Radboud University Nijmegen

Exam Symplectic Geometry
April 8, 2011, 14-17, Room HG03.054, Nijmegen

Give precise arguments and answers. Make your reasoning as clear as possible.

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1. Let (M, ω) be a symplectic manifold.
   (a) Give the definition of a compatible almost complex structure J on (M, ω).
   (b) Show that for each (M, ω) a compatible almost complex structure always exists.
   (c) When is (M, ω, J) called a Kähler manifold?
   (d) Show that for (V, h) a Hermitian vector space the projective space $\mathbb{P}(V)$ has a natural Kähler structure.
   (e) Show that a smooth complex projective manifold is always Kähler.

2. Let (M, ω) be a symplectic manifold.
   (a) Give the definition of the Hamilton vector field v_f of a smooth function f on M.
   (b) Define the Poisson bracket $\{f, g\}$ of two smooth functions f and g on M.
   (c) Show that $[v_f, v_g] = v_{\{f,g\}}$ for any two smooth functions f and g on M.
   (d) Derive the Jacobi identity for the Poisson bracket.
   (e) What was the motivation of Jacobi to derive this "Jacobi identity"?

3. Let (M, ω) be a symplectic manifold of dimension $2n$.
   (a) When is a set of n smooth functions $(f_1, \cdots, f_n)$ on M called an integrable system.
   (b) Let $f = (f_1, \cdots, f_n) : M \to \mathbb{R}^n$ be an integrable system with $f(M) = \mathcal{R} \sqcup \mathcal{D}$ the disjoint union of the locus of regular values $\mathcal{R}$ and of discriminant $\mathcal{D}$. Under the assumption that $f : M \to \mathbb{R}^n$ is proper and has connected fibers show that $M_c = f^{-1}(c)$ for $c \in \mathcal{R}$ has the structure of a Lagrangian torus.
   (c) What are action-angle coordinates in this situation?
   (d) Explain the definition of the monodromy representation of the fundamental group $\Pi_1(\mathcal{R}, c)$

4. Suppose (M, ω) is a connected symplectic manifold, and $G \times M \to M$ is a Hamiltonian action of a connected Lie group G with moment map $\mu : M \to \mathfrak{g}^*$. Give the definitions of the phrases Hamiltonian action and moment map. Explain the proces of symplectic reduction. Suppose H is a Hamiltonian (so a smooth function) on M that is invariant under G. What is the reduced Hamiltonian H_ξ at a strongly regular value $\xi \in \mathfrak{g}^*$?