A.7. Since Γ is neat, the Γ-stabilizer of I, Γ_I, is contained in this Heisenberg group and is in fact a cocompact subgroup of it. So the center of Γ_I is infinite cyclic and acts faithfully by purely imaginary translations in the fibers of π_I, whereas the quotient of Γ_I by its center acts faithfully on the affine space $\mathbb{B}(I)$ as a lattice of maximal rank. Hence

$$\Gamma_I\backslash\mathbb{B} \to \Gamma_I\backslash\mathbb{B}(I)$$

is a punctured disc bundle whose base is a principal homogeneous space for the complex torus $\Gamma_I\backslash I^\perp/I$. The associated disc bundle can be understood as the Γ_I-orbit space of $\mathbb{B} \sqcup \mathbb{B}(I)$ endowed with a suitable topology with the bundle projection given by the obvious retraction

$$\Gamma_I\backslash(\mathbb{B} \sqcup \mathbb{B}(I)) \to \Gamma_I\backslash\mathbb{B}(I).$$

The associated line bundle over $\Gamma_I\backslash\mathbb{B}(I)$ has a Riemann form which is the negative of the form ψ induced on the translation space $I^\perp/I$. This implies that the dual of this line bundle is ample. So $\Gamma_I\backslash\mathbb{B}(I)$ can be contracted analytically in $\Gamma_I\backslash(\mathbb{B}\sqcup\mathbb{B}(I))$. The result of this contraction is that we added a singleton to $\Gamma_I\backslash\mathbb{B}$. This is the local model of the Baily-Borel compactification near the cusp attached to I (the added point is that cusp). The contraction mapping itself is the local model of a well-known (orbifold) resolution of the Baily-Borel compactification, one that apparently has the zero section $\Gamma_I\backslash\mathbb{B}(I)$ as exceptional divisor.

Any K-linear subspace $W \subset I^\perp$ which contains I defines an intermediate contraction and hence an intermediate modication of the cusp as follows. The image of W in $I^\perp/I$ defines a subtorus of $\Gamma_I\backslash I^\perp/I$. This subtorus gives rise to a torus fibration:

$$\Gamma_I\backslash\mathbb{B}(I) \to \Gamma_I\backslash\mathbb{B}(W).$$

That fibration is the restriction of a contraction

$$\Gamma_I\backslash(\mathbb{B}\sqcup\mathbb{B}(I)) \to \Gamma_I\backslash(\mathbb{B}\sqcup\mathbb{B}(W))$$

which leaves $\Gamma_I\backslash\mathbb{B}$ unaltered. It can be performed in the analytic category for the same reason as for the full contraction. So $W = I^\perp$ gives the Baily-Borel model and $W = I$ the natural resolution. We still have a natural retraction

$$\Gamma_I\backslash(\mathbb{B}\sqcup\mathbb{B}(W)) \to \Gamma_I\backslash\mathbb{B}(W)$$

and this retraction is locally analytically trivial.

10.3. Compactifications of arrangement type. Now let us return to the more specific situation of 6.2 (we continue to assume that Γ is neat). For every $H \in \mathcal{H}$, $\mathbb{B}(H)$ is totally geodesic subball of $\mathbb{B}$ and the collection of these is locally finite on $\mathbb{B}$. So

$$\mathbb{B}(\mathcal{H}) := \cup_{H\in\mathcal{H}}\mathbb{B}(H)$$

is closed in $\mathbb{B}$ and defines a closed analytic hypersurface $\mathbb{B}(\mathcal{H})_\Gamma$ of $\mathbb{B}_\Gamma$. This hypersurface is arrangementlike in the sense of 10.1 and hence determines a blowup $\tilde{\mathbb{B}} \to \mathbb{B}$. This blowup is Γ-invariant and hence defines a blowup $\tilde{\mathbb{B}}_\Gamma \to \mathbb{B}_\Gamma$ of orbit spaces. We explain how this blowup naturally extends across the Baily-Borel compactification. For every isotropic K-line $I \subset V$, let us denote by $I_\mathcal{H}$ the intersection of $I^\perp$ and the $H \in \mathcal{H}$ containing I. So $I \subset I_\mathcal{H} \subset I^\perp$, with $I_\mathcal{H} = I^\perp$ in case no $H \in \mathcal{H}$ passes through I. The preceding construction attaches to the collection $\{I_\mathcal{H}\}_I$ an intermediate modification of the cusps of $\mathbb{B}^*_\Gamma$. Let us denote this blowup

$$\mathbb{B}^\mathcal{H}_\Gamma \to \mathbb{B}^*_\Gamma.$$

Each member H of $\mathcal{H}$ passing through I defines an affine hyperplane in $\mathbb{B}(I_\mathcal{H})$ and hence an orbit in $\Gamma_I\backslash\mathbb{B}(I_\mathcal{H})$ under a complex subtorus of codimension one. The closure of the image of H in $\Gamma_I\backslash(\mathbb{B}\sqcup\mathbb{B}(I_\mathcal{H}))$ is the preimage of that orbit under the retraction of $\Gamma_I\backslash(\mathbb{B}\sqcup\mathbb{B}(I_\mathcal{H}))$ onto $\Gamma_I\backslash\mathbb{B}(I_\mathcal{H})$. In other words the closure of the divisor $\mathbb{B}(\mathcal{H})_\Gamma$ in $\mathbb{B}^\mathcal{H}_\Gamma$ is in an obvious sense locally trivial near the boundary of $\mathbb{B}_\Gamma$ in $\mathbb{B}^\mathcal{H}_\Gamma$. This implies that the normal crossing resolution of this divisor naturally extends across $\mathbb{B}^\mathcal{H}_\Gamma$ to give the sought for extension of the blowup:

$$\tilde{\mathbb{B}}^\mathcal{H}_\Gamma \to \mathbb{B}^\mathcal{H}_\Gamma \to \mathbb{B}^*_\Gamma.$$

The closure $\mathbb{B}(\mathcal{H})^*_\Gamma$ of $\mathbb{B}(\mathcal{H})_\Gamma$ in $\mathbb{B}^*_\Gamma$ is a hypersurface and the blowup above has the virtue that the strict transforms of the irreducible components of this hypersurface get separated. (This strict transform also supports an effective Cartier divisor.)

There is a topological contraction of the exceptional locus of $\tilde{\mathbb{B}}_\Gamma^{\mathcal{H}} \to \mathbb{B}_\Gamma^*$ which is of a very similar nature as our compactification of the hyperplane complement $\mathbb{P}(V) - D$ described in 10.1 (and is also related to the construction described in [**21**]): topologically it is gotten as the Γ-orbit space of a stratified extension $\hat{\Omega}$ of $\Omega := \mathbb{B} - \mathbb{B}(\mathcal{H})$ as a Γ-space. The strata $\Omega(W)$ of this extension are indexed by certain subspaces W of V: if $\mathcal{I}$ denotes the collection of K-hyperplanes of V that are isotropic, then W is an intersection of members of $\mathcal{H} \cup \mathcal{I}$. We require that W is *not* positive definite or what amounts to the same, that $\mathbb{P}(W) \cap \mathbb{B}^* \neq \emptyset$. The corresponding stratum $\Omega(W)$ is the image of Ω in $\mathbb{P}(V/W)$. If the algebra of Γ-automorphic forms on $\mathbb{B}$ with arbitrary poles along $\mathbb{B}(\mathcal{H})$ is zero in negative degrees and of finite type, then we believe that the proj of this algebra has $\hat{\Omega}_\Gamma$ as underlying topological space, thus endowing the latter with the structure of a projective variety that makes the contraction map $\tilde{\mathbb{B}}_\Gamma^{\mathcal{H}} \to \hat{\Omega}_\Gamma$ a morphism.

Almost all the compactifications we encountered in this paper appear to be of this type, as the following examples illustrate (proofs of these statements are omitted).

EXAMPLE 10.2. The Deligne-Mumford modification $\overline{\mathcal{D}} = S_{12} \backslash \overline{\mathcal{M}}_{0,12}$ of $\mathcal{D}^* = \mathcal{D}_{12}^*$ fits in a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbb{B}}_{\mathrm{U}(\Lambda)}^{\mathcal{H}} & \longrightarrow & \mathbb{B}_{\mathrm{U}(\Lambda)}^* \\ \Big\downarrow & & \Big\downarrow \\ \overline{\mathcal{D}} & \longrightarrow & \mathcal{D}^* \end{array}$$

with horizontal arrows the modifications, vertical arrows the inverse period mapping, and $\mathcal{H}$ the collection of mirrors (hyperplanes perpendicular to a 3-vector). The right vertical arrow is the Deligne-Mostow isomorphism. The left vertical arrow however is not an isomorphism, but it factors through a 'minimal' ball quotient modification relative to $\mathcal{H}$, which is a variation of the construction described in this section where only those subspaces in the intersection lattice of $\mathcal{H}$ are blown up for which the orthogonal sublattice is irreducible.

EXAMPLE 10.3. The modification

$$\mathcal{M}^{M*} \to \mathcal{M}^*$$

is via the period mapping identified with the modification

$$\tilde{\mathbb{B}}_{o,\mathrm{U}(\Lambda_o)}^{\mathcal{H}_o(6,9)} \to \mathbb{B}_{o,\mathrm{U}(\Lambda_o)}^*,$$

where $\mathcal{H}_o$ is the restriction of $\mathcal{H}$ above to the complexification of Λ_o and $\mathcal{H}_o(6,9) \subset \mathcal{H}_o$ the subcollection of hyperplanes of d-invariant 6 or 9.

In this case we have a contraction of the exceptional locus that gives the Miranda compactification of the $\mathrm{U}(\Lambda_o)$-orbit space of $\Omega := \mathbb{B}_o - \mathbb{B}_o(\mathcal{H}_o(6,9))$. The strata $\Omega(W)$ of the extension $\hat{\Omega}$ for which W has hyperbolic signature are listed in Proposition A.11. Using the obvious notation, we find the following cases:

(0) For a hyperplane W of d-invariant 6 resp. 9, $\Omega(W)$ is a singleton. This corresponds in $\mathcal{M}^M$ to the single isomorphism class of a rational elliptic surface with a I_9-fiber, resp. a I_4^*-fiber.
(1) For a codimension two intersection W of d-invariant $(6,9)$ resp. $(9,9)$ we get a one-dimensional stratum $\Omega(W)$ parametrizing rational elliptic surfaces of type I_8' (resp. I_8'').
(2) For a codimension three intersection W of d-invariant $(6,9,9)$ we get a two-dimensional stratum $\Omega(W)$ parametrizing rational elliptic surfaces with a I_7-fiber.

The maximal strata come from the cases when W is positive degenerate: if we take for W the intersection of all members of $\mathcal{H}_o(6,9)$ containing an isotropic line of type (θ) resp. (0), then $\Omega(W)$ is of dimension 3 resp. 1 and parametrizes rational elliptic surfaces with an I_6-fiber resp. I_0^*-fiber.

Appendix A. Unitary lattices over the Eisenstein ring

In this appendix we collect and prove some properties concerning the lattice Λ. We advise the reader first to browse through the text and then to consult it when the need arises.

The lattice Λ is among the lattices considered by Allcock in [**2**]. Let us begin with an observation implicit in his paper. Suppose L is a $\mathbb{Z}$-lattice equipped with an even symmetric bilinear form $(\ \cdot\ ) : L \times L \to \mathbb{Z}$ and an orthogonal automorphism τ of order 6 that has only *primitive* 6th roots of unity as eigen values (in other words, τ satisfies $\tau^2 - \tau + 1 = 0$). Then L becomes in an obvious manner a torsion free $\mathcal{O}$-module. Since $\mathcal{O}$ is a principal ideal domain, this module will be free also. We shall call the order 3 automorphism $-\tau$ a *triality* of L (for this notion naturally extends Cartan's use of that term—see below). A skew-hermitian $\mathcal{O}$-valued form ϕ on L is then defined by

$$\phi(x,y) := \omega(x \cdot y) - (x \cdot \tau y).$$

Using

$$\begin{aligned} 2(\tau x \cdot x) &= -((\tau x - 1) \cdot (\tau x - 1)x) + (\tau x \cdot \tau x) + (x \cdot x) \\ &= -(\tau^2 x \cdot \tau^2 x) + (\tau x \cdot \tau x) + (x \cdot x) = (x \cdot x), \end{aligned}$$

we see that $\phi(x,x) = \frac{1}{2}\theta(x \cdot x)$. So for the associated Hermitian form $\psi := -\theta\phi$ on L we have $\psi(x,x) = \frac{3}{2}(x \cdot x)$. In other words, $(\ \cdot\ ) = \frac{1}{3}(\psi + \bar{\psi})$.

A remarkable fact is that orthogonal reflections in L (relative to $(\ \cdot\ )$) determine certain unitary reflections relative to ψ: recall or note that any vector $r \in L$ with $(r \cdot r) = 2$ (a 'root') defines an orthogonal $\mathbb{Z}$-linear reflection in L that sends r to $-r$; likewise, the $\mathcal{O}$-linear transformation

$$s_r(x) := x - \omega^{-1}\phi(x,r)r.$$

is the identity on the ψ-orthogonal complement of r and since $\phi(r,r) = \theta$, it is immediate that s_r multiplies r by the third root of unity $-\omega$. So s_r is a unitary reflection in L of order 3, which is why such a transformation is called a *triflection*. Note that the triflections generate a normal subgroup $\mathrm{G}(L)$ of the unitary group $\mathrm{U}(L)$ of L.

Conversely, every finitely generated torsion free $\mathcal{O}$-module L equipped with a $\theta\mathcal{O}$-valued Hermitian form ψ (or equivalently, a $\mathcal{O}$-valued skew-hermitian form ϕ) so arises, reason for us to call such data an *$\mathcal{O}$-lattice*. The associated (anti-linear) map $x \in L \mapsto \phi(-,x) \in \mathrm{Hom}_{\mathcal{O}}(L,\mathcal{O})$ is bijective precisely when the underlying even symmetric bilinear form $(\cdot)$ is unimodular.

Let us call $x \in L$ an *n-vector* if $\psi(x,x) = n$ (so then 3 divides n). If a positive definite $\mathcal{O}$-lattice L is spanned by its 3-vectors, then the underlying even integral lattice decomposes canonically into an orthogonal sum of root lattices of type A_k, D_k or E_k. This decomposition is unique and hence respected by τ. Since τ cannot interchange summands (otherwise it would have eigen values of order 2 or 3), this decomposition is in fact one of $\mathcal{O}$-lattices. So the indecomposable cases must be of type A_{even}, $D_{\mathrm{even}\geq 4}$, E_6 and E_8. On the other hand, it is easy to see that a triality cannot exist inside the Weyl groups $W(A_k)$ or $W(B_k)$ for k even and at least 4. So the possible indecomposable $\mathbb{Z}$-lattices with a triality are of type A_2, D_4, E_6 and E_8. For example, a type D_4 root lattice admits a triality in $W(F_4)$ (which is in fact the automorphism group of the underlying $\mathbb{Z}$-lattice). By inspecting Carter's description of conjugacy classes in exceptional Weyl groups [**7**] we find that for a root lattice of type A_2, D_4, E_6 and E_8 a triality exists and is unique up to conjugacy. They can be gotten in a uniform manner as follows: let Λ^k be the $\mathcal{O}$-lattice with basis $r_1,\dots,r_k$, such that each r_i is a 3-vector, $\psi(r_i,r_{i+1}) = \theta$ for $i = 1,\dots k-1$ and $\psi(r_i,r_j) = 0$ when $j > i+1$. So Λ^{10} is the $\mathcal{O}$-lattice encountered in Section 4. One may verify that Λ^k is positive definite iff $k = 1,2,3,4$ and that in these cases the underlying root lattice are of type A_2, D_4, E_6, E_8 respectively. (For $k = 2$, we get the classical triality on D_4.) By means of Coxeter [**9**] we identify $G(\Lambda^k)$ in Shephard and Todd's Table VII in [**33**]. The associated triflection group $\mathrm{G}(\Lambda^k)$ appears there with number 4) for $k = 2$, 25) for $k = 3$ and 32) for $k = 4$. (The group $\mathrm{G}(\Lambda^3)$ is the Hesse group of symmetries of the Hesse pencil $\lambda(x^3+y^3+z^3)+\mu(xyz)$; $\mathrm{G}(\Lambda^4)$ is sometimes called the *Witting group*.)

A.1. The lattice Λ^4. The case E_8 is of particular interest: following [**2**], τ is then realizable as the 5th power of a Coxeter transformation. (A Coxeter transformation of such a root lattice has order 30 and its eigen values are the eight primitive 30th roots of unity.)

The 3-vectors of Λ^4 are the roots of the E_8-lattice, hence there are 240 of them. If we identify $\mathcal{O}/\theta\mathcal{O}$ with $\mathbb{F}_3$, then

$$\Lambda^4_{\mathbb{F}_3} := \mathbb{F}_3 \otimes_{\mathcal{O}} \Lambda^4 \cong \Lambda^4/\theta\Lambda^4$$

gets the structure of a vector space of dimension 4 over $\mathbb{F}_3$. The skew hermitian $\mathcal{O}$-valued form ϕ on Λ_4 induces a symplectic $\mathbb{F}_3$-valued form on $\Lambda^4_{\mathbb{F}_3}$. It turns out to be nondegenerate. There results a homomorphism

$$\mathrm{U}(\Lambda^4) \to \mathrm{Sp}(\Lambda^4_{\mathbb{F}_3}) \cong \mathrm{Sp}(4, \mathbb{F}_3)$$

which Allcock shows to be surjective with kernel the scalar subgroup μ_3. (Note that $\omega + 1$ is divisible by θ, so that ω acts as minus the identity in $\Lambda^4_{\mathbb{F}_3}$.) In particular, $U(\Lambda^4)$ is transitive on $\Lambda^4_{\mathbb{F}_3} - \{0\}$. He further observes that every nonzero element of $\Lambda^4_{\mathbb{F}_3}$ has in its preimage precisely three 3-vectors (a μ_3-orbit). Allcock uses this to prove:

Lemma A.1 ([**2**], Theorem 5.2). *The group* $\mathrm{U}(\Lambda^4)$ *acts transitively on the set of* 6*-vectors and on the set of* 3*-vectors in* Λ^4.

We shall further exploit this reduction to study 3- and 6-vectors in the Λ^4-lattice. We begin with noting that it remembers the relative position of the μ_6-orbits of two nonproportional 3-vectors r, r': the fact that these two span a positive definite lattice of rank two implies that $|\psi(r, r')| < 3$, and as $\psi(r, r')$ is divisible by θ, we have either $\psi(r, r') = 0$ or $\psi(r, r') \in \mu_6\theta$. This means that their images in $\Lambda^4_{\mathbb{F}_3}$ span an isotropic resp. nondegenerate rank two sublattice.

The description of the 6-vectors in terms of this reduction must be less straightforward, witness the fact that there are $80 \cdot 27$ 6-vectors and 80 nonzero elements in $\Lambda^4_{\mathbb{F}_3}$. The next lemma offers one such description.

Lemma A.2. *A* 6*-vector* $z \in \Lambda^4$ *can be written in exactly three ways as the sum of two* 3*-vectors* $z = r_1 + r_2$ *with* $\psi(r_1, r_2) = \theta$. *All such pairs* r_1, r_2 *span the same rank two sublattice* L_z *of* Λ^4. *The image of* L_z *in* $\Lambda^4_{\mathbb{F}_3}$ *is a nondegenerate plane and assigning to* z *the mod* θ *reduction of the pair* (z, L_z) *defines a bijection between the set of* μ_3*-orbits of* 6*-vectors in* Λ^4 *and the set of pairs* (v, P)*, where* $P \subset \Lambda^4_{\mathbb{F}_3}$ *is a nondegenerate plane and* $v \in P - \{0\}$.

Proof. Consider the set S of pairs of 3-vectors (r, r') in Λ^4 with $\psi(r, r') = \theta$. The mod θ reduction of a pair $(r, r') \in S$ is pair of vectors (v, v') in $\Lambda^4_{\mathbb{F}_3}$ with symplectic product 1. The number of such pairs of vectors is $80 \cdot 27$. The 3-vectors mapping to v are the elements of the μ_3-orbit of r and likewise for r'. So the preimage of (v, v') in S is the μ_3-orbit of the *pair* (r, r'). Hence S has $80 \cdot 27 \cdot 3$ elements. The image of the map $(r, r') \in S \mapsto r + r' \in \Lambda^4$ consists of 6-vectors, hence is the set of all 6-vectors, since it is $U(\Lambda^4)$-invariant. As there are $80 \cdot 27$ 6-vectors, we see that each 6-vector occurs precisely three times. If $(r, r') \in S$, then $(\omega r', r + (1 - \omega) r')$ and $((1 - \omega) r, \omega r + r')$ are two other elements of S with the same sum. So there are no more elements in S with that property. Hence the span of r and r' only depends on $r + r'$. All the assertions of the lemma now have been proved. □

Allcock's result says that mod θ reduction gives a bijective correspondence between the μ_6-orbits of 3-vectors and the lines $\ell \subset \Lambda^4_{\mathbb{F}_3}$. Lemma A.2 can be understood as asserting a similar relationship between the μ_6-orbits of 6-vectors and the flags (ℓ, P) in $\Lambda^4_{\mathbb{F}_3}$, where ℓ is a line in a nondegenerate plane P. Since symplectic geometry over a finite field is a priori a lot simpler than unitary geometry over the Eisenstein ring, such an interpretation is helpful when determining the relative position of a 3-vector and 6-vector in Λ^4. To see this, note that for a nondegenerate flag (ℓ, P) in $\Lambda^4_{\mathbb{F}_3}$ and a line ℓ' in $\Lambda^4_{\mathbb{F}_3}$ the following possibilities present themselves:

(a) $\ell' = \ell$,
(b) $\ell + \ell' = P$,
(c) $\ell' \not\subset P$, ℓ' not perpendicular to ℓ,
(d) $\ell' \not\subset P$, ℓ' perpendicular to ℓ but not to P,
(e) ℓ' perpendicular to P.

By elementary symplectic geometry, each of these cases represents a single orbit under the symplectic group. Let us see what this tells us about the relative position of a 3-vector r and a 6-vector z. From the preceding it follows that the unitary group of Λ^4 has precisely five orbits in the set of pairs of μ_6-orbits $(\mu_6.r, \mu_6.z)$. We give in each of the five cases above a representative example with $z = r_1 + r_2$ (so that $L_z = \mathcal{O}r_1 + \mathcal{O}r_2$ and hence P is the image of L_z in $\Lambda^4_{\mathbb{F}_3}$).

(a) $r = \omega^2 r_1 + r_2$ (so $\psi(r, z) = 0$),
(b) $r = r_1$ (so $\psi(r, z) = 3 + \theta$),
(c) $r = r_3$ (so $\psi(r, z) = -\theta$),
(d) $r = \omega r_2 + r_3$ (so $\psi(r, z) = 3$),
(e) $r = r_4$ (so $\psi(r, z) = 0$).

The case (a) is somewhat special: then r and z are perpendicular and span an imprimitive sublattice. We also see that the orthogonal complement of z in L_z is spanned by r. So any 3-vector with the same mod θ-reduction as z lies in L_z and spans with z a subgroup of finite index in L_z.

Corollary A.3. *Let $z = r_1 + r_2$ be the standard 6-vector in Λ^4. Then the set of those 3-vectors in Λ^4 which have a fixed nonzero Hermitian inner product with z make up a single $\mathrm{U}(\Lambda^4)_z$-orbit. The 3-vectors perpendicular to z span a lattice of type $\Lambda^1 \times \Lambda^2$, with basis $(\omega^2 r_1 + r_2, r_1 - \theta r_2 - 2r_3 + \theta r_4, r_4)$. A 3-vector perpendicular to z spans with z a primitive sublattice if and only if it belongs to the Λ^2-summand (hence any two such are in the same $\mathrm{U}(\Lambda^4)_z$-orbit).*

Proof. Let r' be a 3-vector in Λ^4 with $\psi(r', z) \neq 0$. It follows from the preceding that r' is $\mathrm{U}(\Lambda^4)_z$-equivalent to $\omega^i r$, with $i \in \mathbb{Z}/6$ and r a vector mentioned in one of the cases (b), (c), (d). In these cases the exponent $i \in \mathbb{Z}/6$ is determined by the inner product of r' with z. The last part of the corollary is straightforward. $\square$

In case (d), we have that $z-r$ is a 3-vector perpendicular to r. We shall need to know in how many ways z can be written as a sum of two perpendicular 3-vectors.

Corollary A.4. *A* 6*-vector* z *is written in exactly* 4 *distinct ways as a sum of two perpendicular* 3*-vectors in* Λ_4*. These vectors are orthogonal to the orthogonal complement of* z *in* L_z*, and so span with* z *a rank* 3 *sublattice of* Λ^4*. (For* $z=r_1+r_2$ *these sum decompositions are* $z=(z-r)+r$ *with* $r=\omega r_2+r_3$*,* $\omega r_2+r_3+\omega^{-1}r_4$*,* $\omega r_2+r_3+\omega^{-2}r_4$*,* $\omega r_1+2r_2-\theta r_3-r_4$*.)*

Proof. We begin with noting that a line ℓ' in $\Lambda^4_{\mathbb{F}_3}$ has property (d) if and only if it is the graph ℓ_f of a nonzero homomorphism $f:\ell\to P^\perp$. It is clear that there are 8 such lines. They come in 4 pairs: we have $\ell\subset\ell_f+\ell_{f'}$ if and only if $f+f'=0$. In that case $\ell_f+\ell_{f'}$ is isotropic and so $\ell_f,\ell_{f'}$ correspond to μ_6-orbits of 3-vectors that are perpendicular. There are unique 3-vectors $r_f,r_{f'}$ in these orbits with sum z. Let r be a 3-vector with the same mod θ-reduction as z. Since $\ell-\ell_f$ is isotropic, r is also orthogonal to r_f. We noted that r spans the orthogonal complement of z in L_z, and so the second assertion of the corollary follows. □

The following is proved in a similar fashion as A.3. The proof is in fact easier and so we omit it.

Lemma A.5. *Let* $r\in\Lambda^4$ *be a* 3*-vector. Then the stabilizer group* $\mathrm{U}(\Lambda^4)_r$ *acts transitively on the set of those* 3*-vectors in* Λ^4 *which have a fixed inner product with* r*.*

A.2. The lattice Λ**.** A *hyperbolic* $\mathcal{O}$*-lattice* is obtained as follows: let M be a free finitely generated $\mathcal{O}$-module. Regard M as $\mathbb{Z}$-module. The module $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})\oplus M$ has the natural quadratic form $q(\xi,x)=\xi(x)$ for which it is an unimodular $\mathbb{Z}$-lattice. Now let $\mathcal{O}$ act on M as before and on $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$ contragradiently. Then the preceding construction turns $\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Z})\oplus M$ into a nonsingular $\mathcal{O}$-lattice H_M. So $H_{\mathcal{O}}\cong\mathcal{O}^2$ with Hermitian form $\psi(z,w)=\theta(z_1\bar w_2-z_2\bar w_1)$. Notice that the skew-hermitian form $\phi=-\theta^{-1}\psi$ has discriminant 1. We shall denote the given basis of $H_{\mathcal{O}}$ by (e,f). Consider the lattice $\Lambda^4\perp\Lambda^4\perp H_{\mathcal{O}}$ and denote the first two summands Λ' and Λ'' (with basis $(r_i')_{i=1}^4$ resp. $(r_i'')_{i=1}^4$). We shall identify $\Lambda=\Lambda^{10}$ with $\Lambda'\perp\Lambda''\perp H_{\mathcal{O}}$ by means of the unitary isomorphism

$$(r_1,\dots,r_{10})\mapsto(r_1'',\dots,r_4'',s+e,\omega e+\theta f,\omega^{-1}e+r_1',r_2',r_3',r_4'),$$

where $s\in\Lambda''$ is characterized by the fact that it is perpendicular to r_1'',r_2'',r_3'' and $\psi(s,r_4'')=\theta$. This shows in particular that Λ has signature $(9,1)$, as asserted earlier. Notice that this isomorphism also identifies Λ^6 (the span of $r_1,\dots,r_6$) with $\Lambda''\perp H_{\mathcal{O}}$.

Allcock [**2**] proves that $\mathrm{U}(\Lambda)$ acts transitivily on the set of primitive 0-vectors of Λ, in other words, every primitive 0-vector of $\Lambda^4\perp\Lambda^4\perp H_{\mathcal{O}}$ can be transformed by a unitary transformation into e. We derive from this the corresponding statement for the set of 6-vectors:

PROPOSITION A.6. *Each 3-vector in Λ is perpendicular to a primitive null vector and the group $\mathrm{U}(\Lambda)$ acts transitively on the set of pairs (r,n) with r a 3-vector and n a primitive 0-vector perpendicular to r (in particular, $\mathrm{U}(\Lambda)$ is transitive on the set of 3-vectors in Λ).*

Similarly, each 6-vector in Λ is perpendicular to a primitive null vector. The group $\mathrm{U}(\Lambda)$ acts transitively on the set of 6-vectors, but has two orbits in the set of pairs (z,n) with z a 6-vector and n a primitive 0-vector perpendicular to z. These two orbits are represented by $(r_1'+r_2',e)$ (type (θ)) and $(r_1'+r_1'',e)$ (type (0)).

A.7. Before we begin the proof it is useful to make a few general observations. Let V be a finite dimensional complex vector space equipped with a skew-hermitian form ϕ. Let also be given a nonzero isotropic vector $e\in V$. For every $v\in V$ with $\phi(v,e)=0$ we define the transformation $T_{e,v}$ in V by

$$T_{e,v}(x)=x+\phi(x,e)v+\phi(x,v)e+\tfrac{1}{2}\psi(v,v)\psi(x,e)e$$

One checks that $T_{e,v}$ is unitary and fixes e. Its action in $e^\perp$ is simply given by $x\in e^\perp\mapsto x+\phi(x,v)e$. Notice that $T_{e,v}$ only depends on the image of v in $e^\perp/\mathbb{R}\sqrt{-1}e$. We have

$$T_{e,u}T_{e,v}=T_{e,u+v+\frac{1}{2}\phi(v,u)e}.$$

These transformations make up the unipotent radical of the stabilizer of e in the unitary group $\mathrm{U}(V)$. It is a Heisenberg group with center the transformations $T_{e,\lambda e}$ with λ real. Suppose that $L\subset V$ is a discrete $\mathcal{O}$-submodule in V of maximal rank such that ϕ takes on $L\times L$ values in $\mathcal{O}$. If e and v lie in L and $\phi(v,v)$ is even, then clearly $T_{e,v}$ preserves L. So if $x\in e^\perp\cap L$, then $x+\mathcal{O}e$ is contained in a $U(L)_e$-orbit if $\phi(x,v)=1$ for some $v\in L\cap e^\perp$ with $\phi(v,v)$ even. Or what amounts to the same, if $v\in L\cap e^\perp$ with $\psi(x,v)=-\theta$ and $\psi(v,v)\in 6\mathbb{Z}$.

PROOF OF A.6. We only prove the statements involving a 6-vector, the proof of the one about a 3-vector is similar and easier. We begin with the last clause. Let (z,n) be as in the proposition. By Allcock's result, a unitary transformation will map this into a pair with second component e and so we may assume that $n=e$. Then z can be written $x'+x''+\lambda e$ with $x'\in\Lambda'$, $x''\in\Lambda''$ and $\lambda\in\mathcal{O}$. We must have $\psi(x',x')+\psi(x'',x'')=6$. Since the two terms must be nonnegative multiples of three they are $(6,0)$, $(3,3)$ or $(0,6)$. The stabilizer of e contains the interchange of Λ' and Λ'' as well as the unitary group of each of these summands. So we can eliminate the last case and by A.1 assume that $(x',x'')=(r_1'+r_2',0)$ or $(x',x'')=(r_1',r_1'')$. In either case, there exists a 6-vector $v\in\Lambda'$ with $\psi(z,v)=\theta$ and so by the discussion (A.7) there exists a unitary transformation fixing e that sends z to $x'+x''$. The last assertion follows.

We next show that any 6-vector z is perpendicular to a primitive null vector. The orthogonal complement Λ_z of z is a free $\mathcal{O}$-module of signature $(8,1)$. So its complexification $\mathbb{C}\otimes_{\mathcal{O}}\Lambda_z=\mathbb{R}\otimes_{\mathbb{Z}}\Lambda_z$ represents zero.

Its real dimension is ≥ 5 and a theorem of Meyer [**31**] then implies that Λ_z also represents zero. In other words, there exists a null vector perpendicular to z.

It remains to see that $r_1' + r_2'$ and $r_1' + r_1''$ are in the same $U(\Lambda)$-orbit. This is left to the reader. □

A.3. The lattice Λ_o. Let us now fix a sublattice $\Lambda_o \subset \Lambda$ that is the orthogonal complement of a 6-vector $z_o \in \Lambda$. In view of A.6 all such sublattices are unitary equivalent.

PROPOSITION A.8. *The* $\mathrm{U}(\Lambda)$*-stabilizer of* Λ_o *maps isomorphically to the unitary group* $\mathrm{U}(\Lambda_o)$ *of* Λ_o*.*

The proof is a modification of a standard argument in lattice theory. In order to make it transparent we begin with a general discussion. Given an $\mathcal{O}$-lattice L, let us simply write L^* for $\mathrm{Hom}_{\mathcal{O}}(L, \mathcal{O})$. The skew-hermitian form $\phi_L := -\theta^{-1}\psi_L$ on L induces an antilinear map $a_L : L \to L^*$, $x \mapsto \phi(\ ,x)$. Suppose that ϕ_L is nondegenerate (i.e., has nonzero discriminant). Then a_L maps L bijectively onto a sublattice of L^* of finite index, so that $C(L) := L^*/a_L(L)$ is a finite $\mathcal{O}$-module. The order of $C(L)$ is then the square absolute value of the discriminant of L. For instance, if L is spanned by a $3n$-vector, then $C(L) \cong \mathcal{O}/(n\theta)$, which has indeed order $3n^2$. The form ϕ_L determines a skew-hermitian form ϕ_{L^*} on L^* such that $\psi_{L^*}(a_L(x), a_L(y)) = \psi_L(y, x)$. This form now takes values in the field $\mathbb{Q}(\omega)$. If however one of its arguments lies in the image of a_L, then it takes values in $\mathcal{O}$. So ψ_{L^*} induces a skew-hermitian form $\phi_{C(L)} : C(L) \times C(L) \to \mathbb{Q}(\omega)/\mathcal{O}$. It is clear that every unitary transformation of L induces a unitary transformation in $C(L)$.

Suppose now L of discriminant ± 1 and let $M \subset L$ be a primitive nondegenerate submodule with orthogonal complement N. So $M \perp N$ sits in L as a submodule of finite index. Composing a_L with restriction to $M \perp N$ gives an embedding of $L/(M+N)$ in $C(M \perp N) = C(M) \perp C(N)$. This image is isotropic for the skew-hermitian $\mathbb{Q}(\omega)/\mathcal{O}$-valued form on $C(M) \perp C(N)$. Since L has discriminant ± 1, it is a maximal sublattice in $\mathbb{Q} \otimes_{\mathbb{Z}} L$ on which ϕ is $\mathcal{O}$-valued, an so its image in $C(M) \perp C(N)$ is maximally isotropic. It is clear that the projection of this image in either summand is a bijcetion. In other words, the image is the graph of an isomorphism $\alpha : C(M) \cong C(N)$ which changes the sign of the forms.

It is clear that an automorphism of $M \perp N$ preserves L if and only if is preserves the image of L in $C(M) \perp C(N)$. So a pair of unitary transformations of $M \perp N$ of the form (u_M, u_N) preserves L if and only if α commutes with the unitary transformations in $C(M)$ and $C(N)$ induced by u_M and u_N.

PROOF OF A.8. We apply this to the case at hand: $L = \Lambda$, $M = \Lambda_o$ and N spanned by the 6-vector z_o. Then $C(\Lambda_o) \cong C(\mathcal{O}z_o) \cong \mathcal{O}/(2\theta)$, where in the latter case the skew form takes the value $\frac{1}{2}\theta^{-1}$ on a generator. One easily verifies that the group of unitary transformations of $\mathcal{O}/(2\theta)$ is μ_6. As this is also the group of unitary transformations of $\mathcal{O}z_o$, it follows that every unitary transformation of Λ_o extends uniquely to unitary transformation of Λ. □

In order to classify the 3-vectors in Λ relative to z_o, we first consider the abstract $\mathcal{O}$-lattices spanned by a 3-vector and a 6-vector.

LEMMA A.9. *Let L be a positive definite $\mathcal{O}$-lattice of rank two spanned by a 6-vector z and 3-vectors. Then we are in one of the following four cases: L has a basis (e_1, e_2) such that*

(δ_6) $z = e_1 + e_2$ *and ψ has the matrix* $\left(\begin{smallmatrix} 3 & \theta \\ \bar\theta & 3 \end{smallmatrix}\right)$ *so that L has discriminant* 6 *or*

(δ_9) $z = e_1 + e_2$ *and ψ has the matrix* $\left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ *so that L has discriminant* 9 *or*

(δ_{15}) $z = e_1$ *and ψ has the matrix* $\left(\begin{smallmatrix} 6 & \theta \\ \bar\theta & 3 \end{smallmatrix}\right)$ *so that L has discriminant* 15 *or*

(δ_{18}) $z = e_1$ *and ψ has the matrix* $\left(\begin{smallmatrix} 6 & 0 \\ 0 & 3 \end{smallmatrix}\right)$ *so that L has discriminant* 18.

Moreover, is $M \supset L$ a rank two $\mathcal{O}$-lattice that strictly contains L, then we are in case δ_{18} and M is isomorphic to the lattice of case δ_6.

PROOF. Suppose first that L is spanned by the 6-vector z and a 3-vector r. We have $\psi(z, r) = \theta u$ for some $u \in \mathcal{O}$. Since L is positive definite, we must have $|u|^2 < 6$. Since $u \in \mathcal{O}$, this implies that up to a unit u equals 0, 1, θ or 2. By multiplying r with a unit we may assume that u acually equals one of these values. For $u = 0$ we get case δ_{18}, and for $u = 1$ we get case δ_{15}. For $u = \theta$ we get case δ_9 by taking $(e_1, e_2) = (z + r, -r)$ and for $u = 2$ we get case δ_6 by taking $(e_1, e_2) = (z - \omega r, \omega r)$.

For the last part of the lemma, we observe that for an overlattice $M \supset L$ we must have that the quotient of the discriminant of M by the discriminant of L must be the norm of an element of $\mathcal{O}$. Since the discriminant of M is also divisible by 3, this implies that L is of type δ_9 or δ_{18}. The case δ_9 has as underlying integral lattice a root lattice of type $A_1 \perp A_1$. This admits no even overlattice and hence cannot occur. There remains the case that L is of type δ_{18} with M of discriminant 6. It is then not hard to see that M is as asserted.

□

If $r \in \Lambda$ is a 3-vector, which together with z_o spans a primitive positive definite sublattice of Λ, then according to Lemma A.9, the discriminant of this sublattice can take 4 values : 6, 9, 15 or 18. We call this value the *d-invariant* of r. Proposition A.6 shows that the primitive isotropic lines $I \subset \Lambda_o$ come in two types (types (0) and (θ)) and that each type is represented by a single $\mathrm{U}(\Lambda_o)$-orbit.

PROPOSITION A.10. *Let $I \subset \Lambda_o$ be a primitive isotropic line and denote by $I(6)$ resp. $I(9)$ the span of I and the 3-vectors $r \in I^\perp$ with d-invariant 6 resp. 9. Then:*

(θ) *If I is type (θ), then $I(6)/I$ and $I(9)/I$ are perpendicular sublattices of $I^\perp/I$ of rank 1 and 2 respectively. Moreover, there are precisely 4 rank one sublattices of $I^\perp/I$ spanned by the image of a 3-vector in $I^\perp$ of d-invariant 9.*

(0) *If I is type* (0)*, then* $I(6) = I$ *and* $I(9)/I$ *is of rank* 1.

PROOF. By Proposition A.6 we may assume that I is spanned by e and that $z_o = r_1' + r_2'$ in case (θ) and $z_o = r_1' + r_1''$ in case (0). This identifies $I^\perp/I$ with $\Lambda' \perp \Lambda''$. A 3-vector in $I^\perp$ maps to a 3-vector in $I^\perp/I \cong \Lambda' \perp \Lambda''$ of the same d-invariant and the 3-vector of $\Lambda' \perp \Lambda''$ lies in Λ' or in Λ''.

In case (θ) it is clear that any 3-vector in Λ'' has d-invariant 18, so if we are after the 3-vectors of d-invariant 6 or 9, then we only have to deal with Λ'. Assertion (θ) then follows from Lemma A.2 and Corollary A.4.

Case (0) follows from the simple observation that $r_1' + r_1''$ cannot be written in any other way as a sum of two 3-vectors in $\Lambda' \perp \Lambda''$. $\square$

PROPOSITION A.11. *Let $L \subset \Lambda$ be a primitive sublattice containing z_o. Then $L^\perp$ is isomorphic to an orthogonal product of lattices $\Lambda^{k_1} \perp \Lambda^{k_2} \perp \cdots$ with $k_1 \geq 6$ if and only if L is spanned by z_o and 3-vectors of d-invariant 6 and 9 and we are then in one of the following cases:*

(6) $(L, z_o) \cong (\Lambda^2, r_1 + r_2)$*,* $L^\perp \cong \Lambda^8$ *and L is spanned by z_o and a 3-vector of d-invariant 6,*

(9) $(L, z_o) \cong (\Lambda^1 \perp \Lambda^1, r + r')$*,* $L^\perp \cong \Lambda^7 \perp \Lambda^1$ *and is spanned by z_o and a 3-vector of d-invariant 9,*

(6, 9) $(L, z_o) \cong (\Lambda^3, r_1 + r_2)$*,* $L^\perp \cong \Lambda^7$ *and L is spanned by z_o and two 3-vectors of d-invariant $6, 9$.*

(9, 9) $(L, z_o) \cong (\Lambda^3, r_1 + r_3)$*,* $L^\perp \cong \Lambda^7$ *and L is spanned by z_o and two 3-vectors of d-invariant $9, 9$.*

(6, 9, 9) $(L, z_o) \cong (\Lambda^4, r_1 + r_2)$*,* $L^\perp \cong \Lambda^6$ *and L is spanned by z_o and three 3-vectors of d-invariant $6, 9, 9$.*

Each of these possibilities respresents a single $\mathrm{U}(\Lambda_o)$-equivalence class and this is also the complete list of $\mathrm{U}(\Lambda_o)$-equivalence classes of positive definite sublattices of Λ spanned by z_o and 3-vectors of d-invariant 6 and 9.

PROOF. Let us first assume that $L^\perp$ is isomorphic to an orthogonal product $\Lambda^{k_1} \perp \Lambda^{k_2} \perp \cdots$ with $k_1 \geq 6$. Since $\Lambda \cong \Lambda^6 \perp \Lambda^4$ and $\Lambda^{k_1} \cong \Lambda^6 \perp \Lambda^{k_1 - 6}$, we see that it is enough to investigate the corresponding issue in Λ^4. The 6-vectors in Λ^4 are all unitary equivalent, and so we can assume that $z = r_1 + r_2$. The assertions regarding the classification now follow from Corollary A.3.

Assume now that $L \subset \Lambda$ is a positive definite sublattice and spanned by z_o and 3-vectors of d-invariant 6 and 9. Assume also that its rank is ≤ 5. Then the orthogonal complement of the lattice L is hyperbolic of sufficiently high rank and so by Meyer's theorem contains a primitive null vector. We may assume that this null vector is e and that z_o is either $r_1' + r_2'$ or $r_1' + r_1''$. So L projects isomorphically to a sublattice $\bar{L} \subset \Lambda' \perp \Lambda''$ spanned by 3-vectors. Since the 3-vectors helping to span L are of d-invariant 6 or 9, A.10 implies that $\bar{L} \subset \Lambda'$ when $z_o = r_1' + r_2'$ and $\bar{L} \subset \mathcal{O}r_1' + \mathcal{O}r_1''$

when $z_o = r_1' + r_1''$. In particular, L is of rank ≤ 4. All the assertions now follow in a straightforward manner from A.3, A.10 and A.7. $\square$

PROPOSITION A.12. *The 3-vectors in Λ of fixed d-invariant form a single orbit under the action of $U(\Lambda_o)$.*

PROOF. For $d = 6$ or 9 this is part of the statement of the previous proposition. The cases $d = 15$ and $d = 18$ are handled in a similar way. $\square$

References

[1] D. Abramovich, A. Vistoli: *Complete moduli for fibered surfaces*, in: Recent Progress in Intersection Theory, Proc. Intern. Conf. on Intersection Theory, Bologna 1997, G. Ellingsrud, W. Fulton, A. Vistoli (eds.), Birkhäuser 2000, see also `arXiv.org/abs/math.AG/9804097`, `/math.AG/9811059`, `/math.AG/9908167`.

[2] D. Allcock: *The Leech Lattice and Complex Hyperbolic Reflections*, Invent. Math. 140 (2000), 283–301.

[3] D. Allcock, J.A. Carlson, D. Toledo: *The complex hyperbolic geometry for moduli of cubic surfaces*, 55 pp., `arXiv.org/abs/math/0007048`, see also *A complex hyperbolic structure for moduli of cubic surfaces*, C. R. Acad. Sci. Paris, t. 326, Série I (1998), 49–54.

[4] W. Barth, C. Peters, A. Van de Ven: *Compact Complex Surfaces*, Erg. der Math. u. i. Grenzgebiete, 3e Folge, 4, Springer-Verlag, Berlin etc. 1984.

[5] A. Borel: *Introduction aux groupes Arithmétiques* Hermann, Paris 1969.

[6] N. Bourbaki: *Groupes et algèbres de Lie, Ch. 4,5,6*, Masson, Paris 1981.

[7] R.W. Carter: *Conjugacy classes in Weyl groups,* in: Seminar on algebraic groups and related finite groups, Lect. Notes in Math. 131, Springer-Verlag, Berlin etc. 1970.

[8] W. Couwenberg: *Complex Reflection Groups and Hypergeometric Functions*, Thesis (123 p.), Katholieke Universiteit Nijmegen, 1994.

[9] H.S.M. Coxeter: *Finite groups generated by unitary reflections*, Abh. Math. Sem. Hamburg 31 (1967), 125–135.

[10] P. Deligne, G.D. Mostow: *Monodromy of hypergeometric functions and non-lattice integral monodromy*, Publ. Math. IHES 63 (1986), 58–89.

[11] P. Deligne, G.D. Mostow: *Commensurabilities among lattices in* $\mathrm{PU}(1, n)$, Ann. of Math. Study 132, Princeton U.P., Princeton 1993.

[12] M. Demazure: *Surfaces de Del Pezzo, I-V.* In: Séminaire sur les Singularités des Surfaces, Lecture Notes in Math. 777, Springer-Verlag Berlin etc. 1980.

[13] I. Dolgachev: *Rationality of fields of invariants*, in: Algebraic Geometry Bowdoin 1985 Proc. Symp. Pure Math. 46 Part 2, 3–16. AMS, Providence (RI) 1987.

[14] I. Dolgachev, D. Ortland: *Point sets in projective spaces and theta functions* Astérisque 165 (1988), Soc. Math. de France.

[15] E. Fadell, J. van Buskirk: *The braid groups of E^2 and S^2*, Duke Math. J. 29 (1962), 243–258.

[16] S. Kondo: *A complex hyperbolic structure on the moduli space of curves of genus three*, J. reine u. angew. Math. 525 (2000) 219–232.

[17] S. Kondo: *The moduli space of curves of genus 4 and Deligne-Mostow's complex reflection groups*, preliminary version (2000), 10 pp.

[18] M. Kontsevich: *Enumeration of rational curves via torus actions*, in: The Moduli of Curves, 335–368, R. Dijkgraaf, C. Faber, G. van der Geer eds., Progress in Math. 129, Birkhaüser (1995).

[19] Yu.I. Manin: *Cubic Forms*, 2nd ed., North-Holland Math. Library, Elsevier Sc. Pub. 1986.

[20] E. Looijenga: *Rational surfaces with an anti-canonical cycle*, Ann. of Math. 114 (1981), 267–322.

[21] E. Looijenga: *New compactifications of locally symmetric varieties*, in: Proceedings of the 1984 Conference in Algebraic Geometry, 341–364, J. Carrell, A.V. Geramita, P. Russell eds. CMS Conference Proceedings, vol. 6, Amer. Math. Soc. Providence RI (1984).

[22] I. Macdonald: *Affine root systems and Dedekind's η-function*, Invent. Math. 15 (1972), 91–143.

[23] R. Miranda: *On the stability of pencils of cubic curves*, Amer. J. Math. 102 (1980), 1172–1202.

[24] R. Miranda: *The Moduli of Weierstrass Fibrations over* $\mathbb{P}^1$, Math. Ann. 255 (1981), 379–394.

[25] R. Miranda: *Persson's list of singular fibers for a rational elliptic surface*, Math. Z. 205 (1990), 191–211.

[26] I. Morrison, U. Persson: *Numerical sections on elliptic surfaces*, Comp. Math. 59 (1986), 323–337.

[27] G.D. Mostow: *Generalized Picard lattices arising from half-integral conditions*, Publ. Math. IHES 63 (1986), 91–106.

[28] D. Mumford, J. Fogarty, F. Kirwan: *Geometric Invariant Theory*, 3rd ed., Erg. der Math. u. i. Grenzgebiete 34, Springer-Verlag Berlin etc. 1994.

[29] U. Persson: *Configurations of Kodaira fibers on rational elliptic surfaces*, Math. Z. 205 (1990), 1–47.

[30] F. Pham: *Formules de Picard-Lefschetz généralisées et ramification des intégrales*, Bull. Soc. Math. de France 93 (1965), 333–367.

[31] J.-P. Serre: *Cours d'Arithmétique*, Presses Univ. de France, Paris 1970.

[32] H. Sterk: *Compactifications of the period space of Enriques surfaces I, II*, Math. Z. 207 (1991), 1–36 and Math. Z. 220 (1995), 427–444.

[33] G.C. Shephard, J.A. Todd: *Finite unitary reflection groups*, Canadian J. Math. 6 (1954), 274–304.

[34] R. Vakil: *Twelve points on the projective line, branched covers, and rational elliptic fibrations.* To appear.

[35] E.B. Vinberg: *Hyperbolic reflection groups*, Russian Math. Surveys 40 (1985), 31–75.

Faculteit Wiskunde en Informatica, Catholic University of Nijmegen, Postbus 9010, NL-6500 GL Nijmegen, The Netherlands

E-mail address: `heckman@sci.kun.nl`

Faculteit Wiskunde en Informatica, University of Utrecht, Postbus 80.010, NL-3508 TA Utrecht, The Netherlands

E-mail address: `looijeng@math.uu.nl`