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INVARIANTS OF THREE-DIMENSIONAL MANIFOLDS

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0. Introduction

There are several approaches to the construction of invariants of a three-dimensional manifold using quantum groups, and, more generally, monoidal categories. In this paper we consider the combinatorial approach which was suggested by Turaev and Viro (see [TV] and later generalizations in [T2], [KS], [Po], [Ro] among others) and construct an invariant starting from a monoidal category $\mathbf{C}$ without any braiding conditions.

The starting point of a combinatorial approach is a triangulation of a given three-dimensional manifold M, and the invariant is constructed using the combinatorics of this triangulation. More precisely, fix a monoidal category $\mathbf{C}$ (a category with the multiplication functor $X \otimes Y$ on objects) over a field k, which is semisimple, has a finite number of isomorphism classes of simple objects, and possesses a duality $X \mapsto X^*$ such that X^{**} is isomorphic to X. To define $I_{\mathbf{C}}(M)$, we must fix a *balancing*, i.e. functorial isomorphisms $X \to X^{**}$, $X \in \operatorname{Ob}\mathbf{C}$, satisfying certain natural conditions. On the other hand, we do not assume that $\mathbf{C}$ satisfies any kind of braiding conditions, i.e. commutativity conditions of the form $X \otimes Y \simeq Y \otimes X$. A similar construction, which also does not use braiding, was recently suggested by Kuperberg [Ku], who used the language of Hopf algebras.

For any three-dimensional manifold M with boundary S and a triangulation D of M, we construct a finite-dimensional linear space $W(S, D')$ depending on the restriction D' of D to S (a triangulation of S) and a vector $I_{\mathbf{C}}(M, S, D) \in W(S, D')$. The main result of the paper (Theorem 1) is that $I_{\mathbf{C}}(M, S, D)$ depends only on D' and not on D itself. Taking the inductive limit of spaces $W(S, D')$ over all triangulations D' of S, we construct a space $K(S)$ and a vector $I_{\mathbf{C}}(M, S) \in K(S)$, which is our invariant of a three-dimensional manifold M with boundary S. (See §6 of the paper, where this is explained in slightly different terms.) In particular, when M is a closed manifold, $S = \varnothing$, we have $K(\varnothing) = k$, and our construction gives an invariant $I_{\mathbf{C}}(M) \in k$.

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In constructing the invariant $I_{\mathbf{C}}$ of a three-dimensional manifold M using a triangulation of M, the main problem is, of course, to prove that the $I_{\mathbf{C}}(M, S, D)$ depends only on the restriction of the triangulation D to S. Our proof of this fact is parallel, sometimes closely parallel, to the original proof of Turaev and Viro [TV], see also [T2]. In particular, we also use the so-called Pachner moves to obtain triangulations of M from one another, and show how the invariance of $I_{\mathbf{C}}$ under these moves follows from axioms of balanced monoidal categories. However, our proof differs from that in [T2] in that we do not use the braiding and also certain other conditions in [T2], which allows us to treat more general monoidal categories $\mathbf{C}$. To make the proof more transparent, we develop in §2 a technique of marked diagrams, which allows us to prove equalities between composite morphisms in balanced categories.

Let us also mention another approach to the construction of invariants using monoidal categories. This approach, first suggested by Reshetikhin and Turaev [RT1,2] and later developed and generalized by many authors (see bibliography in [T2]), starts with a presentation of a three-dimensional manifold as a surgery of the three-dimensional sphere S^3, and the corresponding invariants are closely related to similar invariants of knots, links, and tangles. It would be interesting to find out whether one can also eliminate braiding conditions on $\mathbf{C}$ using this approach.

We would like to thank Thomas Kerler who read the manuscript and made several useful remarks.

1. Balancing

Let $\mathbf{C} = (\mathcal{C}, \otimes, \varphi)$ be a strict rigid monoidal category over an algebraically closed field k, with unit object $\mathbf{1}$ and duality (X^*, i_X, e_X). The second dual $\delta(X) = X^{**}$ is a functor $\delta : \mathcal{C} \to \mathcal{C}$ (for definitions of all these notions see [DMi] or [CPr, Ch. 5]). Recall that there exist functorial isomorphisms

$$\lambda_{X,Y} : (X \otimes Y)^* \to Y^* \otimes X^*$$

that are compatible with morphisms i and e. Denoting $\alpha_{X,Y} = \lambda_{X^*,Y^*} \circ (\lambda^*_{X,Y})^{-1}$, we obtain functorial isomorphisms

$$\alpha_{X,Y} : (X \otimes Y)^{**} \to X^{**} \otimes Y^{**} \ ,$$

which make δ a monoidal functor.

Definition 1: *Balancing* in a strict rigid monoidal category $\mathbf{C}$ is a monoidal isomorphism of functors $\beta : \mathrm{Id} \to \delta$.

Therefore, a balancing in $\mathbf{C}$ is a family of isomorphisms $\beta_X : X \to X^{**}$ such that for any $f : X \to Y$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{\beta_X} & X^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ Y & \xrightarrow{\beta_Y} & Y^{**} \end{array} \tag{1}$$

commutes, and for any $X, Y \in \operatorname{Ob} \mathcal{C}$ the diagram

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{\beta_{X \otimes Y}} & (X \otimes Y)^{**} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \alpha_{X,Y}} \\ X \otimes Y & \xrightarrow{\beta_X \otimes \beta_Y} & X^{**} \otimes Y^{**} \end{array} \tag{2}$$

commutes.

A strict rigid monoidal category with fixed balancing is called a *balanced* category.

Let $\mathbf{C}$ be a balanced category. The ring $\operatorname{Hom}(\mathbf{1}, \mathbf{1})$ is a commutative semisimple algebra over k. For simplicity, we assume that $\operatorname{Hom}(\mathbf{1}, \mathbf{1}) = k$ and will identify $\operatorname{Hom}(\mathbf{1}, \mathbf{1})$ with k.

DEFINITION 2: (i) For a morphism $f : X \to X$ in $\mathcal{C}$, its trace $\operatorname{tr} f \in k$ is defined as the element of k corresponding to the the composition

$$\mathbf{1} \xrightarrow{i_X} X \otimes X^* \xrightarrow{f \otimes \mathrm{id}} X \otimes X^* \xrightarrow{\beta_X \otimes \mathrm{id}} X^{**} \otimes X^* \xrightarrow{e_{X^*}} \mathbf{1} .$$

(ii) The dimension of an object $X \in \operatorname{Ob} \mathcal{C}$ is $\dim X = \operatorname{tr}(\mathrm{id}_X)$.

Trace and dimension satisfy the standard properties listed in the following proposition.

PROPOSITION 1. (i) Additivity. *For $f_1 : X_1 \to X_1$, $f_2 : X_2 \to X_2$ we have* $\operatorname{tr}(f_1 \oplus f_2) = \operatorname{tr} f_1 + \operatorname{tr} f_2$. *In particular,* $\dim(X_1 \oplus X_2) = \dim X_1 + \dim X_2$.

(ii) Multiplicativity. *For $f_1 : X_1 \to X_1$, $f_2 : X_2 \to X_2$ we have* $\operatorname{tr}(f_1 \otimes f_2) = \operatorname{tr} f_1 \cdot \operatorname{tr} f_2$. *In particular,* $\dim(X_1 \otimes X_2) = \dim X_1 \cdot \dim X_2$.

(iii) *If, in addition, $\mathcal{C}$ is a semisimple category, then for $f : X \to Y$, $g : Y \to X$ we have* $\operatorname{tr}(fg) = \operatorname{tr}(gf)$.

Proof. (i) This follows from the fact that $(X \oplus Y)^* = X^* \oplus Y^*$ with

$$i_{X \oplus Y} : \mathbf{1} \to (X \oplus Y) \otimes (X \oplus Y)^* = (X \otimes X^*) \oplus (X \otimes Y^*) \oplus (Y \otimes X^*) \oplus (Y \otimes Y^*)$$

being given by $i_{X \oplus Y} = i_X \oplus 0 \oplus 0 \oplus i_Y$, and similarly for $e_{X \oplus Y}$.

(ii) This follows from the fact that $i_{X \otimes Y} : \mathbf{1} \to (X \otimes X^*) \otimes (Y \otimes Y^*)$ is the composition

$$\mathbf{1} \xrightarrow{i_X} X \otimes X^* \xrightarrow{\mathrm{id} \otimes i_Y \otimes \mathrm{id}} X \otimes Y \otimes Y^* \otimes X^* \xrightarrow{\mathrm{id} \otimes (\lambda_{X,Y})^{-1}} (X \otimes Y) \otimes (X \otimes Y)^* ,$$

with $\lambda_{X,Y}$ as at the beginning of this section, a similar formula for $e_{X\otimes Y}$, and the commutativity of the diagram (2).

(iii) Let $\mathcal{E}$ be the set of isomorphism classes of simple objects in $\mathcal{C}$. For each class α, select a representative $E_\alpha \in \mathrm{Ob}\,\mathcal{C}$. Since $\mathcal{C}$ is a semisimple category, the canonical morphism

$$\bigoplus_{\alpha\in\mathcal{E}} \mathrm{Hom}(E_\alpha, X) \otimes E_\alpha \to X \tag{3}$$

is an isomoprhism. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. For $\alpha \in \mathcal{E}$, define the linear map

$$f_\alpha : \mathrm{Hom}(E_\alpha, X) \to \mathrm{Hom}(E_\alpha, Y)$$

by the formula $f_\alpha : \varphi \mapsto f \circ \varphi$. Then, under the isomorphism (3) and a similar isomorphism for Y,

$$f : \bigoplus_{\alpha\in\mathcal{E}} \mathrm{Hom}(E_\alpha, X) \otimes E_\alpha \to \bigoplus_{\alpha\in\mathcal{E}} \mathrm{Hom}(E_\alpha, Y) \otimes E_\alpha$$

is given by

$$f = \bigoplus_{\alpha\in\mathcal{E}} (f_\alpha \otimes \mathrm{id}_{E_\alpha}) \ .$$

A similar decomposition holds for $g : Y \to X$:

$$g = \bigoplus_{\alpha\in\mathcal{E}} (g_\alpha \otimes \mathrm{id}_{E_\alpha}) \ ,$$

with the linear maps

$$g_\alpha : \mathrm{Hom}(E_\alpha, Y) \to \mathrm{Hom}(E_\alpha, X)$$

given by $g_\alpha : \psi \mapsto g \circ \psi$. By parts (i) and (ii),

$$\mathrm{tr}(fg) = \sum_\alpha \mathrm{tr}(f_\alpha g_\alpha) \dim(E_\alpha) \ ,$$

where $\mathrm{tr}(f_\alpha g_\alpha)$ is the ordinary trace of the automorphism $f_\alpha g_\alpha$ of the linear space $\mathrm{Hom}(E_\alpha, Y)$. Similarly,

$$\mathrm{tr}(gf) = \sum_\alpha \mathrm{tr}(g_\alpha f_\alpha) \dim(E_\alpha) \ .$$

Therefore, part (iii) follows from the corresponding property of the trace of a linear map.

2. Calculus of Diagrams

In this section we describe a technique that allows us to prove equalities of composite morphisms in a balanced category $\mathbf{C} = (\mathcal{C}, \otimes, \varphi)$.

By a *diagram* we mean a directed graph G with simple edges, imbedded into the coordinate plane $\mathbb{R}^2$ in such a way that the projection of each edge to the x-axis is one-to-one and points in the positive direction. We assume that the graph is located in the strip $0 \le x \le 1$.

Further, we assume that each vertex of G located on the line $x = 0$ has exactly one outgoing edge, and, of course, no incoming edges. These edges are called input edges of G. Similarly, each vertex on the line $x = 1$ has exactly one incoming edge and no outgoing edges. These edges are called output edges of G.

A *marked diagram* is a diagram G such that to any edge in G there corresponds an object from $\mathcal{C}$, and to any intermediate vertex in G (that is, a vertex with the x-coordinate strictly between 0 and 1) there corresponds a certain morphism in $\mathcal{C}$ according to the following convention. Let v be a vertex in G. Then the corresponding morphism f_v acts from $X_1 \otimes \cdots \otimes X_k$ to $Y_1 \otimes \cdots \otimes Y_l$, where $X_1, \ldots, X_k$ are the objects corresponding to edges entering v, numbered from top to bottom, and $Y_1, \ldots, Y_l$ are objects corresponding to objects exiting v, also numbered from top to bottom (see Fig. 1).

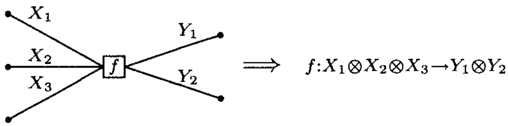


Figure 1. A morphism $f_v : X_1 \otimes \cdots \otimes X_k \to Y_1 \otimes \cdots \otimes Y_l$

We assume that if one of the numbers k or l is 0, then the corresponding product is the unit object $\mathbf{1}$ in $\mathcal{C}$.

On the figures below, we will write the object corresponding to an edge in G, and mark vertices of G by small squares, with the corresponding morphism written inside or near this vertex. Two classes of morphisms have special notation. If the morphism corresponding to a vertex v is one of the morphisms i_X or e_X for $X \in \operatorname{Ob}\mathcal{C}$, we will mark this vertex by a crossed square and usually omit the name of the corresponding morphism in the diagram. If the morphism corresponding to a vertex v is one of the isomorphisms β_X or $(\beta_X)^{-1}$, we will mark this vertex by a black square and also omit the name of the corresponding morphism in the diagram. See examples in Fig. 2.

Let G be a marked diagram. Denote $X_s = X_s(G) = X_1 \otimes \cdots \otimes X_k$, where $X_1, \ldots, X_k$ are the objects associated to all edges from source vertices of G, numbered from top to bottom, and $X_t = X_t(G) = Y_1 \otimes \cdots \otimes Y_l$, where

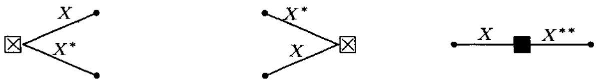


Figure 2. Morphisms i_X, e_X, and β_X

$Y_1, \ldots, Y_l$ are objects associated to edges entering target vertices of G, also numbered from top to bottom. We want to associate to G a morphism $f(G) : X_s(G) \to X_t(G)$ in $\mathbf{C}$. To define such a morphism, we first introduce elementary diagrams G and the morphisms associated to them.

DEFINITION 3: (i) An elementary diagram is a marked diagram of one of the following types:

(a) the diagram of the form $\circ \xrightarrow{X} \circ$;
(b) the diagram in Fig. 1.

In the second case k and/or l can be 0.

(ii) The morphism $f(G)$ associated to an elementary diagram is defined as follows:

If G is a diagram of type (a), then $f(G) = \mathrm{id}_X$.
If G is a diagram of type (b), then $f(G) = f : X_s \to X_t$.

Now we introduce the gluing operations for marked diagrams.

The first operation is defined as follows. Let G, G' be two marked diagrams with k, k' input edges and l, l' output edges respectively. We assume that $l = k'$ and that the marks on output edges of G coincide with marks of input edges of G': $Y_1 = X'_1$, $Y_2 = X'_2, \ldots, Y_l = X'_l$. The *composition* $G \circ G'$ is the marked diagram with k input edges and l' output edges obtained by putting G and G' side by side, removing vertices at $l = k'$ intermediate points, and squeezing the resulting diagram into the strip $0 \leq x \leq 1$ (see Fig. 3, where $k = 3$, $l = k' = 2$, $l' = 1$).

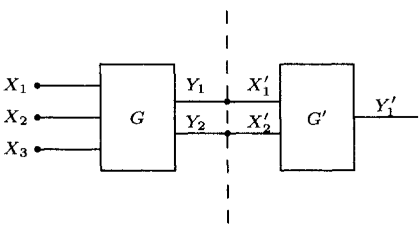


Figure 3. Composition of two diagrams

The second operation is defined as follows. Again let G, G' be two marked diagrams with k, k' input edges and l, l' output edges, respectively. Their *product* $G \times G'$ is the marked diagram with $k + k'$ input edges and $l + l'$ output edges obtained by putting G' on top of G, see Fig. 4.

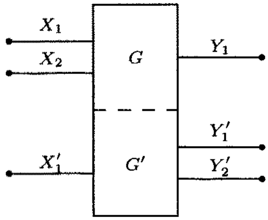


Figure 4. Product of two diagrams

PROPOSITION 2. *Any marked diagram G can be represented as the composition of products of elementary diagrams, i.e.*

$$G = (G_{11} \times \cdots \times G_{1n_1}) \circ \cdots \circ (G_{m1} \times \cdots \times G_{mn_m}) , \tag{4}$$

where all G_{ij} are elementary diagrams.

Proof. Clear.

Now we define composition rules for morphisms $f(G)$ as follows:

$$\begin{aligned} f(G \circ G') &= f(G) \circ f(G') , \\ f(G \times G') &= f(G) \otimes f(G') . \end{aligned} \tag{5}$$

PROPOSITION 3. *Formulas* (5) *determine $f(G)$ uniquely for any marked diagram G.*

Proof. By Proposition 2, we can associate to G a morphism $f(G)$ using the decomposition (4) and formulas (5). To prove that different decompositions of a marked diagram G yield equal morphisms, it suffices to consider the case when G is itself elementary. But in this case the assertion is evident since at most one of the morphisms $f(G_{ij})$ is not an identity morphism.

Now we present the reduction rules of the calculus of diagrams.

Two internal vertices v_1, v_2 with x-coordinates $x_1 < x_2$ are called incompatible if there are no oriented paths from v_1 to v_2 in G, and neighboring, if there are no vertices in G with x-coordinate strictly between x_1 and x_2.

PROPOSITION 4. (i) *Let v_1 and v_2 be two incompatible neighboring vertices in G. Let G' be the marked diagram obtained from G by moving v_2 to the right and v_1 to the left, so that their x-coordinates switch, see Fig. 5. Then $f(G) = f(G')$.*

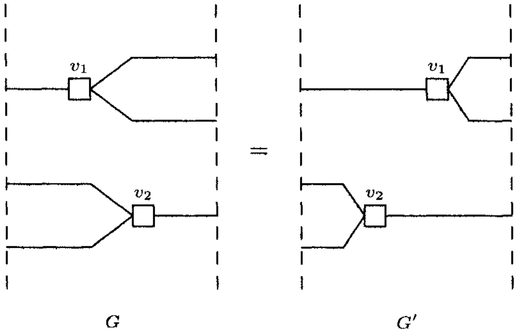


Figure 5. Switching of vertices v_1 and v_2

(ii) *For any $X \in \mathrm{Ob}\,\mathcal{C}$, $f(G) = \mathrm{id}_X$, $f(G') = \mathrm{id}_{X^*}$, where G is the diagram in Fig. 6 and G' is the diagram in Fig. 7.*

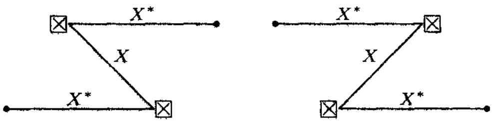


Figure 6 Figure 7

(iii) *Let $\varphi : X_1 \otimes \cdots X_k \to Y_1 \otimes \cdots \otimes Y_l$ be an arbitrary morphism and G, G' be the marked diagrams in Fig. 8. Then $f(G) = f(G')$.*

Proof. Part (i) follows from the functoriality of $\otimes$, part (ii) expresses properties of morphisms e_X and i_X, and part (iii) expresses the fact that β is an isomorphism of functors.

3. Pairing of Hom Spaces

Let $\mathbf{C}$ be a semisimple balanced category. For two objects $X, Y \in \mathcal{C}$ define the pairing

$$\langle \cdot, \cdot \rangle : \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,X) \to k$$

by the formula $\langle f, g \rangle = \mathrm{tr}(gf)$.

LEMMA 1. *The pairing $\langle \cdot, \cdot \rangle$ is nondegenerate.*

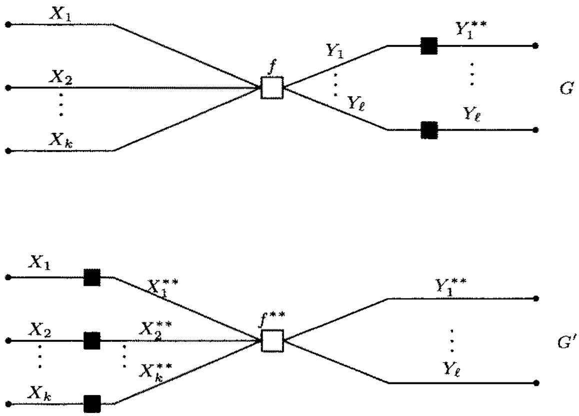


Figure 8. Balancing

Proof. Since $\mathcal{C}$ is semisimple, we can assume that $X = Y$ is a simple object in $\mathcal{C}$. Since $\langle \mathrm{id}_X, \mathrm{id}_X \rangle = \dim X$, it suffices to prove that $\dim X \neq 0$ for a simple $X \in \mathrm{Ob}\,\mathcal{C}$. For a simple X, we have $\dim \mathrm{Hom}(\mathbf{1}, X \otimes X^*) = \dim \mathrm{Hom}(X^* \otimes X, \mathbf{1}) = 1$. Since, evidently, $i_X \neq 0$, $e_X \neq 0$, $\beta_X \neq 0$, the lemma follows.

LEMMA 2. *For $f : X \to Y$, $g : Y \to X$ each of the following compositions*

$$\begin{array}{l} \mathbf{1} \xrightarrow{i_X} X \otimes X^* \xrightarrow{f \otimes g^*} Y \otimes Y^* \xrightarrow{\beta_Y \otimes \mathrm{id}} Y^{**} \otimes Y^* \xrightarrow{e_{Y^*}} \mathbf{1} \\ \mathbf{1} \xrightarrow{i_Y} Y \otimes Y^* \xrightarrow{g \otimes f^*} X \otimes X^* \xrightarrow{\beta_X \otimes \mathrm{id}} X^{**} \otimes X^* \xrightarrow{e_{X^*}} \mathbf{1} \end{array} \tag{6}$$

is the multiplication by $\langle f, g \rangle$.

Proof. Let us prove that the first composition in (6) equals $\mathrm{tr}(fg)$. We must compare morphisms $\mathbf{1} \to \mathbf{1}$ corresponding to the diagrams in Figures 9 and 10.

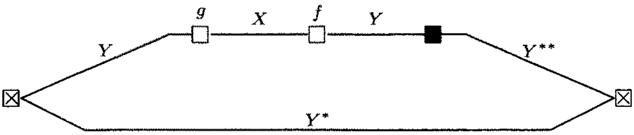


Figure 9. $\mathrm{tr}(fg)$

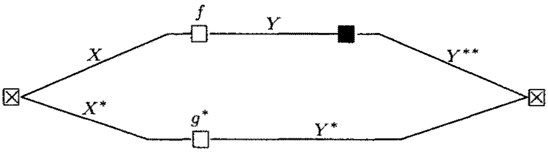


Figure 10. The first morphism in (6)

By the definition of g^*, the diagram in Fig. 10 is equivalent to the diagram in Fig. 11. Applying to this diagram the reduction rule 1 for X, we get the required diagram for $\operatorname{tr}(fg)$.

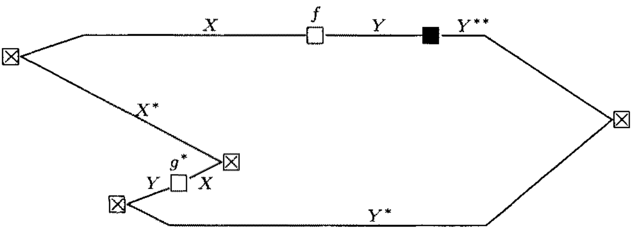


Figure 11

Similarly, the second morphism in (6) equals $\operatorname{tr}(gf)$. Now Lemma 2 follows from Proposition 1(iii).

LEMMA 3. *Let Y be an object of $\mathcal{C}$, and $f : \mathbf{1} \to Y$, $g : 1 \to Y^*$ be two morphisms. Then the compositions*

$$\mathbf{1} \xrightarrow{g \otimes f} Y^* \otimes Y \xrightarrow{e_Y} \mathbf{1}$$

and

$$\mathbf{1} \xrightarrow{f \otimes g} Y \otimes Y^* \xrightarrow{\beta_Y \otimes \mathrm{id}} Y^{**} \otimes Y^* \xrightarrow{e_{Y^*}} \mathbf{1}$$

coincide.

Proof. As in the proof of Lemma 2, it is easy to see that the first composition equals $f^* \circ g : \mathbf{1} \to \mathbf{1}$, and the second composition equals $g^* \circ f^{**} : \mathbf{1} \to \mathbf{1}$. Therefore, the lemma follows from the fact that for any $\theta : \mathbf{1} \to \mathbf{1}$, we have $\theta^* = \theta$.

4. Spaces $V(X_1, \ldots, X_n)$

For a family of objects $X_1, X_2, \ldots, X_n$ in a balanced category $\mathbf{C}$ define the vector space $V(X_1, \ldots, X_n)$ by

$$V(X_1, \ldots, X_n) = \operatorname{Hom}(\mathbf{1}, X_1 \otimes \ldots \otimes X_n) \, .$$

The mapping $(X_1,\dots,X_n)\mapsto V(X_1,\dots,X_n)$ extends to the functor V from the category $\mathcal{C}\times\cdots\times\mathcal{C}$ to the category of vector spaces.

Define the linear map $\sigma_n=\sigma_n(X_1,X_2,\dots,X_n):V(X_1,X_2,\dots,X_n)\to V(X_2,\dots,X_n,X_1)$ as follows. For $f:\mathbf{1}\to X_1\otimes X_2\otimes\ldots\otimes X_n$, $\sigma_n f$ is the composition

$$\mathbf{1}\xrightarrow{i_{X_1^*}}X_1^*\otimes X_1^{**}\xrightarrow{\mathrm{id}\otimes f\otimes\mathrm{id}}X_1^*\otimes X_1\otimes X_2\otimes\ldots\otimes X_n\otimes X_1^{**}\longrightarrow$$
$$\xrightarrow{e_{X_1}\otimes\mathrm{id}\otimes\beta_{X_1}^{-1}}X_2\otimes\ldots\otimes X_n\otimes X_1\ .$$

The family of linear maps $\sigma_n(X_1,X_2,\dots,X_n)$ defines a morphism of functors $\sigma_n:V\Rightarrow V\circ s_n$, where s_n is the cyclic shift of factors in the product $\mathcal{C}\times\cdots\times\mathcal{C}$.

PROPOSITION 5. (i) *σ_n is an isomorphism of functors.*

(ii) *$(\sigma_n)^n$ is the identity morphism of the functor V.*

(iii) *For $(X_1,\dots,X_n)\in\mathrm{Ob}(\mathcal{C}\times\cdots\times\mathcal{C})$ define $c(X_1,\dots,X_n)$: $V(X_1,\dots,X_n)\otimes V(X_n^*,\dots,X_1^*)$ by the formula $c(v_1,v_2)=\langle v_2,v_1^*\rangle$, where the $\langle\cdot,\cdot\rangle$ is the pairing from Lemma 1. Then c defines a non-degenerate pairing between $V(X_1,\dots,X_n)$ and $V(X_n^*,\dots,X_1^*)$, which commutes with isomorphisms in $\mathcal{C}\times\cdots\times\mathcal{C}$.*

Proof. The first assertion follows from the second one, which is equivalent to the statement that for each $(X_1,X_2,\dots,X_n)\in\mathrm{Ob}(\mathcal{C}\times\cdots\times\mathcal{C})$, $\sigma_n(X_1,X_2,\dots,X_n)$ is the identity automorphism of the space $V(X_1,X_2,\dots,X_n)$. We give the proof for $n=2$, leaving the general case to the reader.

First, we prove the following

LEMMA 4. *Let $f:\mathbf{1}\to X$ be a morphism in $\mathcal{C}$. The composition*

$$\mathbf{1}\xrightarrow{i_{X^*}}X^*\otimes X^{**}\xrightarrow{\mathrm{id}\otimes f\otimes\mathrm{id}}X^*\otimes X\otimes X^{**}\xrightarrow{e_X}X^{**}\xrightarrow{\beta_X^{-1}}X \tag{7}$$

coincides with f.

Proof of the lemma. Since $\mathbf{1}^*$ is canonically isomorphic to $\mathbf{1}$, f^* is, by definition, the composition

$$X^*\xrightarrow{\mathrm{id}\otimes f}X^*\otimes X\xrightarrow{e_X}\mathbf{1}\ ,$$

and f^{**} is the composition

$$\mathbf{1}\xrightarrow{i_{X^*}}X^*\otimes X^{**}\xrightarrow{\mathrm{id}\otimes f\ \mathrm{id}}X^*\otimes X\otimes X^{**}\xrightarrow{e_X}X^{**}\ .$$

It remains to apply the commutativity of the diagram (1) for $f:\mathbf{1}\to X$.

Now let $f:\mathbf{1}\to X_1\otimes X_2$. Then $(\sigma_2)^2f:\mathbf{1}\to X_1\otimes X_2$ is the following composition:

$$\begin{aligned}\mathbf{1}&\xrightarrow{i_{X_2}} X_2^*\otimes X_2^{**}\xrightarrow{\mathrm{id}\otimes i_{X_1}\otimes \mathrm{id}} X_2^*\otimes X_1^*\otimes X_1^{**}\otimes X_2^{**}\longrightarrow\\ &\xrightarrow{\mathrm{id}\otimes f\otimes \mathrm{id}} X_2^*\otimes X_1^*\otimes X_1\otimes X_2\otimes X_1^{**}\otimes X_2^{**}\longrightarrow\\ &\xrightarrow{e_{X_1}} X_2^*\otimes X_2\otimes X_1^{**}\otimes X_2^{**}\xrightarrow{e_{X_2}} X_1^{**}\otimes X_2^{**}\xrightarrow{\beta_{X_1}^{-1}\otimes\beta_{X_1}^{-1}} X_1\otimes X_2\ .\end{aligned}\tag{8}$$

Using the compatibility of morphisms $\lambda_{X,Y}$ with morphisms i and e and the fact that β is a monoidal functor (formula (1)), we see that the composition (8) coincides with the composition (7) for $X = X_1 \otimes X_2$, i.e. equals f.

The third assertion of Proposition 5 follows from Lemma 2.

5. Colored Tetrahedra

In this section we introduce the notion of a coloring γ of the standard tetrahedron Δ, associate to each coloring γ a linear space $V(\gamma)$, and construct a functional $L(\gamma)$ on this space. The functionals $L(\gamma)$ are the main building blocks in constructing invariants $I_{\mathbf{C}}(M, S)$ of three-dimensional manifolds (see the introduction), and essentially coincide with these invariants when M is a tetrahedron with the standard triangulation.

From now on $\mathbf{C} = (\mathcal{C}, \otimes, \varphi)$ will be a semisimple balanced category over a field k with a finite number of isomorphism classes of simple objects. Denote by $\mathcal{E}$ the set of these isomorphism classes. Also, denote by $\widetilde{\mathcal{C}}$ the full subcategory of $\mathcal{C}$ consisting of all simple objects. Clearly, any nonzero morphism in $\widetilde{\mathcal{C}}$ is an isomorphism. In the category $\widetilde{\mathcal{C}}$ we have the duality $a \mapsto a^*$ induced by the duality in $\mathcal{C}$.

Let Δ be the tetrahedron with vertices numbered $\{0, 1, 2, 3\}$. Introduce the orientation of Δ by the condition that the basis $\{\overrightarrow{01}, \overrightarrow{02}, \overrightarrow{03}\}$ be positive. Given an object $a = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathrm{Ob}(\widetilde{\mathcal{C}})^6$, we associate to each oriented edge l of Δ one of the objects a_j or a_j^* as follows:

$$\begin{array}{lll}\overrightarrow{03}\Longleftrightarrow a_1\ , & \overrightarrow{01}\Longleftrightarrow a_2\ , & \overrightarrow{12}\Longleftrightarrow a_3\ ,\\ \overrightarrow{23}\Longleftrightarrow a_4\ , & \overrightarrow{02}\Longleftrightarrow a_5\ , & \overrightarrow{13}\Longleftrightarrow a_6\ ,\\ \overrightarrow{30}\Longleftrightarrow a_1^*\ , & \overrightarrow{10}\Longleftrightarrow a_2^*\ , & \overrightarrow{21}\Longleftrightarrow a_3^*\ ,\\ \overrightarrow{32}\Longleftrightarrow a_4^*\ , & \overrightarrow{20}\Longleftrightarrow a_5^*\ , & \overrightarrow{31}\Longleftrightarrow a_6^*\ ,\end{array}\tag{9}$$

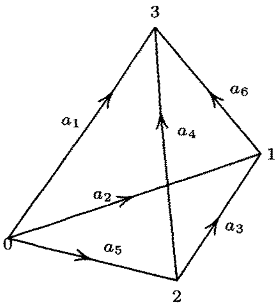
so that to an edge with the opposite orientation corresponds the dual object from $\mathcal{C}$.

Let F_1, F_2, F_3, F_4 be the faces of Δ with the orientation induced from the orientation of Δ (see Fig. 12). For each face F_i, let $\langle l_{1,i}, l_{2,i}, l_{3,i}\rangle$ be its oriented boundary, and $a(l_{1,i}), a(l_{2,i}), a(l_{3,i})$ be the objects of $\widetilde{\mathcal{C}}$ associated to the edges $l_{k,i}$ by (9), so that each $a(l_{k,i})$ is either one of the components

a_j of a, or a dual a_j^*. Denote

$$V(F_i)(a) = V\big(a(l_{1,i}), a(l_{2,i}), a(l_{3,i})\big) \ .$$

Using the isomorphism σ_3 from §4, we can identify the spaces V_i corresponding to three choices of the initial edge $l_{1,i}$ on the boundary of the face F_i, and Proposition 5(ii) shows that this identification is well defined.



Faces:
023, 012
132, 031

Figure 12

For a specific choice of initial edges on faces F_i as in (9), we have

$$\begin{aligned} V(F_1)(a) &= \mathrm{Hom}(\mathbf{1}, a_5 \otimes a_4 \otimes a_1^*) \ , \\ V(F_2)(a) &= \mathrm{Hom}(\mathbf{1}, a_2 \otimes a_3 \otimes a_5^*) \ , \\ V(F_3)(a) &= \mathrm{Hom}(\mathbf{1}, a_6 \otimes a_4^* \otimes a_3^*) \ , \\ V(F_4)(a) &= \mathrm{Hom}(\mathbf{1}, a_1 \otimes a_2^* \otimes a_6^*) \ . \end{aligned}$$

Define the vector space $V(a)$ by

$$V(a) = \bigotimes_{i=1}^{4} V(F_i)(a) \ .$$

Now let $\kappa : a \to \widehat{a}$ be a morphism in $(\widetilde{\mathcal{C}})^6$, so that $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6)$ and each $\kappa_i : a_i \to \widehat{a}_i$ is an isomorphism in $\mathcal{C}$. Define the linear map $V(\kappa) : V(a) \to V(\widehat{a})$ as the product

$$V(\kappa) = \bigotimes_{i=1}^{4} V(F_i)(\kappa) \ ,$$

where $V(F_1)(\kappa) : V(F_1)(a) \to V(F_1)(\widehat{a})$ is given by

$$V(F_1)(\kappa)(v) = \big(\kappa_5 \otimes \kappa_4 \otimes (\kappa_1^*)^{-1}\big) \circ v \quad \text{for} \quad v : \mathbf{1} \to a_5 \otimes a_4 \otimes a_1^* \ ,$$

and similarly for $V(F_2)(\kappa)$, $V(F_3)(\kappa)$, $V(F_4)(\kappa)$.

LEMMA 5. *With these definitions, V becomes a functor from the category $(\widetilde{\mathcal{C}})^6$ to the category* **Vect** *of finite-dimensional vector spaces over k.*

Proof. Clear.

For each $a \in \mathrm{Ob}(\widetilde{\mathcal{C}})^6$ we define a linear functional $L(a)$ in the space $V(a)$ as follows. For $v_i \in V(F_i)(a)$, $i = 1, 2, 3, 4$, let $L(a)(v_1 \otimes v_2 \otimes v_3 \otimes v_4) \in k$ be the element corresponding to the morphism $\mathbf{1} \to \mathbf{1}$ given by the composition

$$\mathbf{1} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \xrightarrow{v_4 \otimes v_2 \otimes v_1} a_1 \otimes a_6^* \otimes a_2^* \otimes a_2 \otimes a_3 \otimes a_5^* \otimes a_5 \otimes a_4 \otimes a_1^* \to \tag{10}$$
$$\xrightarrow{e_{a_2}, e_{a_5}} a_1 \otimes a_6^* \otimes a_3 \otimes a_4 \otimes a_1^* \xrightarrow{v_3} a_1 \otimes a_6^* \otimes a_6 \otimes a_4^* \otimes a_3^* \otimes a_3 \otimes a_4 \otimes a_1^* \to$$
$$\xrightarrow{e_{a_6}, e_{a_4}, e_{a_3}} a_1 \otimes a_1^* \xrightarrow{\beta_{a_1}} a_1^{**} \otimes a_1^* \xrightarrow{e_{a_1^*}} \mathbf{1}$$

or using the calculus of diagrams from §2 by the diagram in Fig. 13.

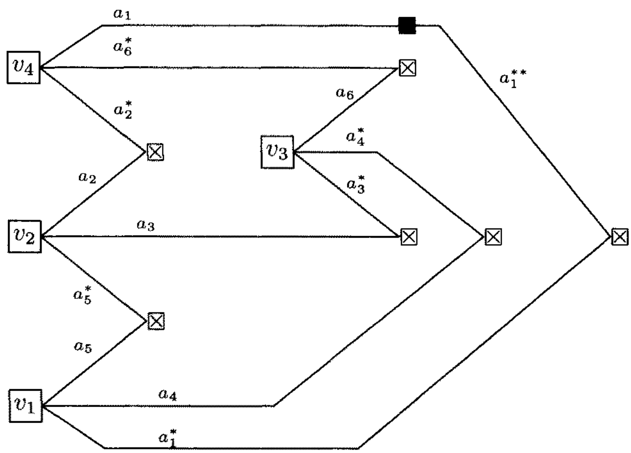


Figure 13

Denote by Triv the trivial functor $(\widetilde{\mathcal{C}})^6 \to$ **Vect**; it associates to each $a \in \mathrm{Ob}(\widetilde{\mathcal{C}})^6$ the one-dimensional vector space k, and to each morphism in $(\widetilde{\mathcal{C}})^6$ the identity map $k \to k$.

LEMMA 6. *The linear functional $L(a) : V(a) \to k$ determines a morphism of functors $L : V \Rightarrow$* Triv.

In other words, for any morphism $\kappa : a \to \widehat{a}$ in $(\widetilde{\mathcal{C}})^6$ we have

$$L(a) = L(\widehat{a}) \circ V(\kappa) \; .$$

Proof. The proof follows from the fact that for each i, $1 \le i \le 6$, the object a_i and its dual a_i^* occur exactly once among the objects associated to edges of Δ.

Lemma 6 shows that we can consider the space $V(a)$ and the linear functional $L(a)$ on $V(a)$ as depending only on isomorphism classes of simple objects a_i in $\mathcal{C}$. More precisely, we give the following definition.

DEFINITION 4: A *coloring* of an oriented tetrahedron Δ is a function γ that associates an element $\gamma(l)$ of the set $\mathcal{E}$ to each oriented edge l of Δ in such a way that if l^* is the edge l with the opposite orientation, then $\gamma(l^*) = \gamma(l)^*$.

Let n be a numbering of vertices of a colored tetrahedron Δ by indices $\{0,1,2,3\}$ that is compatible to the orientation of Δ in the sense described at the beginning of §5. Using Lemma 6, to such a numbering we associate a vector space $V^n(\gamma)$ and a linear functional $L^n(\gamma)$ on it as follows.

Let $W^n(\gamma) = \bigoplus V(a)$, where the direct sum is taken over all $a = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathrm{Ob}(\widetilde{\mathcal{C}})^6$ such that each object a_i corresponding to the edge l of Δ as in (9), belongs to the equivalence class $\gamma(l)$, so that

$$a_1 \in \gamma(\overrightarrow{03}) , \qquad a_2 \in \gamma(\overrightarrow{01}) , \qquad a_2 \in \gamma(\overrightarrow{12}) ,$$
$$a_4 \in \gamma(\overrightarrow{23}) , \qquad a_5 \in \gamma(\overrightarrow{02}) , \qquad a_6 \in \gamma(\overrightarrow{13}) .$$

Denote $V^n(\gamma) = W^n(\gamma)/R$, where R is the subspace generated by elements of the form $(v|_a, V(\kappa)v|_{\kappa(a)})$, where $v \in V(a)$ sits in the position a of the direct sum $\bigoplus V(a)$ and $V(\kappa)v$ sits in the position $\kappa(a)$. By Lemma 6, the functional $L(a)$ vanishes on R and we get the functional $L^n(\gamma)$ on $V^n(\gamma)$.

Proposition 5(ii, iii) provides canonical isomorphisms of the spaces $V^n(\gamma)$ corresponding to different numberings n of the vertices of Δ. Therefore, we can identify all these spaces and denote the resulting space by $V(\gamma)$. The next proposition shows that the functionals $L^n(\gamma)$ are compatible with these identifications.

PROPOSITION 6. *The linear functional $L^n(\gamma)$ does not depend on the orientation compatible numbering n of vertices of the tetrahedron Δ.*

Proof. Since the group of orientation preserving motions of the tetrahedron is generated by two permutations of vertices, it suffices to prove that

$$L^n(\Delta,\gamma) = L^{n'}(\Delta,\gamma) = L^{n''}(\Delta,\gamma) ,$$

where the numbering n' is obtained from n by the substitution

$$0 \to 0 , \quad 1 \to 2 , \quad 2 \to 3 , \quad 3 \to 1 , \tag{11}$$

and n'' is obtained from n by the substitution

$$0 \to 1\,, \quad 1 \to 2\,, \quad 2 \to 0\,, \quad 3 \to 3\,.$$

We consider the first case in detail, leaving the second to the reader.

Under the substitution (11), simple objects $a_1, a_2, a_3, a_4, a_5, a_6$ are replaced by $A_1, A_2, A_3, A_4, A_5, A_6$, where

$$A_1 = a_2\,, \quad A_2 = a_5\,, \quad A_3 = a_4\,, \quad A_4 = a_6^*\,, \quad A_5 = a_1\,, \quad A_6 = a_3^*\,.$$

Therefore, we must prove that the morphism $\mathbf{1} \to \mathbf{1}$ corresponding to the composition (10) coincides with the morphism $\mathbf{1} \to \mathbf{1}$ corresponding to the composition

$$\begin{aligned} &\mathbf{1} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \xrightarrow{v_4 \otimes v_2 \otimes v_1} a_2 \otimes a_3 \otimes a_5^* \otimes a_5 \otimes a_4 \otimes a_1^* \otimes a_1 \otimes a_6^* \otimes a_2^* \to \qquad (12)\\ &\xrightarrow{e_{a_5}, e_{a_1}} a_2 \otimes a_3 \otimes a_4 \otimes a_6^* \otimes a_2^* \xrightarrow{v_3} a_2 \otimes a_3 \otimes a_3^* \otimes a_6 \otimes a_4^* \otimes a_4 \otimes a_6^* \otimes a_2^* \to\\ &\quad \xrightarrow{\beta_{a_3}, \beta_{a_6}} a_2 \otimes a_3^{**} \otimes a_3^* \otimes a_6^{**} \otimes a_4^* \otimes a_4 \otimes a_6^* \otimes a_2^* \xrightarrow{e_{a_3^*}, e_{a_6^*}, e_{a_4}} a_2 \otimes a_2^* \to\\ &\quad \xrightarrow{\beta_{a_2}} a_2^{**} \otimes a_2^* \xrightarrow{e_{a_2^*}} \mathbf{1}\,. \end{aligned}$$

In the proof we use a pictorial interpretation of compositions (10) and (12), so that we must compare morphisms $\mathbf{1} \to \mathbf{1}$ corresponding to the diagrams in Fig. 13 and in Fig. 14.

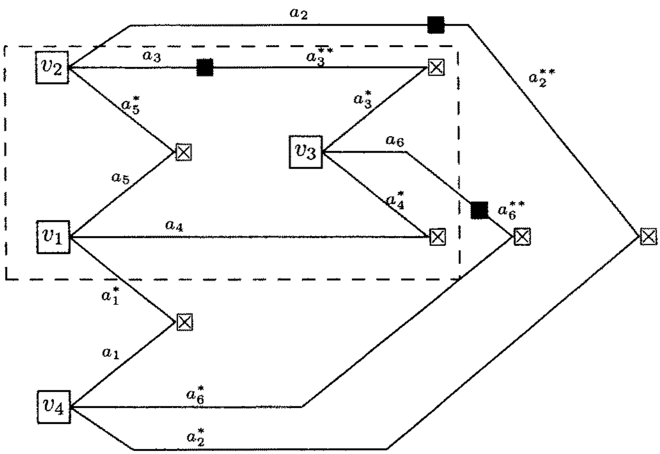


Figure 14

The first step is to move the box v_3 in the diagram in Fig. 13 below the box v_2. Namely, Lemma 3 implies that the morphism $\mathbf{1} \to \mathbf{1}$ represented by the diagram in Fig. 13 coincides with the morphism $\mathbf{1} \to \mathbf{1}$ represented by the diagram in Fig. 15.

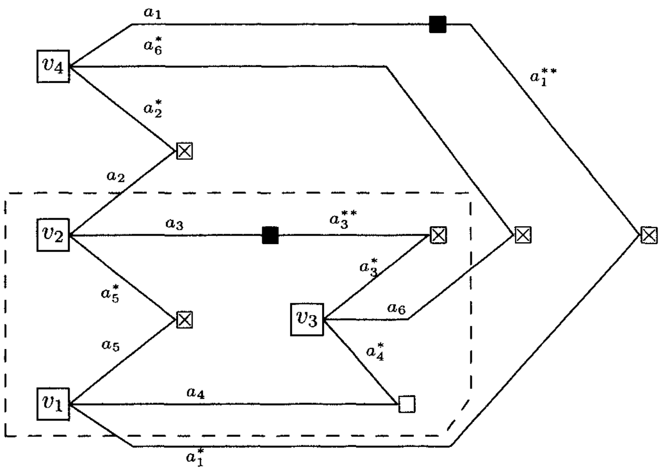


Figure 15

Next we see that the diagrams in Fig. 14 and in Fig. 15 have a common part (in the dotted box), which is a morphism $\mathbf{1} \to a_2 \otimes a_6 \otimes a_1^*$. Denoting this morphism by w we see that to prove the proposition it suffices to establish the equality of the morphisms $\mathbf{1} \to \mathbf{1}$ represented by the diagrams in Fig. 16 and in Fig. 17.

This equality follows from Lemma 3 above. Proposition 6 is proved.

Hence, for every coloring γ of the tetrahedron Δ we constructed a linear space $V(\gamma)$ and a functional $L(\gamma)$ on it.

6. Definition of the Invariant

To define the invariant, we first define the space where this invariant lives.

Let X be a manifold, D a finite triangulation of X, and Γ the set of oriented edges of D. A *coloring* of D is a function $\gamma : \Gamma \to \mathcal{E}$ such that $\gamma(l^*) = (\gamma(l))^*$, where l^* is the edge l with the opposite orientation. In this

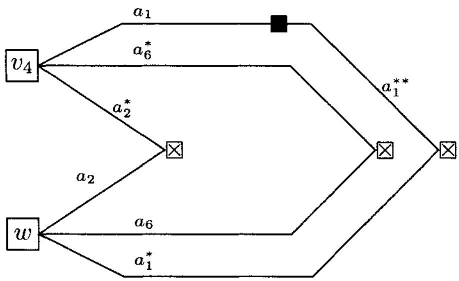


Figure 16

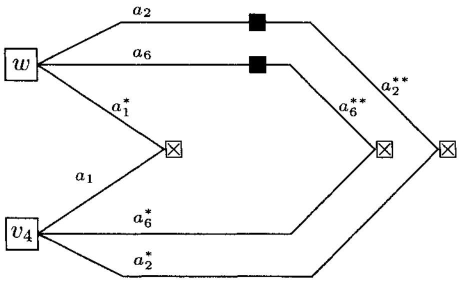


Figure 17

paper we consider either oriented three-dimensional manifolds with boundary, or closed oriented two-dimensional manifolds. We associate a vector space $W(X, D, \gamma)$ to a colored triangulated manifold (X, D, γ). The definition is different for two- and three-dimensional manifolds.

A. $\dim X = 3$. For each tetrahedron Δ of D let $W(\Delta, \gamma|_\Delta)$ be the space introduced in the previous section, i.e.

$$W(\Delta, \gamma|_\Delta) = V(F_1)(\gamma|_\Delta)^* \otimes V(F_2)(\gamma|_\Delta)^* \otimes V(F_3)(\gamma|_\Delta)^* \otimes V(F_4)(\gamma|_\Delta)^* \ ,$$

where F_1, F_2, F_3, F_4 are the faces of Δ with the orientation induced by the orientation of Δ.

Next, define

$$W(X, D, \gamma) = \bigotimes_\Delta W(\Delta, \gamma|_\Delta) \ ,$$

where the product is taken over all tetrahedra Δ of the triangulation D.

B. $\dim X = 2$, X is a closed oriented triangulated surface. In this case, define

$$W(X,D,\gamma) = \bigotimes_F V(F)(\gamma|_F)^* \ ,$$

where the product is taken over all triangles F of the triangulation D. Similarly to Lemma 6, we note that the right-hand side of this formula is well-defined since each edge l of D occurs among the sides of faces D twice, with opposite orientations.

Now let (M,D) be a three-dimensional oriented triangulated manifold with boundary $S = \partial M$. We denote by D_S the induced triangulation of S. By Proposition 6, for each tetrahedron Δ of D we have an element $L(\gamma|_\Delta) \in W(\Delta, \gamma|_\Delta)$. Denote

$$L(M,D,\gamma) = \otimes L(\gamma|_\Delta) \in W(M,D,\gamma) \ .$$

Since each inner face F of D belongs to two tetrahedra of the triangulation D, it occurs in the tensor product defining the space $W(M,D,\gamma)$ twice, with opposite orientations. By Proposition 5(iii), the corresponding spaces $V(F)(\gamma|_F)$ are dual to each other. Denote by $c_F : (V(F)(\gamma|_F))^* \otimes V(F)(\gamma|_F) \to k$ the corresponding pairing. Since each boundary face of D is a face of D_S, the tensor product $c_M = \bigotimes_F c_F$ over all inner faces F of D is a linear map

$$c_M : W(M,D,\gamma) \to W(S, D_S, \gamma|_S) \ .$$

To define the invariant associated to the given balanced rigid triangulated category we must choose, for each $a \in \mathcal{E}$, a square root $(\dim a)^{1/2} \in k$. Also, we assume that $\sum_{a\in\mathcal{E}} (\dim a)(\dim A^*) \neq 0$ and choose a square root

$$N = \left(\sum_{a\in\mathcal{E}} (\dim a)(\dim A^*)\right)^{1/2} \in k \ .$$

Definition 5: Let γ_S be a coloring of D_S. Denote

$$\begin{aligned} &I(M,D,\gamma_S) \qquad (13)\\ &= N^{-2v-v'} \sum_\gamma \left(\prod_l (\dim \gamma(l))\right)\left(\prod_{l'} (\dim \gamma(l'))^{1/2}\right) c_M(L(M,D,\gamma))\\ &\in W(S, D_S, \gamma_S) \ . \end{aligned}$$

In this formula the sum is taken over all coloring γ of D that extend γ_S, l' runs over all edges on the boundary of M (i.e. edges of D_S), l runs over all edges inside M, v' is the number of vertices on the boundary of M (vertices of S), v is the number of vertices inside M.

Let us note that if M is a closed manifold, then $I(M,D)$ does not depend on the above choices of square roots.

The main result of this paper is the following:

Theorem 1. *The element $I(M, D, \gamma_S)$ does not depend on the extension of the triangulation D_S of S to the triangulation D of the manifold M.*

DEFINITION 6: For a three-dimensional manifold M with triangulated boundary (S, D_S), a given balanced rigid semisimple monoidal category $\mathbf{C}$ with the finite number of equivalence classes of simple objects, and a coloring γ_S of D_S, we define $I_{\mathbf{C}}(M, D_S, \gamma_S)$ to be the element $I(M, D, \gamma_S)$ for arbitrary extension of the triangulation D_S of S to the triangulation D of the manifold M.

In particular, if M is a closed manifold, $\partial M = \varnothing$, we obtain a number $I_{\mathbf{C}}(M) \in k$ that depends only on M.

Theorem 1 will be proved in §§8–12. In the next section we interpret the invariant $I_{\mathbf{C}}(M, D_S, \gamma_S)$ in terms of cobordisms between closed oriented surfaces and study what happens when we glue together two manifolds M.

7. Cobordisms and Gluing

For a closed oriented triangulated surface (S, D) define the space $W(S, D)$ by the formula

$$W(S, D) = \bigoplus_{\gamma} W(S, D, \gamma) ,$$

where the sum is taken over all coloring γ of edges of D. If M is a three-dimensional oriented manifold with triangulated boundary (S, D), we denote

$$I_{\mathbf{C}}(M, S, D) = \bigoplus_{\gamma} I_{\mathbf{C}}(M, D, \gamma),$$

where again the sum is taken over all coloring γ of edges of D.

Recall that a cobordism between two closed oriented surfaces S_1 and S_2 is a three-dimensional oriented manifold N with boundary $\partial N = (S_1) \cup (-S_2)$ (disjoint union), where $-S_2$ is the surface S_2 with the opposite orientation. We denote such a cobordism by $N : S_1 \to S_2$. In particular, an oriented three-dimensional manifold with boundary S can be considered as a cobordism $M : S \to \varnothing$.

Now assume that S_1, S_2 are triangulated surfaces, with triangulations D_2 and D_2 respectively. Colorings γ_1 of (S_1, D_1) and γ_2 of (S_2, D_2) define the coloring γ of the triangulated boundary $(S_1 \cup (-S_2), D_1 \cup D_2)$ of the manifold N. Definition 6 provides an element $I_{\mathbf{C}}(N, D_1 \cup D_2, \gamma) \in W(S_1 \cup (-S_2), D_1 \cup D_2, \gamma)$. Since S_1 and S_2 are disjoint,

$$W\big(S_1 \cup (-S_2), D_1 \cup D_2, \gamma\big) = W(S_1, D_1, \gamma_1) \otimes W(S_2, D_2, \gamma_2)^* ,$$

so that $I_{\mathbf{C}}(N, D_1 \cup D_2, \gamma)$ gives a linear map

$$\Theta : W(S_2, D_2, \gamma_2) \to W(S_1, D_1, \gamma_1) .$$

By Theorem 1, this map depends (for fixed triangulations and colorings of S_1 and S_2) only on the isotopy class of the cobordism $N : S_1 \to S_2$. The direct sum of these maps Θ over all colorings γ_1 and γ_2 defines a linear map

$$\Theta(N) : W(S_2, D_2) \to W(S_1, D_1) ,$$

which again depends only on the isotopy class of N. If M with $\partial M = S$ is considered as a cobordism $M : S \to \varnothing$, then the image of $1 \in k = W(\varnothing)$ in $W(S, D)$ under $\Theta(M)$ coincides with $I_{\mathbf{C}}(M, S, D) \in W(S, D)$.

Now let (S_i, D_i), $i = 1, 2, 3$, be three closed oriented surfaces and $N : S_1 \to S_2$, $N' : S_2 \to S_3$ be two cobordisms. Their composition is defined as a cobordism $N' \circ N : S_1 \to S_3$, given by the gluing of N and N' along S_2.

COROLLARY 1 (from Theorem 1). *We have* $\Theta(N' \circ N) = \Theta(N) \circ \Theta(N')$.

COROLLARY 2. *For any M with $\partial M = S$, any cobordism $N : S' \to S$, and any triangulations D and D' of S and S' respectively, we have*

$$\Theta(N) I_{\mathbf{C}}(M, S, D) = I_{\mathbf{C}}(N \circ M, S', D') .$$

The trivial cobordism of a closed oriented surface S is defined by $N_S = S \times I$, where I is the closed unit interval. Since $N_S \circ N_S = N_S$, we have

COROLLARY 3. *$\Theta(N_S) : W(S, D) \to W(S, D)$ is a projection.*

Denote $K(S, D) = \operatorname{im} \Theta(N_S) \subset W(S, D)$.

COROLLARY 4. *For any M with $\partial M = S$ and any triangulation D of S we have $I_{\mathbf{C}}(M, S, D) \in K(S, D)$.*

COROLLARY 5. *For any cobordism $N : S_1 \to S_2$ we have*

$$\Theta(N)\big(K(S_1, D_1)\big) \subset K(S_2, D_2) .$$

Proof. There exists an isotopy $N_{S_2} \circ N \to N$ identical on $\partial(N_{S_2} \circ N) = S_1 \cup (-S_2) = \partial N$. Therefore, Corollary 5 follows from Corollary 1.

In particular, consider the trivial cobordism $N_S = S \times I : S \to S$ between (S, D_1) and (S, D_2) for two triangulations D_1 and D_2 of S. Since $N_S \circ N_S = N_S$, we have

COROLLARY 6. *The restriction of $\Theta(N_S)$ to $K(S, D_2)$ is an isomorphism*

$$\alpha_{D_1 D_2} : K(S, D_2) \to K(S, D_1)$$

with the following properties:

$$\alpha_{D_1 D_2} \circ \alpha_{D_2 D_3} = \alpha_{D_1 D_3} , \qquad \alpha_{DD} = \mathrm{id} ,$$
$$\alpha_{D_1 D_2}\big(I_{\mathbf{C}}(M, S, D_2)\big) = I_{\mathbf{C}}(M, S, D_1) \quad \textit{for any } M,\ \partial M = S .$$

DEFINITION 7: For a closed surface S, denote by $K(S)$ the direct limit $\lim_D K(S,D)$ over all triangulations D of S (with respect to isomorphisms $\alpha_{D_1 D_2}$).

For a three-dimensional manifold M with boundary S denote

$$I_{\mathbf{C}}(M,S) = \lim_{D} I_{\mathbf{C}}(M,S,D) \in K(S) .$$

COROLLARY 7. *Any cobordism $N : S_1 \to S_2$ defines a linear map $\Theta(N) : K(S_2) \to K(S_1)$, such that $\Theta(N \circ N') = \Theta(N') \circ \Theta(N)$ for $N : S_1 \to S_2$, $N' : S_2 \to S_3$; in particular, $\Theta(N)(I_{\mathbf{C}}(M,S)) = I_{\mathbf{C}}(N \circ M, S)$ for a three-dimensional manifold M with $\partial M = S$ and a cobordism $N : S \to S'$.*

The family of spaces $K(S)$ and vectors $I_{\mathbf{C}}(M,S)$ is one of the main ingredients of the Topological Quantum Field Theory associated to the invariant $I_{\mathbf{C}}$ of three-dimensional manifolds.

In the remaining part of this section we present certain results and conjectures about the spaces $K(S)$.

The proof of the next two propositions is left to the reader.

PROPOSITION 7. *If S is the sphere S^2, then $\dim K(S) = 1$.*

Let G be a finite group, ζ a 3-cocycle on G, $\mathbf{C}(G,\zeta)$ the monoidal category associated to (G,ζ) (see [CPr, Example 5.1.6]) with trivial duality and balancing.

PROPOSITION 8. *Let $\mathbf{C} = \mathbf{C}(G,1)$ be the category associated to the trivial 3-cocycle on the finite group G and S a closed oriented surface with the fundamental group π. Then $\dim K(S)$ equals the number of equivalence classes of homomorphisms $\pi \to G$ modulo conjugations by elements of G.*

The following conjecture describes $\dim K(T)$, where T is the two-dimensional torus.

Denote by $D(\mathbf{C})$ the Drinfeld double of the monoidal category $\mathbf{C}$ (see [M]).

CONJECTURE 1. *Assume that $D(\mathbf{C})$ is a semisimple category. Then $\dim K(T)$ is equal to the number of equivalence classes of simple objects in $D(\mathbf{C})$.*

If $D(\mathbf{C})$ is not semisimple, then the number of equivalence classes of simple objects in $D(\mathbf{C})$ should, conjecturally, be replaced by the dimension of a certain space introduced by Lyubashenko [L], see also Kerler [Ke].

Let us remark that if $\mathbf{C} = \mathbf{C}(G,1)$, then $D(\mathbf{C})$ is the category of G-equivariant (under the action of G on itself by conjugations) vector bundles on G. A simple object of $D(\mathbf{C})$ is a pair (A,ρ) consisting of a conjugacy class $A \subset G$ and an irreducible representation ρ of the centralizer $Z(a)$ of an element $a \in A$. Therefore, the number of simple objects in $D(\mathbf{C})$ equals the number of G-conjugacy classes of pairs (g_1, g_2) of commuting elements in G, and the corresponding special case of Conjecture 1 agrees with the special case of Proposition 8.

8. Pachner's Theorem

Now we proceed with the proof of Theorem 1. For simplicity, we will consider only the case of a closed three-dimensional manifold M, so that $I_{\mathbf{C}}(M, \varnothing)$ is an element $I_{\mathbf{C}}(M) \in k$. All arguments can be easily extended to manifolds with boundary (see similar arguments in [TV]).

The proof of Theorem 1 is based on the possibility of passing from any triangulation of M to any other triangulation by a sequence of simple changes. More precisely, consider the following elementary changes of a triangulation D of a closed three-dimensional manifold M.

(A) Replace two tetrahedra with a common face by three tetrahedra with a common edge, as shown in Fig. 18. This change increases the number of tetrahedra in the triangulation by 1, the number of faces by 2, the number of edges by 1, and does not change the number of vertices.

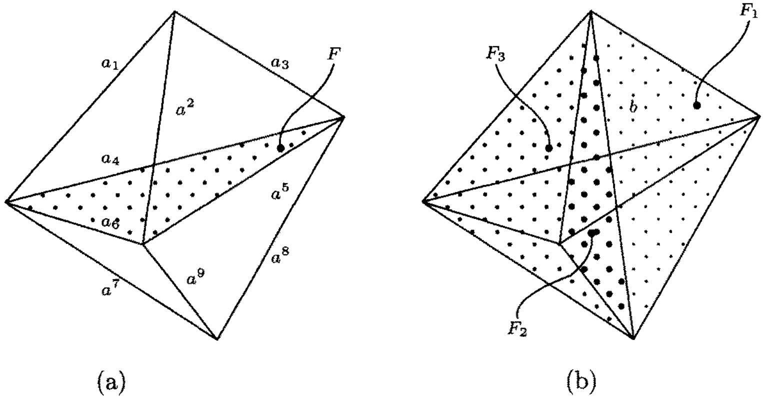


Figure 18. Pachner's change (A): (a) before the change, (b) after the change

(B) Replace a tetrahedron Δ of D by four tetrahedra forming the stellar subdivision of Δ with the center inside Δ. This change increases the number of tetrahedra in the triangulation by 3, the number of faces by 6, the number of edges by 4, and the number of vertices by 1.

Theorem 2 (Pachner, [P]). *Any two finite triangulations of a closed three-dimensional manifold can be obtained from one another by a sequence of changes of types* (A), (B) *and their inverses.*

REMARK: There is a version of Pachner's theorem for manifolds with boundary; it allows one to compare trangulations of a given manifold M that agree on its boundary ∂M.

In view of Pachner's theorem, it suffices to prove that $I(M, D)$ does not change under transformations (A) and (B). We begin with case (A).

9. Invariance Under Change (A)

Let $K \subset M$ be the body formed by two simplices with a common face (see Fig. 18(a)). Since the two triangulations we want to compare coincide outside K, it suffices to consider only the part of the sum defining $I(M, D)$ for tetrahedra inside K. Let Δ_1, Δ_2 be the tetrahedra of the triangulation (a), and $\Delta_3, \Delta_4, \Delta_5$ be the tetrahedra of the triangulation (b). Denote by F the internal face in the triangulation of K shown in Fig. 18(a). Similarly, denote by F_1, F_2, F_3 the three internal faces of the triangulation of K shown in Fig. 18(b). Consider colorings γ' of the triangulation (a) of K and γ'' of the triangulation (b) such that colors corresponding to exterior edges (those on the boundary of K) coincide. Denote these colors by $a_1, \dots, a_9$, as in Fig. 18. The colors $a_1, \dots, a_9$ determine γ' uniquely. Denote by b the color corresponding to the internal edge in the triangulation (b). The invariance of the number $I(M, D)$ under the change (A) in a consequence of the following formula:

$$\begin{aligned} &c_F\big(L(\Delta_1, \gamma') \otimes L(\Delta_2, \gamma')\big) \\ &\qquad = \sum_{b \in \mathcal{E}} (\dim b)(c_{F_1} c_{F_2} c_{F_3})\big(L(\Delta_3, \gamma'') \otimes L(\Delta_4, \gamma'') \otimes L(\Delta_5, \gamma'')\big)\ , \end{aligned} \tag{14}$$

which is sometimes called the Biederharn formula.

To prove formula (14), we give another interpretation for the functional $L(\Delta, \gamma)$, which relates it to associativity morphisms in $\mathbf{C}$, and show that (14) is essentially equivalent to the pentagon axiom in $\mathbf{C} = (\mathcal{C}, \otimes, \varphi)$.

Recall that the associativity morphisms and the pentagon axiom in a semisimple monoidal category can be expressed as follows. For any objects a_1, a_2, a_3, a_4, a_5 in $\mathcal{C}$, the composition of morphisms defines a linear map

$$\mathrm{Hom}(a_1, a_5 \otimes a_4) \otimes \mathrm{Hom}(a_5, a_2 \otimes a_3) \to \mathrm{Hom}(a_1, a_2 \otimes a_3 \otimes a_4)\ . \tag{15}$$

Fix a_1, a_2, a_3, a_4, and let a_5 run over the set $\mathcal{E}$ of all isomorphism classes of simple objects in $\mathcal{C}$. Since $\mathcal{C}$ is a semisimple category, maps (15) combine to an isomorphism

$$\bigoplus_{a_5\in\mathcal{E}} \mathrm{Hom}(a_1, a_5\otimes a_4)\otimes \mathrm{Hom}(a_5, a_2\otimes a_3) \xrightarrow{\sim} \mathrm{Hom}(a_1, a_2\otimes a_3\otimes a_4) .$$

Similarly, we have the isomorphism

$$\bigoplus_{a_6\in\mathcal{E}} \mathrm{Hom}(a_1, a_2\otimes a_6)\otimes \mathrm{Hom}(a_6, a_3\otimes a_4) \xrightarrow{\sim} \mathrm{Hom}(a_1, a_2\otimes a_3\otimes a_4) .$$

Therefore, for each 6-tuple of elements $a_1,\dots,a_6\in\mathcal{E}$ we have a linear map

$$\lambda : \mathrm{Hom}(a_1, a_5\otimes a_4)\otimes \mathrm{Hom}(a_5, a_2\otimes a_3) \to \mathrm{Hom}(a_1, a_2\otimes a_6)\otimes \mathrm{Hom}(a_6, a_3\otimes a_4) ,$$

which will be denoted $\lambda(a_1,a_2,a_3,a_4,a_5,a_6)$ in the case when we will need to specify $a_1,a_2,a_3,a_4,a_5,a_6\in\mathcal{E}$.

The invariance of the sum (13) under the Pachner move (B) is a result of the pentagon axiom in $\mathbf{C}$, which, in terms of maps λ, is equivalent to the following relation. For any $a_1,\dots,a_9\in\mathcal{E}$ we have

$$\begin{aligned}&\lambda(a_1,a_9,a_4,a_5,a_7,a_8)\circ\lambda(a_1,a_2,a_3,a_7,a_6,a_9)\\ &\quad=\sum_{b\in\mathcal{E}}\lambda(a_8,a_2,a_3,a_4,b,a_9)\circ\lambda(a_1,a_2,b,a_5,a_6,a_8)\circ\lambda(a_6,a_3,a_4,a_5,a_7,b) .\end{aligned}$$

Here both sides are morphisms:

$$\begin{aligned}&\mathrm{Hom}(a_1, a_2\otimes a_6)\otimes \mathrm{Hom}(a_6, a_3\otimes a_7)\otimes \mathrm{Hom}(a_7, a_4\otimes a_5)\\ &\quad\to \mathrm{Hom}(a_1, a_8\otimes a_5)\otimes \mathrm{Hom}(a_8, a_9\otimes a_4)\otimes \mathrm{Hom}(a_9, a_2\otimes a_3) .\end{aligned}$$

Now we use the duality

$$\begin{aligned}\big(\mathrm{Hom}(a_1, a_2\otimes a_6)\big)^* &= \mathrm{Hom}(a_2\otimes a_6, a_1) ,\\ \big(\mathrm{Hom}(a_6, a_3\otimes a_4)\big)^* &= \mathrm{Hom}(a_3\otimes a_4, a_6) ,\end{aligned}\tag{16}$$

given by the formula $\langle\varphi,\psi\rangle = \mathrm{tr}(\varphi\psi)$ (see Lemma 1). It allows us to consider the map λ as a linear functional on the space

$$\mathrm{Hom}(a_1, a_5\otimes a_4)\otimes \mathrm{Hom}(a_5, a_2\otimes a_3)\otimes \mathrm{Hom}(a_2\otimes a_6, a_1)\otimes \mathrm{Hom}(a_3\otimes a_4, a_6) .$$

The next lemma gives an explicit formula for this functional.

LEMMA 7. *For*

$$\begin{aligned}&\varphi_1 : a_1\to a_5\otimes a_4 , &&\varphi_2 : a_5\to a_2\otimes a_3 ,\\ &\varphi_3 : a_3\otimes a_4\to a_6, &&\varphi_4 : a_2\otimes a_6\to a_1 ,\end{aligned}\tag{17}$$

we have

$$\lambda(\varphi_1\otimes\varphi_2\otimes\varphi_3\otimes\varphi_4) = (\dim a_6)\,\mathrm{tr}\,\Phi ,$$

where $\Phi : a_1\to a_1$ *is the composition*

$$a_1\xrightarrow{\varphi_1} a_5\otimes a_4\xrightarrow{\varphi_2\otimes\mathrm{id}} a_2\otimes a_3\otimes a_4\xrightarrow{\mathrm{id}\otimes\varphi_3} a_2\otimes a_6\xrightarrow{\varphi_4} a_1 .\tag{18}$$

Proof. Take arbitrary morphisms $\psi_3 : a_6 \to a_3 \otimes a_4$, $\psi_4 : a_1 \to a_2 \otimes a_6$ and compare $\operatorname{tr}(\varphi_3\psi_3)\operatorname{tr}(\varphi_4\psi_4)$ with the trace of the composition

$$\theta : a_1 \xrightarrow{\psi_4} a_2 \otimes a_6 \xrightarrow{\psi_3 \otimes \mathrm{id}} a_2 \otimes a_3 \otimes a_4 \xrightarrow{\varphi_3 \otimes \mathrm{id}} a_6 \otimes a_4 \xrightarrow{\varphi_3} a_1 .$$

Since a_6 is a simple object, $\varphi_3\psi_3 = \mu\, \mathrm{id}_{a_6}$ for some number μ, and $\operatorname{tr}(\varphi_3\psi_3) = \mu \dim a_6$. Therefore,

$$\theta = \mu\varphi_4\psi_4$$

and

$$\operatorname{tr}\theta = \mu \operatorname{tr}(\varphi_4\psi_4) = \operatorname{tr}(\varphi_3\psi_3)\operatorname{tr}(\varphi_4\psi_4)/\dim a_6 .$$

Lemma 7 immediately follows from this formula.

The canonical isomorphisms $\operatorname{Hom}(X,Y) = \operatorname{Hom}(\mathbf{1}, Y \otimes X^*)$, $X, Y \in \operatorname{Ob}\mathcal{C}$, yield isomorphisms of linear spaces

$$\begin{aligned} \operatorname{Hom}(a_1, a_5 \otimes a_4) &= V(a_5, a_4, a_1^*) , \\ \operatorname{Hom}(a_5, a_2 \otimes a_3) &= V(a_2, a_3, a_5^*) , \\ \operatorname{Hom}(a_2 \otimes a_6, a_1) &= V(a_1, a_6^*, a_2^*) , \\ \operatorname{Hom}(a_3 \otimes a_4, a_6) &= V(a_6, a_4^*, a_3^*) . \end{aligned} \tag{19}$$

This enables us to consider λ as a linear functional on the space

$$V(a_5, a_4, a_1^*) \otimes V(a_2, a_3, a_5^*) \otimes V(a_6, a_4^*, a_3^*) \otimes V(a_1, a_6^*, a_2^*) ,$$

i.e. on the same space as the functional $L(\Delta, \gamma)$.

LEMMA 8. *In the above notation,*

$$\lambda = (\dim a_6) L(\Delta, \gamma) .$$

Proof. Select $\varphi_1, \varphi_2, \varphi_3, \varphi_4$. We must prove that $L(\Delta,\gamma)(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \operatorname{tr}\Phi$, where $\Phi : a_1 \to a_1$ is defined by (18). The proof is shown in Fig. 19.

To complete the proof of formula (14), we note that the duality (16) is compatible with the duality between the spaces V in Proposition 5(iii) under the isomorphisms (19). Therefore, after the identification (14) the pentagon axiom becomes the required formula (14).

Before proving the invariance of the sum (13) under the stellar transformation (B), we establish two properties of functionals $L(\Delta, \gamma)$, normalization and orthogonality.

10. Normalization Property

The normalization property determines the functional $L(\Delta, \gamma)$ for a degenerate coloring Γ, i.e. such a coloring that one of the objects $\gamma(l)$ is the object $\mathbf{1}$. By Proposition 6, it suffices to consider the case $a_1 = \mathbf{1}$ (we use the notation in (9)).

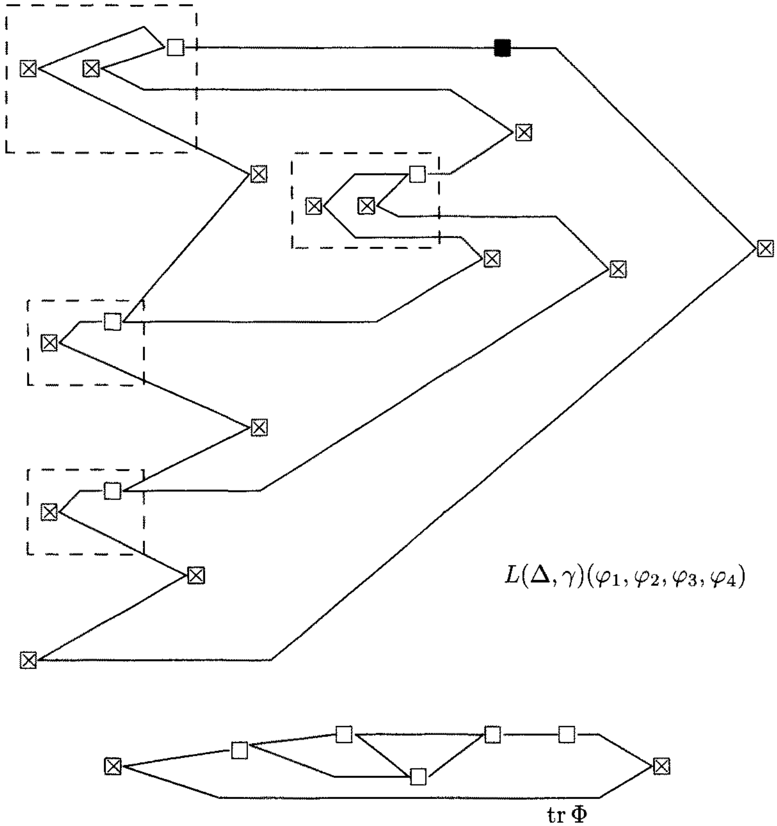


Figure 19. Proof of Lemma 8

PROPOSITION 9 (normalization). *Let γ be a coloring of the tetrahedron Δ such that $a_1 = \gamma(\overrightarrow{01}) = \mathbf{1}$. Then*

(i) *The space $V(\Delta,\gamma)$ is zero unless $a_4 = a_5^*$ and $a_6 = a_2^*$.*

(ii) *If $a_4 = a_5^*$ and $a_6 = a_2^*$, then the spaces*

$$V_1 = \operatorname{Hom}(\mathbf{1}, a_5 \otimes a_5^*) \ ,$$
$$V_4 = \operatorname{Hom}(\mathbf{1}, a_6 \otimes a_6^*)$$

are one-dimensional with generators i_{a_5} and i_{a_6} respectively, while *the spaces*

$$V_2 = \mathrm{Hom}(\mathbf{1}, a_2 \otimes a_3 \otimes a_4) \ ,$$
$$V_3 = \mathrm{Hom}(\mathbf{1}, a_2^* \otimes a_4^* \otimes a_3^*) = \mathrm{Hom}(\mathbf{1}, a_4^* \otimes a_3^* \otimes a_2^*)$$

are dual to each other.

(iii) The functional $L(\Delta, \gamma)$ is given by the following formula:

$$L(\Delta, \gamma)(i_{a_5}, v_2, v_3, i_{a_6}) = \langle v_2, v_3 \rangle \ ,$$

where $\langle \cdot, \cdot \rangle$ is the pairing between V_2 and V_3 from (ii).

Proof. Part (i) and the first statement in part (ii) follow from the formula

$$\mathrm{Hom}(\mathbf{1}, X \otimes Y^*) = \mathrm{Hom}(X, Y) \ .$$

The second statement in (ii) follow from Proposition 6 and Proposition 5(iii). The proof of part (iii) is clear from the diagram in Fig. 20.

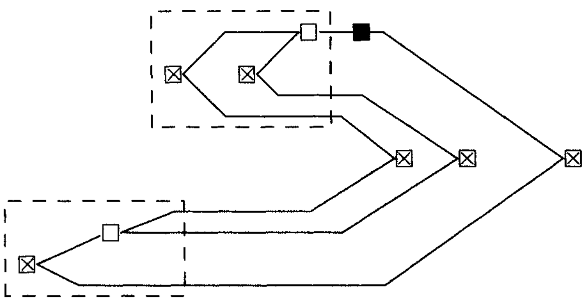


Figure 20. Proof of part (iii) of Proposition 9

11. Orthogonality Property

Let Δ_1, Δ_2 be two tetrahedra glued together along two adjacent faces on each of them. The resulting body B has 4 vertices, 7 edges, 6 faces, and 2 tetrahedra. Of these 6 faces, two are adjacent to both tetrahedra Δ and Δ', and each of the four others is adjacent to only one tetrahedron.

Let γ be a coloring of B, i.e. a function that sends 7 edges of B to elements of $\mathcal{E}$. Equivalently, we can think of γ as a pair of colorings γ_1 of Δ_1 and γ_2 of Δ_2, such that γ_1 and γ_2 take equal values on corresponding edges. More precisely, denote vertices of Δ_1 by $0_1, 1_1, 2_1, 3_1$ and those of Δ_1

by $0_2, 1_2, 2_2, 3_2$ (see Fig. 21). Then γ_1 and γ_2 should satisfy the conditions:

$$\gamma_1(\overrightarrow{0_1 1_1}) = \gamma_2(\overrightarrow{0_2 1_2}) \ (= a_1) \ ,$$
$$\gamma_1(\overrightarrow{1_1 2_1}) = \gamma_2(\overrightarrow{1_2 2_2}) \ (= a_2) \ ,$$
$$\gamma_1(\overrightarrow{0_1 2_1}) = \gamma_2(\overrightarrow{0_2 2_2}) \ (= a_3) \ ,$$
$$\gamma_1(\overrightarrow{1_1 3_1}) = \gamma_2(\overrightarrow{1_2 3_2}) \ (= a_4) \ ,$$
$$\gamma_1(\overrightarrow{2_1 3_1}) = \gamma_2(\overrightarrow{2_2 3_2}) \ (= a_5) \ .$$

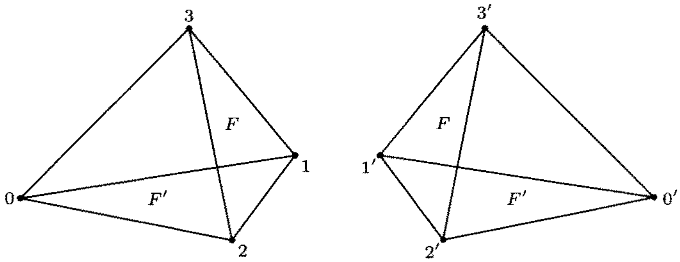


Figure 21.

Denote

$$b_1 = \gamma_1(\overrightarrow{0_1 3_1}), \quad b_2 = \gamma_2(\overrightarrow{0_2 3_2}).$$

To a coloring $\gamma = (\gamma_1, \gamma_2)$ of B we associate the space

$$V(\Delta_1, \gamma_1) \otimes V(\Delta_2, \gamma_2) \ .$$

On this space, consider the functional

$$L(\Delta_1, \gamma_1) \otimes L(\Delta_2, \gamma_2) \ .$$

Note that the space $V(\Delta_1, \gamma_1) \otimes V(\Delta_2, \gamma_2)$ is the product of 8 spaces of the form $V(a, b, c)$, $a, b, c \in \mathcal{E}$, and among these spaces there are two pairs of dual spaces corresponding to faces $F = (0_1 1_1 2_1) = (2_2 1_2 0_2)$, $F' = (1_1 2_1 3_1) = (3_2 2_2 1_2)$. Hence we can define the functional

$$L(B, \gamma) = c_F c_{F'} \big(L(\Delta_1, \gamma_1) \otimes L(\Delta_2, \gamma_2) \big)$$

on the space

$$V(B, \gamma) = V(a_3, a_5, b_1^*) \otimes V(b_2, a_5^*, a_3^*) \otimes V(b_1, a_4^*, a_1^*) \otimes V(a_1, a_4, b_2^*) \ . \quad (20)$$

The space $V(B,\gamma)$ does not depend on a_2. Therefore, we can consider

$$L = \sum_{a_2} (\dim a_2) L(B,\gamma) \ .$$

PROPOSITION 10 (orthogonality). (i) *If $b_1 \neq b_2$, then $L = 0$.*

(ii) *If $b_1 = b_2$, so that four factors in the tensor product $V(B,\gamma)$ (see (20)) are pairwise dual,*

$$V(a_3, a_5, b_1^*) = \big(V(b_2, a_5^*, a_3^*)\big)^* \ ,$$
$$V(b_1, a_4^*, a_1^*) = \big(V(a_1, a_4, b_2^*)\big)^* \ ,$$

then

$$L(\varphi_1 \otimes \varphi_2 \otimes \varphi_3 \otimes \varphi_4) = (\dim b)^{-1} \langle \varphi_1, \varphi_2 \rangle \langle \varphi_3, \varphi_4 \rangle \ . \tag{21}$$

Sketch of the proof. This proposition can be proved in two different ways. The first proof, which is short and formal, goes as follows.

First of all, let V be an arbitrary finite-dimensional linear space V, $i \in V \otimes V^*$ the canonical Casimir element, and $c_V : V \otimes V^* \to k$ the canonical pairing; then

$$c_V(i) = \dim V \ . \tag{22}$$

Now in the Biederharn formula (14) take a_4 (or a_5, or a_6) to be $\mathbf{1}$. Then the normalization property, together with (22), implies (21).

The second proof consists in writing down the definition

$$L(B,\gamma) = c_F c_{F'} \big(L(\Delta_1, \gamma_1) \otimes L(\Delta_2, \gamma_2) \big) \ ,$$

drawing the corresponding diagram in the spirit of Fig. 13, and playing with this diagram to get the required expression. We strongly recommend the reader to carry out this proof in detail. In particular, this might give the feeling of what can be done with diagrams representing morphisms in rigid monoidal categories.

12. Invariance Under Change (B)

Now we prove the invariance of the sum (16) under change (B) of the triangulation of our closed manifold M.

Let Δ be a tetrahedron with vertices $0, 1, 2, 3$, and A be a point inside Δ. The transformation (B) replaces Δ with four tetrahedra Δ_i, $0 \leq i \leq 3$, where Δ_i is the tetrahedron opposite the vertex i (see Fig. 22). Denote by F_{ij}, $0 \leq i, j \leq 3$, the six inner faces of the obtained subdivision (F_{ij} has vertices $\{ijA\}$) and by F_i, $0 \leq i \leq 3$, the four outer faces, which are faces of Δ (F_i is the face opposite the vertex i).

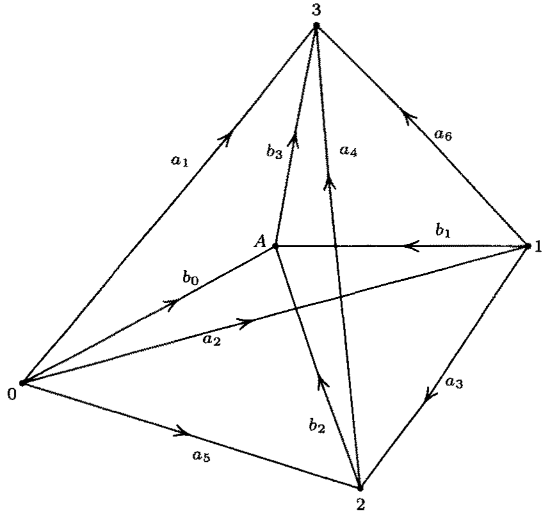


Figure 22

Let γ be a coloring of Δ, and γ' be an extension of γ to a subdivided tetrahedron, so that to specify γ' we must define

$$\gamma'(\overrightarrow{0A}) = b_0 \ , \quad \gamma'(\overrightarrow{1A}) = b_1 \ , \quad \gamma'(\overrightarrow{2A}) = b_2 \ , \quad \gamma'(\overrightarrow{A3}) = b_3$$

(the choice of directions of edges from A will become clear later).

The invariance of $I(M, D)$ under the change (B) is equivalent to the following statement.

PROPOSITION 11. *We have*

$$(\Delta, \gamma) = N^{-2} \sum_{b_0, b_1, b_2, b_3 \in \mathcal{E}} \prod_{i=0}^{3} (\dim b_i)(c_{F_0} c_{F_1} c_{F_2} c_{F_3}) \left(\bigotimes L(\Delta_i, \gamma') \right) \ . \quad (23)$$

Proof. Rewrite the right-hand side of (23) in the form

$$N^{-2} \sum_{b_0, b_1, b_2 \in \mathcal{E}} (\dim b_0 \dim b_1 \dim b_2) c_{F_3} \Bigg\{ L(\Delta_3, \gamma')$$
$$\otimes \left[\sum_{b_3 \in \mathcal{E}} (\dim b_3)(c_{F_0} c_{F_1} c_{F_2}) L(\Delta_0, \gamma') \otimes L(\Delta_1, \gamma') \otimes L(\Delta_2, \gamma') \right] \Bigg\} \ .$$

We see that for fixed b_0, b_1, b_2, the sum over b_3 has the same form as the sum in the Biederharn formula (14). Applying this formula, we replace the expression in brackets by the convolution of the tensor product of two functionals of the form L, say

$$c_F L(\Delta', \gamma') \otimes L(\Delta'', \gamma') ,$$

with one of the functionals, say $L(\Delta', \gamma')$ being just $L(\Delta, \gamma)$ for our initial tetrahedron Δ and the initial coloring γ, while the other, $L(\Delta'', \gamma')$, together with $L(\Delta_3, \gamma')$ forms a setting for applying the orthogonality relation. Using this relation together with the formula

$$\sum_{i,j \in \mathcal{E}} (\dim i)(\dim j) \dim V(i,j,k^*) = N^2 \dim k ,$$

which follows from Lemma 9 below, we complete the proof of Theorem 1.

LEMMA 9. *For $i, j, k \in \mathcal{E}$ denote by*

$$m_{ij}^k = \dim \mathrm{Hom}(k, i \otimes j)$$

the multiplicity of k in $i \otimes j$. Then for any $k \in \mathcal{E}$ we have

$$\sum_{i,j \in \mathcal{E}} (\dim i)(\dim j) m_{ij}^k = N^2 \dim k . \tag{24}$$

Proof. First of all, since the dimension in $\mathbf{C}$ is additive and multiplicative (see Proposition 1(i,ii)) and the category $\mathcal{C}$ is semisimple, we have, for any $k, j \in \mathcal{E}$,

$$\dim k \dim j^* = \sum_i m_{kj^*}^i \dim i . \tag{25}$$

Next, from

$$\mathrm{Hom}(k \otimes j^*, i) = \mathrm{Hom}(k, i \otimes j) ,$$

we get $m_{kj^*}^i = m_{ij}^k$. Substituting into (25), multiplying both parts by $\dim j$, and summing by j over $\mathcal{E}$, we get (24).

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Added in Proof. After this paper was accepted for publication, D. Yetter pointed to his paper [Y], where a generalization of the original Turaev-Viro method is proposed. Also, he brought to our attention the paper [BWe] by J. Barnett and B. Westbury, whose approach to the construction of invariants of three-dimensional manifolds is similar to the approach in this paper. We are grateful to D. Yetter for these comments.

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