Quantum Probability as a Tensor Category

Hans Maassen

QPL Nijmegen, October 28 2011.

Collaboration with:

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Karol Życzkowski (Krakow)

References

The references below are also downloadable from my homepage:
http://www.math.ru.nl/ maassen/

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- Show that this description duely regards classical information as a special case of quantum information;
- Develop a 'DIAGRAMMAR' for this category, unifying Joyal-type diagrams with Nielsen-Chuang-type diagrams.
- Illustrate several features of quantum information.

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- The 'no cloning' theorem;
- The Heisenberg principle of 'no information without perturbation';
- A consequence: the equivalence of protection and darkness for Hilbert subspaces (the Knill-Laflamme condition);
- Entanglement assisted teleportation, including the vital classical message.
- Quantum trajectories and their asymptotic properties (if time permits).

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If someone tells us in what state the system resides, we learn something. This information is carried by the object.

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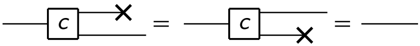

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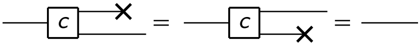

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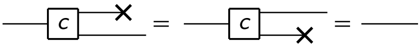

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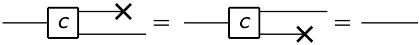

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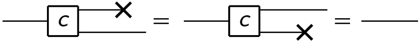

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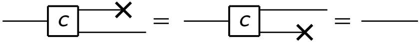

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We see here that cocopiers take a simpler form than copiers. This is a general phenomenon: the "Heisenberg picture" behaves better.

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In quantum mechanics one uses the terms Schrödinger picture (forward maps) and Heisenberg picture (backward maps).

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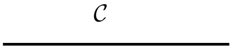

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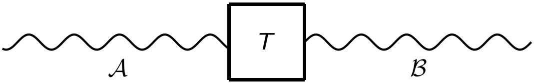

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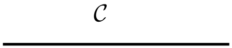

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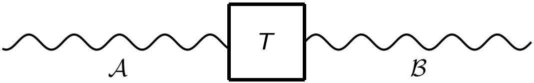

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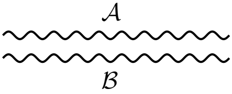

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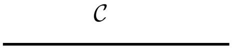

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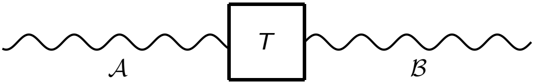

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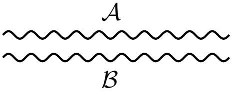

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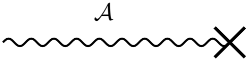

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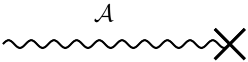

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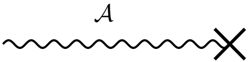

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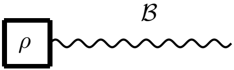

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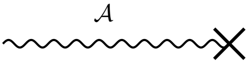

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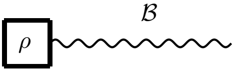

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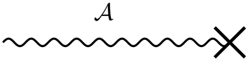

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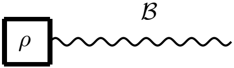

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Hence it is a good thing that QProb has a terminal object, but no initial one.

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Since the information concerns incompatible observables simultaneously, it cannot be copied (as we shall see).

By the Kochen-Specker theorem there is not even a consistent way to answer all the questions that can be asked about them. Yet every bit of it can (possibly) be checked with information elsewhere (in case of entanglement.)

In this case ANY, but not ALL questions can be answered.

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(Multiplication Theorem) *If Cauchy-Schwartz holds with equality, then b is multiplicative: i.e., for all $x \in \mathcal{B}$ we have:*

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Quantum information cannot be copied. We repeat the definition of a copier in the language of *-Alg:

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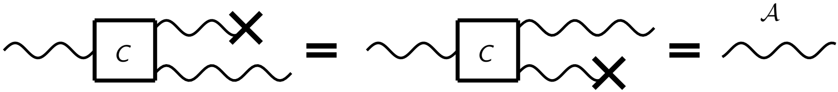

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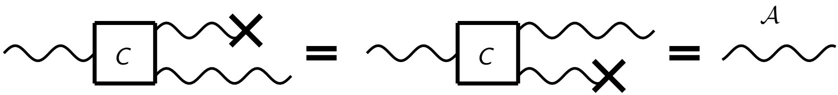

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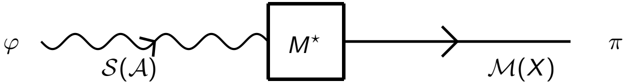

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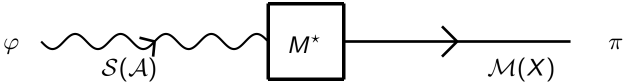

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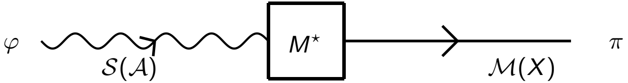

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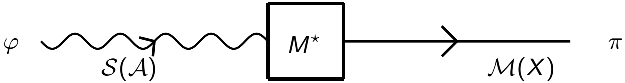

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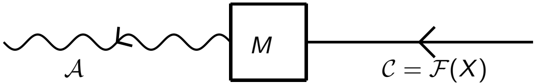

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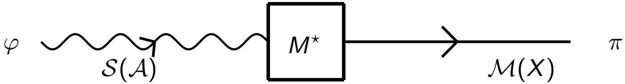

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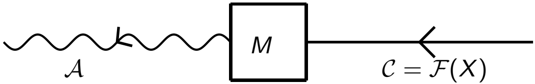

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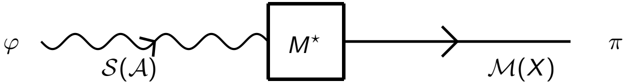

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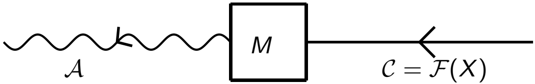

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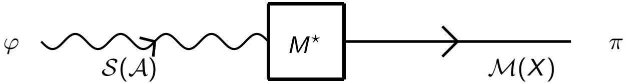

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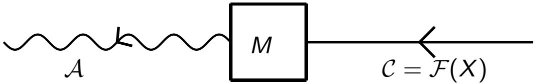

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Positive Operator Valued Measure (POVM)

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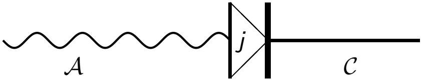

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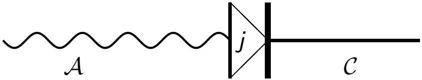

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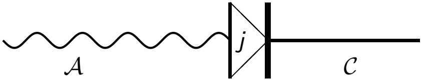

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We conclude that a von Neumann measurement is a right-invertible morphism in QProb from an arbitrary object to an abelian object.

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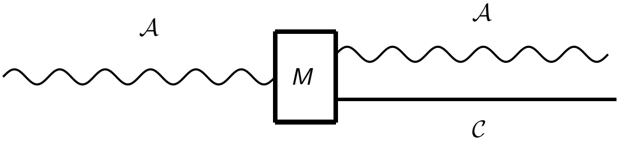

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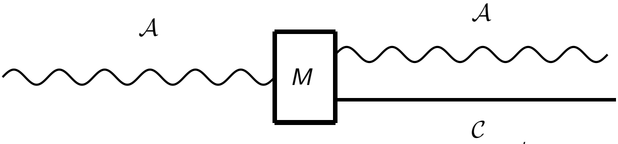

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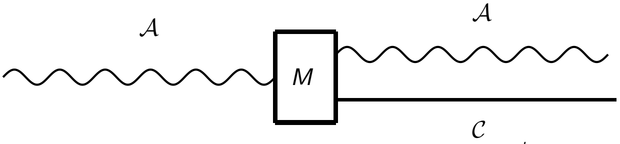

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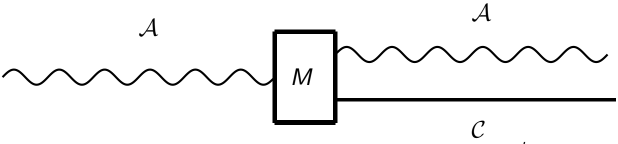

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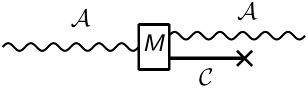

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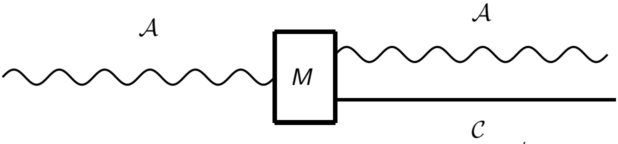

$$M : \mathcal{A} \otimes \mathcal{C} \to \mathcal{A} : \quad M(a \otimes f) = \sum_{i=1}^{k} f(i) T_i(a) \; .$$

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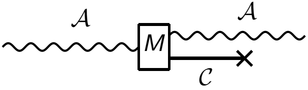

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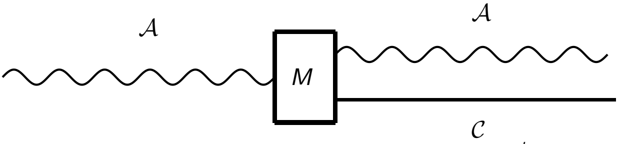

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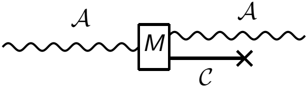

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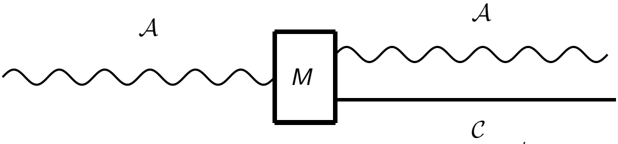

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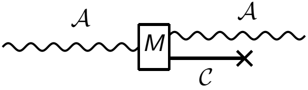

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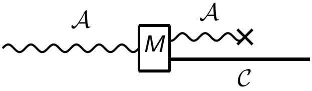

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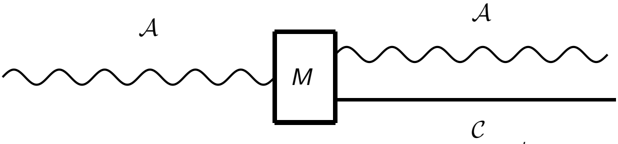

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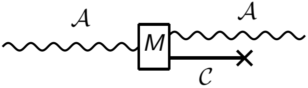

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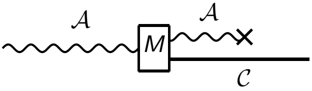

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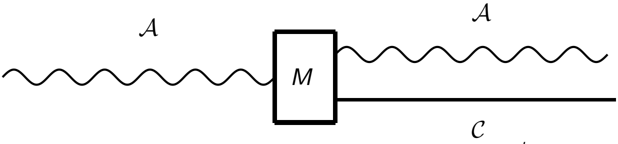

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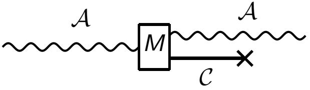

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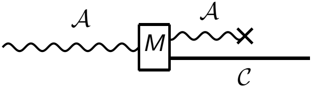

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No Information Without Disturbance

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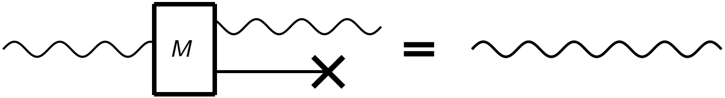

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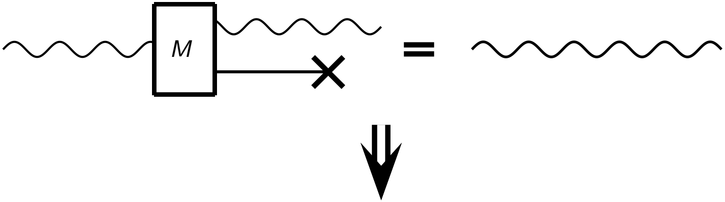

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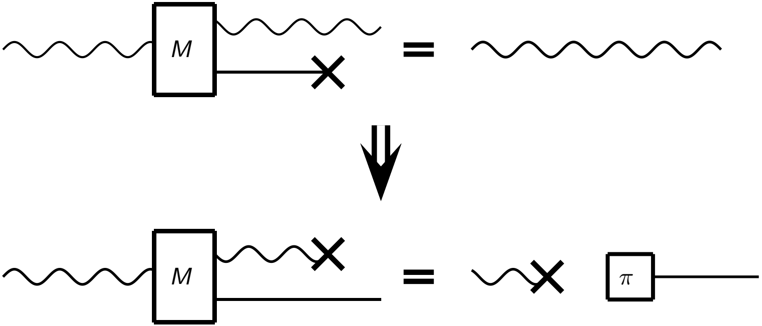

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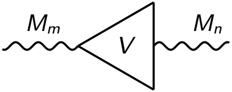

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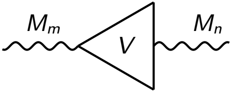

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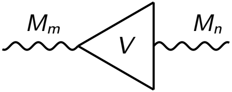

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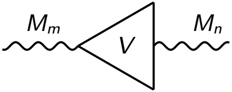

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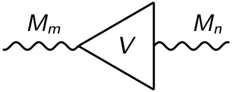

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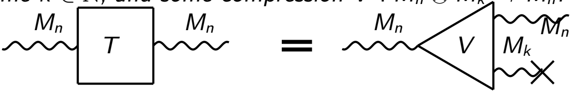

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$$p_{\mathcal{L}} \ a_i^* a_j \ p_{\mathcal{L}} = \lambda_{ij} \ p_{\mathcal{L}} \ .$$

Proof of necessity of Knill-Laflamme condition by diagrams

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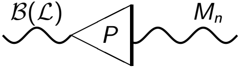

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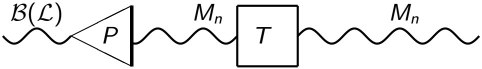

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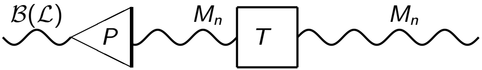

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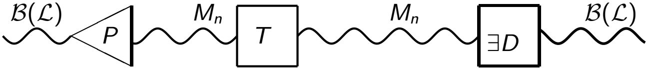

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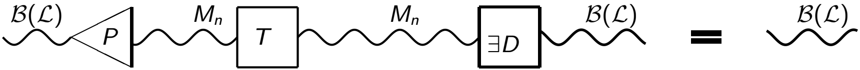

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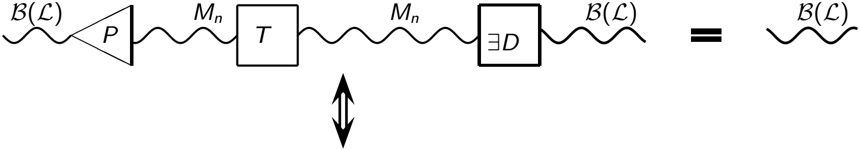

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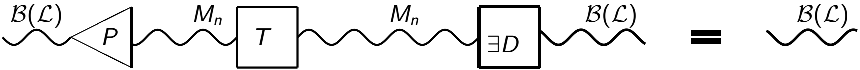

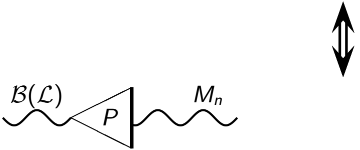

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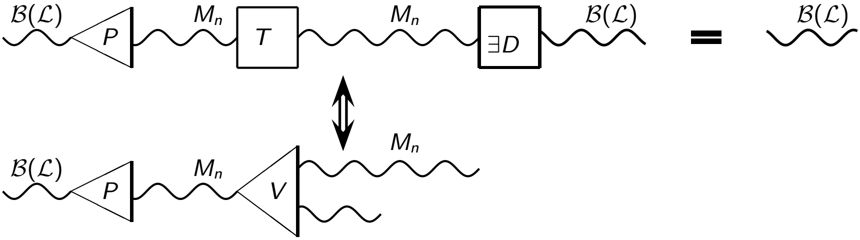

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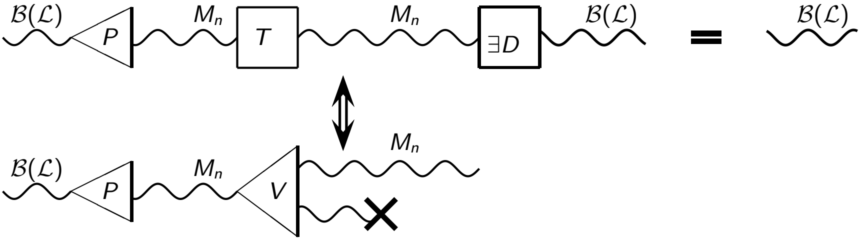

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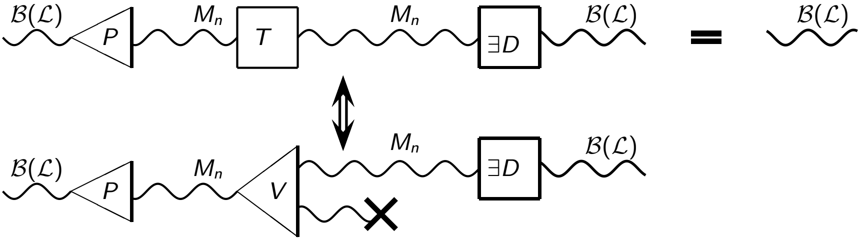

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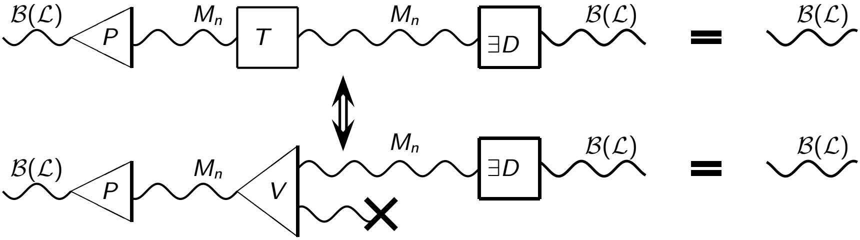

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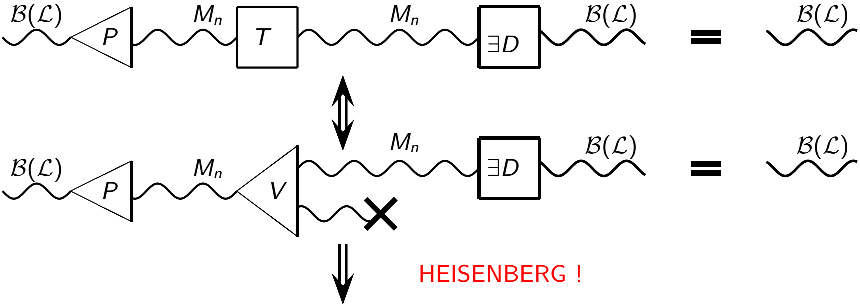

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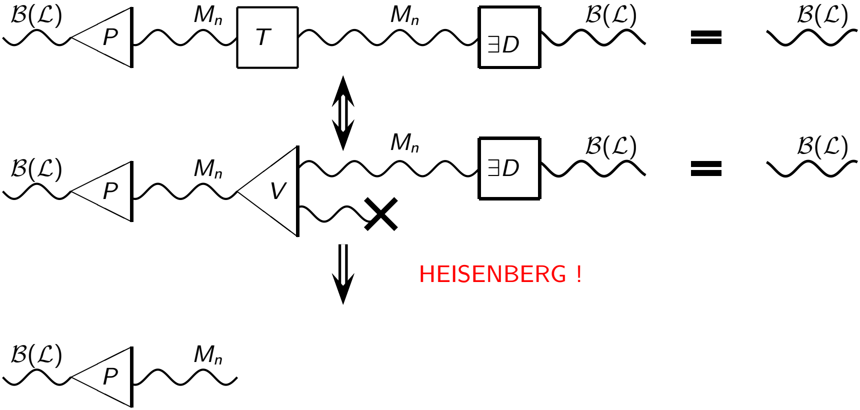

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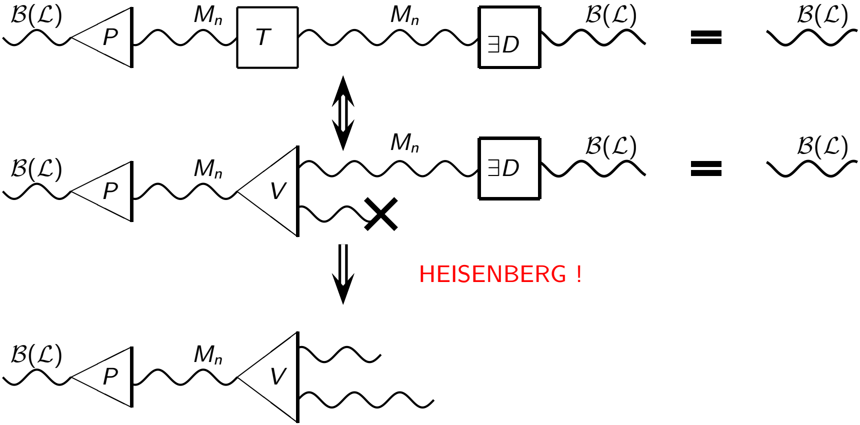

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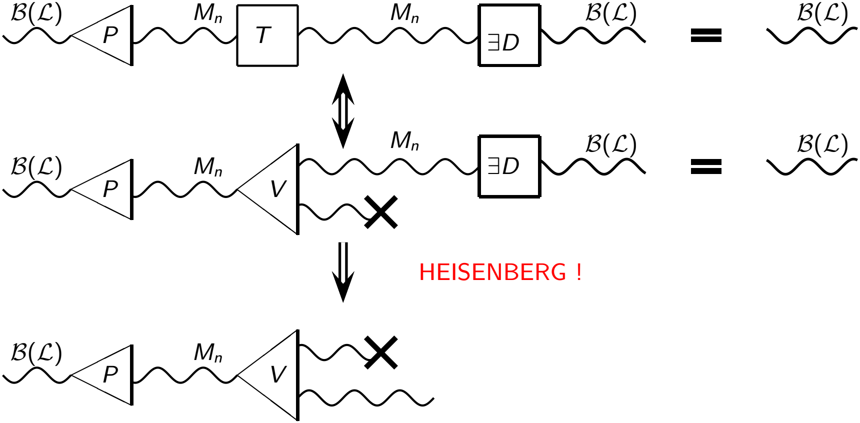

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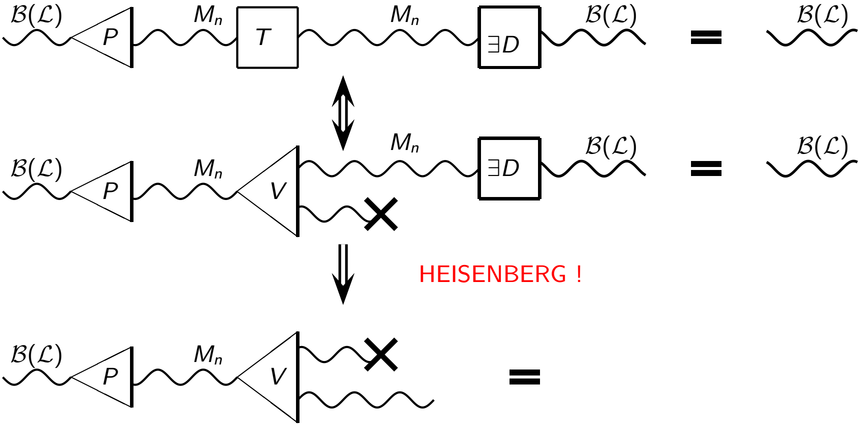

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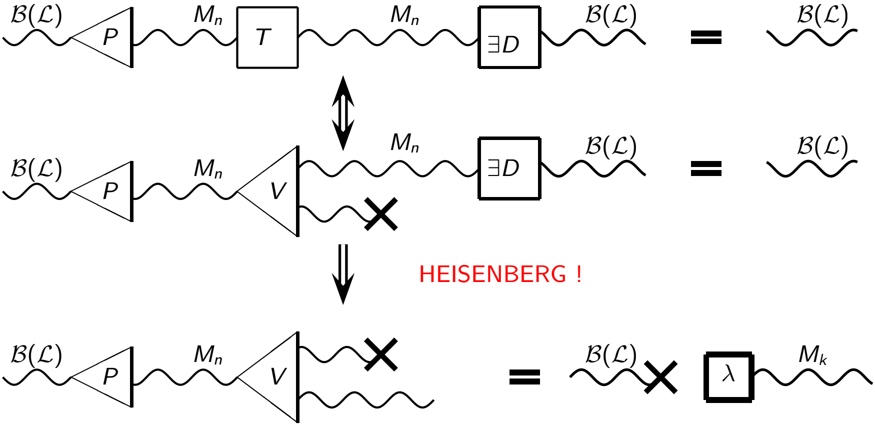

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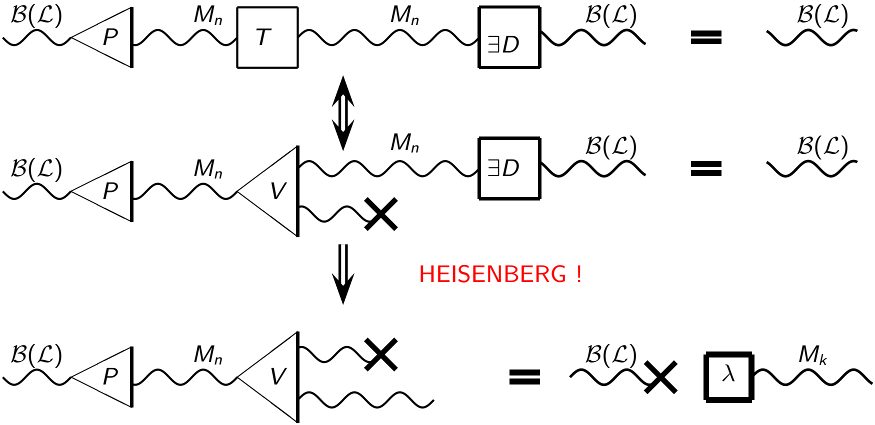

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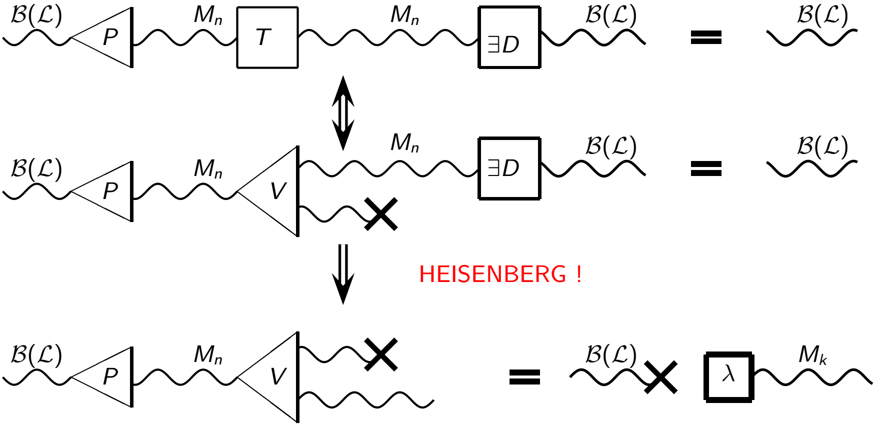

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We shall formulate this well-known fact, which has been experimentally established, and is described for instance in the book by Nielsen and Chuang, as a theorem in *-Alg, combining classical and quantum objects.

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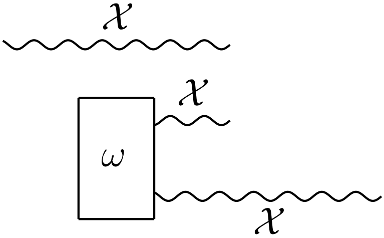

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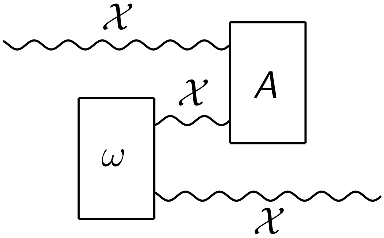

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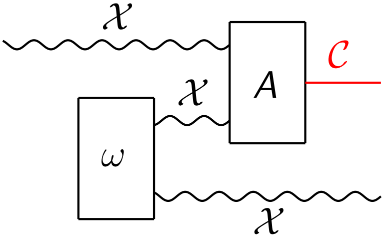

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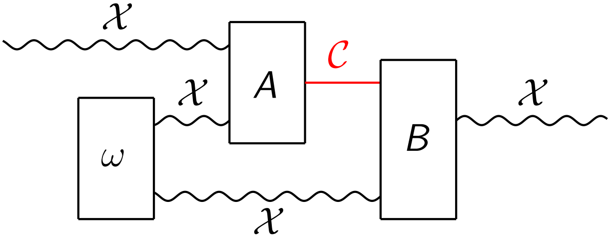

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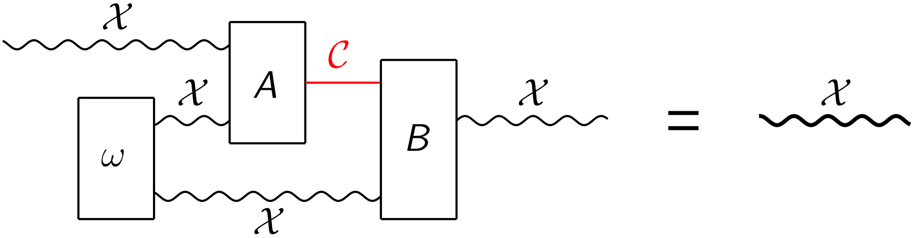

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