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Pullback and Pushout Constructions in C^*-Algebra Theory[1]

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A systematic study of pullback and pushout diagrams is conducted in order to understand restricted direct sums and amalgamated free products of C^*-algebras. Particular emphasis is given to the relations with tensor products (both with the minimal and the maximal C^*-tensor norm). Thus it is shown that pullback and pushout diagrams are stable under tensoring with a fixed algebra and stable under crossed products with a fixed group. General tensor products between diagrams are also investigated. The relations between the theory of extensions and pullback and pushout diagrams are explored in some detail. The crowning result is that if three short exact sequences of C^*-algebras are given, with appropriate morphisms between the sequences allowing for pullback or pushout constructions at the levels of ideals, algebras and quotients, then the three new C^*-algebras will again form a short exact sequence under some mild extra conditions. As a generalization of a theorem of T. A. Loring it is shown that each morphism between a pair of C^*-algebras, combined with its extension to the stabilized algebras, gives rise to a pushout diagram. This result has applications to corona extendibility and conditional projectivity. Finally the pullback and pushout constructions are applied to the class of noncommutative CW complexes defined by (S. Eilers, T. A Loring, and G. K. Pedersen *J. Reine Angew. Math.*, 1998, **499**, 101–143) to show that this category is stable under tensor products and under restricted direct sums.

Key Words: C^*-algebra; pullback diagram; pushout diagram; restricted direct sum; amalgamated free product; extension; tensor product; direct limit; inverse limit; crossed product; noncommutative CW complex.

1. INTRODUCTION

This investigation arose out of a desire to understand some technical problems concerning the class of C^*-algebras labeled "noncommutative *CW* complexes" (or *NCCW* complexes) in [21, Sect. 2.4]. Their existence had been prophesied by Effros with uncanny precision in [18]. Basically these are algebras of matrix-valued continuous functions over topological

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spaces homeomorphic to CW complexes, but the definition allows for all kinds of "automorphism twists" and "dimension drops," so the class is rich. On the other hand, the definition of $NCCW$ complexes is rigid and recursive (patterned after the commutative case), so the class is well suited for axiomatic study. Its importance as the source of "inductive building blocks" for more complicated C^*-algebras is well documented by the Elliott programme. Our main results in this direction are that the category of $NCCW$ complexes with simplicial *-homomorphisms as morphisms is closed under the process of taking kernels, range algebras, and counter-images of subcomplexes and closed under restricted direct sums $A \oplus_C B$ and tensor product $A \otimes B$.

En route to these results we are led to conduct a rather extensive study of algebras that can be constructed as restricted direct sums and as amalgamated free products. This means that we systematically investigate pullback and pushout diagrams in the category of C^*-algebras. Interpreting the theory of general C^*-algebras as "noncommutative topology" (emanating from the category of locally compact Hausdorff spaces), the pullback construction is a perfect generalization of the familiar concept of "glueing" together topological spaces. The pushout construction, by contrast, has no immediate analogue. If performed wholly inside the commutative category the pushout reduces to a construction of filtered closed subspaces of the cartesian product of two given spaces. But this hardly prepares us for the noncommutative generalizations, which involve free product C^*-algebras in the definition of amalgamated free products $A \star_C B$.

A large number of our results, especially about pushout diagrams, are only valid if one or more of the linking morphisms are *proper* (i.e., map an approximate unit of the source algebra into an approximate unit for the range). This is probably no coincidence. As explained in [20, 2.1], the category of (nonunital) C^*-algebras with proper *-homomorphisms as morphisms is the correct noncommutative analogue of the category of locally compact Hausdorff spaces with proper, continuous maps as morphisms. In this light the humble Lemmas 4.6 and 5.2 assume a central position; and certainly they are the most used results in this paper (15 citations).

In the loosely structured Section 2 we gather a number of more or less well known results which we will need in the sequel. We then characterize pullback diagrams abstractly in Section 3, and show that if X_1 and X_2 are C^*-algebras obtained from pullback diagrams, then there are canonical pullback diagrams for $X_1 \otimes X_2$.

In Section 4 we first explore the relations between the kernels of the four morphisms that occur in a pushout diagram. We then consider direct sums of pullback and pushout diagrams, as well as direct and inverse limits. The categorical approach to these problems is heavily influenced by the detailed advice received (with gratitude) from Claude Schochet. We show how to tensor a pushout diagram with a fixed algebra in Section 5. Also, we find abstract characterizations of certain classes of "ideal pushouts" and "hereditary pushouts," i.e., diagrams in which one of the morphisms, $\beta : C \to B$, has an ideal or a hereditary image, while the other, $\alpha : C \to A$, is proper ($\alpha(C)A = A$). We study C^*-algebras of the form $X = A \star_C B$ for ideal pushouts and show that if X_1 and X_2 are in this class, then so is $X_1 \otimes X_2$.

Section 6 is devoted to results about crossed products and Section 7 to multiplier algebra constructions over pullback and pushout diagrams. Due to the universality inherent in these concepts all structures are beautifully conserved, except for the diagram of multiplier algebras over a pushout diagram, which fails spectacularly to be a pushout.

Section 8 begins with some categorical byplay, i.e., results about (large) diagrams that involve only the concepts of pullback and pushout, but no C^*-algebra theory. For all that we need some of these results later. We also revisit the theory from [21] of conditionally projective diagrams. We then in Section 9 consider triples of extensions and morphisms between them, so that one may take the pullback (respectively, the pushout) at the level of ideal, algebra, and quotient. Under some mild extra conditions these three algebras will again form an extension. And if they do, then already the commutativity of the large diagram involved will force the outer squares to be pullbacks, respectively pushouts, if only the middle squares are such.

In Section 10 we study and extend a result due to Loring [33, 6.2.2]. We show that for any proper morphism $\alpha : A \to B$ between σ-unital C^*-algebras the "corner extension" $e_{11} \otimes \alpha : \mathbb{K} \otimes A \to \mathbb{K} \otimes B$ between the stabilized algebras gives rise to an amalgamated free product $\mathbb{K} \otimes B = (\mathbb{K} \otimes A) \star_A B$. In particular, taking $A = \mathbb{C}$ and B unital we obtain the formula $\mathbb{K} \otimes B = \mathbb{K} \star_{\mathbb{C}} B$. These new pushouts have applications to corona extendibility and to conditional projectivity. With generous help from Larry Brown we show that A is a full corner in another σ-unital C^*-algebra B precisely when there is a hereditary embedding of B in $\mathbb{K} \otimes A$ taking A onto $e_{11} \otimes A$. This result allows us to describe amalgamated free products $A \star_C B$, where C is a full corner of B and proper in A. We also survey a recent result by Hjelmborg and Rørdam [25], to discuss whether pullbacks and pushouts of stable C^*-algebras are again stable. Finally, the above-mentioned results on $NCCW$ complexes are contained in Section 11.

The pullback construction entered C^*-algebra theory in Busby's thesis [13], where Peter Freyd is credited for bringing it to that author's attention. However, detailed use of pullback arguments and terminology has been a slow development. An early (and earnest) example, involving the K-theory of a pullback (the Mayer–Vietoris sequence), occurs in book III of Schochet's magnum opus [45]. Another is the thesis of Sheu [46], where the topological stable rank of a surjective pullback is computed. Pushout constructions are even more recent arrivals to C^*-algebra theory;

see [2, 7, 14, 19]. Thanks to the renewed interest in universal constructions during this decade they are now becoming standard tools, see [20–23, 33, 35]. Voiculescu's work on the spatial theory of free products, although formally unrelated, is another source of inspiration; see, e.g., [47]. The earliest traceable result about amalgamated free products of C^*-algebras is probably [1, Theorem 3.1] (cf. Theorems 4.2 and 4.4), and after that Blackadar writes: "It would be interesting to make a systematic study of amalgamations of C^*-algebras." Well, here it is.

2. PREREQUISITES

2.1. *A Bit of Category Theory.* Most of the material in this paper concerns the category $\mathscr{C}^*$ of C^*-algebras with $*$-homomorphisms as morphisms. As pointed out in the introduction many results will only hold in the smaller category with only *proper* morphisms, but we have refrained from making this assumption permanent, to have more freedom. Although the main developments in category theory have been concentrated on abelian categories, cf. [24, 36], the C^*-algebra theory is certainly not immune to the "abstract nonsense" treatment, and we shall use it whenever possible. It shortens some proofs drastically, and even when the proofs rely on special properties of C^*-algebras it clarifies the thinking to frame them in categorical language.

Note first that $\mathscr{C}^*$ is a category with kernels and cokernels. The kernel of a morphism $\varphi : A \to B$ is the embedding $\ker \varphi \to A$, whereas the cokernel of φ is the quotient morphisms $B \to B/I$, where I is the closed ideal of B generated by $\varphi(A)$. In particular, any quotient map φ is the cokernel of the embedding of $\ker \varphi$.

Note also that $\mathscr{C}^*$ has products and coproducts. The product of a family A_i of C^*-algebras is the orthogonal product $\prod A_i$ (distinct from the cartesian product if the family is infinite by containing only the bounded elements). The coproduct is the free product C^*-algebra $\bigstar A_i$ obtained from the free $*$-algebra after completion with respect to the largest C^*-norm whose restriction to each A_i is the original norm. Thus for each family of morphisms $\varphi_i : A_i \to B$ there is a unique morphism $\varphi : \bigstar A_i \to B$ such that $\varphi_i = \varphi \circ \iota_i$ for every i, where ι_i is the embedding of A_i into $\bigstar A_i$.

2.2. *Pullbacks.* A commutative diagram of C^*-algebras

$$\begin{tikzcd} X \arrow[r, "\gamma"'] \arrow[d, "\delta"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & C \end{tikzcd}$$

is a *pullback* if $\ker \gamma \cap \ker \delta = 0$ and if every other *coherent* pair of morphisms $\varphi : Y \to A$ and $\psi : Y \to B$ (where coherence means that $\alpha \circ \varphi = \beta \circ \psi$) from a C^*-algebra Y factors through X; i.e., $\varphi = \delta \circ \sigma$ and $\psi = \gamma \circ \sigma$ for a (necessarily unique) morphism $\sigma : Y \to X$.

It follows that X is isomorphic to the *restricted direct sum*

$$A \oplus_C B = \{(a, b) \in A \oplus B \mid \alpha(a) = \beta(b)\},$$

so that δ and γ can be identified with the projections on first and second coordinates, respectively. In particular, the pullback exists for any triple of C^*-algebras A, B, and C with linking morphisms α and β.

Pullback constructions occur frequently in C^*-algebra theory and are indispensable for the theory of extensions, where they appear in the Busby picture

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook] \arrow[d, equal] & X \arrow[r, "\gamma"'] \arrow[d, "\delta"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & A \arrow[r, hook] & M(A) \arrow[r, "\alpha"] & Q(A) \arrow[r] & 0
\end{tikzcd}$$

Here $X \in \operatorname{ext}(A, B)$, determined by the *Busby invariant* β. (And α is just the quotient map from the multiplier algebra $M(A)$ to the corona algebra $Q(A)$.) Note that $\operatorname{ext}(A, B)$ denotes the full set of extensions (isomorphic to $\operatorname{Hom}(B, Q(A))$), and that our notation is slightly in contravariance with the accepted.

2.3. *Pushouts.* A commutative diagram of C^*-algebras

$$\begin{tikzcd}
C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\
A \arrow[r, "\delta"] & X
\end{tikzcd}$$

is a *pushout* if X is generated by $\gamma(B) \cup \delta(A)$ and if every other coherent pair of morphisms $\varphi : A \to Y$ and $\psi : B \to Y$ (thus $\varphi \circ \alpha = \psi \circ \beta$) into a C^*-algebra Y factors through X; i.e., $\varphi = \sigma \circ \delta$ and $\psi = \sigma \circ \gamma$ for a (necessarily unique) morphism $\sigma : X \to Y$.

Here we find that X is isomorphic to the *amalgamated free product* $A \star_C B$, which is defined as the quotient of the free product C^*-algebra $A \star B$ by the closed ideal generated by $\{\alpha(c) - \beta(c) \mid c \in C\}$. In particular, the pushout exists for any triple of C^*-algebras A, B and C with linking morphisms α and β.

Despite the formal "duality" between pullbacks and pushouts the construction of the amalgamated product $A \star_C B$ is not easy, and frequently the resulting algebra is unwieldly. Nevertheless there is an obvious advantage in describing a given C^*-algebra X as an amalgamated free product, since then all questions of morphisms *out* of X are reduced to—presumably simpler—questions about coherent pairs of morphisms out of A and B.

The best known recipe for producing pushouts is given in [20, Corollary 4.3], cf. [39, Corollary 5.4]. For easy reference we state it here with a short new proof. The necessity of the condition that α be a proper morphism is illustrated by Example 5.4.

2.4. THEOREM. *In a commutative diagram of extensions*

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, hook, "\beta"'] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, hook, "\delta"] & X \arrow[r] & D \arrow[r] & 0 \end{tikzcd}$$

where α is a proper morphism $(A = \alpha(C)\,A)$, the left square is a pushout. Thus,

$$X = A \star_C B = M(A) \oplus_{Q(A)} B/C,$$

where the Busby invariant $\eta : B/C \to Q(A)$ for the extension X is obtained by composing the Busby invariant $B/C \to Q(C)$ for the upper extension B with the induced morphism $\tilde{\alpha} : Q(C) \to Q(A)$.

Proof. Since the morphism $B \to D$ is surjective and $\delta(A)$ is an ideal in X we have a decomposition $X = \delta(A) + \gamma(B)$. If therefore $\varphi : A \to Y$ and $\psi : B \to Y$ is a coherent pair of morphisms into some C^*-algebra Y, and $x = \delta(a) - \gamma(b)$ for some a, b in $A \times B$, we may tentatively set

$$\sigma(x) = \varphi(a) - \psi(b).$$

This actually determines a welldefined *-linear map $\sigma : X \to Y$. For if $\delta(a) = \gamma(b)$, then

$$0 = \gamma(b) + \delta(A) = b + \beta(C) \qquad \text{in } D,$$

so $b = \beta(c)$ for some c in C. Consequently also $\delta(\alpha(c)) = \gamma(\beta(c)) = \gamma(b) = \delta(a)$, whence $a = \alpha(c)$; and thus,

$$\varphi(a) = \varphi(\alpha(c)) = \psi(\beta(c)) = \psi(b).$$

Consider now an arbitrary selfadjoint element $x = \delta(a) - \gamma(b)$ in X. Since α is proper we can write $a = a'\alpha(c)$ for some a' in A and c in C; and since $\beta(C)$ is an ideal in B we have $\beta(c)\, b = \beta(c')$ for some c' in C. Consequently,

$$\delta(a)\,\gamma(b) = \delta(a')\,\gamma(\beta(c)\,b) = \delta(a')\,\gamma(\beta(c')) = \delta(a'\alpha(c')).$$

Similarly, $\varphi(a)\,\psi(b) = \varphi(a'(\alpha(c')))$. Thus we may compute

$$\begin{aligned}\sigma(x^2) &= \sigma((\delta(a) - \gamma(b))^2) = \sigma(\delta(a^2 - 2\,\mathrm{Re}(a'\alpha(c'))) + \gamma(b^2))\\ &= \varphi(a^2 - 2\,\mathrm{Re}(a'\alpha(c'))) + \psi(b^2) = (\varphi(a) - \psi(b))^2 = (\sigma(x))^2.\end{aligned}$$

This shows that σ is multiplicative and therefore a morphism; and evidently $\varphi = \sigma \circ \delta$ and $\psi = \sigma \circ \gamma$. ∎

Another rather general construction is found in [35, Lemma 2.1]. In slightly updated form it reads:

2.5. THEOREM. *In a commutative diagram of extensions*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, "\alpha"] & C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & J \arrow[r, hook] & A \arrow[r, "\delta"] & X \arrow[r] & 0 \end{tikzcd}$$

the right square is a pushout if and only if $\alpha(I)$ generates J as an ideal. Thus,

$$X = A \star_C B = A/\mathrm{Id}(\alpha(\ker\beta)).$$

Proof. Consider a coherent pair of morphisms $\varphi : A \to Y$ and $\psi : B \to Y$. Since $I = \ker\beta$ we must have $I \subset \ker\psi\circ\beta = \ker\varphi\circ\alpha$. Thus $\alpha(I) \subset \ker\varphi$, and since $\mathrm{Id}(\alpha(I)) = J$ it follows that $J \subset \ker\varphi$. Therefore φ induces a morphism $\sigma : X \to Y$ so that $\varphi = \sigma\circ\delta$. Every b in B has the form $b = \beta(c)$ for some c in C, so

$$\psi(b) = \psi(\beta(c)) = \varphi(\alpha(c)) = \sigma(\delta(\alpha(c))) = \sigma(\gamma(\beta(c))) = \sigma(\gamma(b)),$$

whence also $\psi = \sigma\circ\gamma$.

Conversely, if the diagram is a pushout, let $J_0 = \mathrm{Id}(\alpha(I))$(so $J_0 \subset J$), and consider the coherent pair (φ, ψ) consisting of the quotient morphism $\varphi : A \to A/J_0$ and the induced morphism $\psi : B \to A/J_0$ given by $\psi(c + I) = \alpha(c) + J_0$, $c \in C$. By assumption $\varphi = \sigma\circ\delta$ for some morphism $\sigma : X \to A/J_0$, which implies that

$$J = \ker\delta \subset \ker(\sigma\circ\delta) = \ker\varphi = J_0 = \mathrm{Id}(\alpha(I)). \quad ∎$$

We shall consider a common generalization of Theorems 2.4 and 2.5 later (Theorem 5.3).

2.6. *Concatenation and Decatenation.* The definition of pullbacks and pushouts makes sense in any category, and some of the results about them are valid in that generality. The *concatenation* and *decatenation* of diagrams are such examples of (useful) constructions that do not depend on C^*-algebra theory. However, we state them in this category to fix the ideas. The proofs can safely be left to the reader, cf. [36, III.4.Exercise 8].

2.7. PROPOSITION. *If two pullback* (*respectively pushout*) *diagrams of C^*-algebras—written with arrows only going right or down—have a common edge, then the concatenated diagram is again a pullback* (*respectively a pushout*).

2.8. COROLLARY. *For every pullback* (*respectively pushout*) *diagram as below, to the left, and automorphisms π, ρ, σ, and τ of A, B, C, and X, respectively, such that $\alpha \circ \pi = \sigma \circ \alpha$ and $\gamma \circ \tau = \rho \circ \gamma$, the diagram below, to the right, is also a pullback* (*respectively a pushout*).

$$\begin{tikzcd} X \arrow[r, "\gamma"'] \arrow[d, "\delta"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & C \end{tikzcd} \quad \textit{gives} \quad \begin{tikzcd} X \arrow[rr, "\gamma"'] \arrow[d, "\pi\circ\delta\circ\tau"] & & B \arrow[d, "\sigma\circ\beta\circ\rho"] \\ A \arrow[rr, "\alpha"] & & C \end{tikzcd}$$

Proof. Concatenate the original diagram with the diagrams

$$\begin{tikzcd} A \arrow[r, "\alpha"'] \arrow[d, "\pi"] & C \arrow[d, "\sigma"] \\ A \arrow[r, "\alpha"] & C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\gamma"'] \arrow[d, "\tau"] & B \arrow[d, "\rho"] \\ X \arrow[r, "\gamma"] & B \end{tikzcd}$$

which are obviously both pullbacks and pushouts, cf. Example 3.3.B. ▮

2.9. PROPOSITION. *Consider the commutative diagrams of C^*-algebras*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & C_0 \arrow[r] & C \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} C \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A_0 \arrow[r] & X \end{tikzcd}$$

If the concatenated diagram to the left is a pullback and the two morphisms out of B_0 have no common kernel (*in particular if its right square is a pullback*), *then its left square is a pullback.*

If the concatenated diagram to the right is a pushout and A_0 is generated by the images of A and C_0 (in particular if its left square is a pushout), then its right square is a pushout.

2.10. *Adjointable Functors.* Two functors $F: \mathscr{C} \to \mathscr{D}$ and $G = \mathscr{D} \to \mathscr{C}$ between categories $\mathscr{C}$ and $\mathscr{D}$ are said to be *adjoint* to each other if for each C in $\mathscr{C}$ and D in $\mathscr{D}$ there is a natural equivalence

$$\mathrm{Hom}_{\mathscr{D}}(F(C), D) \approx \mathrm{Hom}_{\mathscr{C}}(C, G(D)),$$

cf. [24, II.7] or [36, IV.1]. As shown in [24, Theorem II.7.7] each functor G that has a left adjoint F will preserve products, pullbacks and kernels. Dually, each functor F that has a right adjoint G will preserve coproducts, pushouts and cokernels.

There is a multitude of functors that are adjointable, cf. [36, IV.2] and we shall need a few. To fix the ideas we choose to relate all constructions to the category $\mathscr{C}^*$ of C^*-algebras with *-homomorphisms as morphisms.

2.11. *Examples.* **A.** Let $\mathbb{N}\mathscr{C}^*$ denote the category of sequences (A_n) from $\mathscr{C}^*$ with morphisms given by sequences (φ_n) of morphisms from $\mathscr{C}^*$. The functor $\prod : \mathbb{N}\mathscr{C}^* \to \mathscr{C}^*$ that the each sequence (A_n) associates the product C^*-algebra $\prod A_n$ has a left adjoint, viz., the constant functor that to each element A assigns the constant sequence $(A, A, ...)$.

B. The functor $\bigstar : \mathbb{N}\mathscr{C}^* \to \mathscr{C}^*$ that to each sequence (A_n) assigns the free product $\bigstar A_n$ (the coproduct in $\mathscr{C}^*$) has the constant functor as a right adjoint.

C. Let $\underrightarrow{\mathbb{N}}\mathscr{C}^*$ denote the category of directed sequences from $\mathscr{C}^*$, i.e., sequences (A_n) equipped with morphisms $\varphi_n: A_n \to A_{n+1}$ for every n. A morphism in $\underrightarrow{\mathbb{N}}\mathscr{C}^*$, say from (A_n) to (B_n), is a coherent sequence of morphisms $\alpha_n: A_n \to B_n$, i.e. $\psi_n \circ \alpha_n = \alpha_{n+1} \circ \varphi_n$ for all n. The functor $\varinjlim : \underrightarrow{\mathbb{N}}\mathscr{C}^* \to \mathscr{C}^*$ that to each directed sequence (A_n) assigns the (generalized) direct limit $\varinjlim A_n$ (cf. 4.11) has a right adjoint, viz. the constant functor that to each element A assigns the directed sequence $(A \to A \to \cdots)$.

D. Let $\underleftarrow{\mathbb{N}}\mathscr{C}^*$ denote the category of inversely directed sequences from $\mathscr{C}^*$, i.e., sequences (A_n) equipped with morphisms $\varphi_n : A_{n+1} \to A_n$ for every n. A morphism in $\underleftarrow{\mathbb{N}}\mathscr{C}^*$ from (A_n) to (B_n) is a coherent sequence of morphisms $\alpha : A_n \to B_n$, i.e., $\alpha_n \circ \varphi_n = \psi_n \circ \alpha_{n+1}$ for all n. The functor $\varprojlim : \underleftarrow{\mathbb{N}}\mathscr{C}^* \to \mathscr{C}^*$ that to each inversely directed sequence (A_n) assigns the inverse limit $\varprojlim A_n$ (cf. 4.15) has a left adjoint, viz. the constant functor that to each element A assigns the inversely directed sequence $(\cdots \to A \to A)$.

E. Let $\Delta\mathscr{C}^*$ denote the category of inward directed triples $(A \xrightarrow{\alpha} C \xleftarrow{\beta} B)$ from $\mathscr{C}^*$, equipped with morphisms as indicated. A morphism in $\Delta\mathscr{C}^*$ is a coherent triple (φ, η, ψ) of morphisms from $\mathscr{C}^*$, so that we have a commutative diagram

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi"] & C_1 \arrow[d, "\eta"] & B_1 \arrow[l, "\beta_1"'] \arrow[d, "\psi"] \\ A_2 \arrow[r, "\alpha_2"] & C_2 & B_2 \arrow[l, "\beta_2"'] \end{tikzcd}$$

The pullback functor from $\Delta\mathscr{C}^*$ to $\mathscr{C}^*$ that to each triple $(A \xrightarrow{\alpha} C \xleftarrow{\beta} B)$ assigns the restricted direct sum $A \oplus_C B$ has a left adjoint, viz. the constant functor that to each A assigns the triple $(A \to A \leftarrow A)$.

F. Let $\nabla\mathscr{C}^*$ denote the category of outward directed triples $(A \xleftarrow{\alpha} C \xrightarrow{\beta} B)$ from $\mathscr{C}^*$, equipped with morphisms as indicated, and with the obvious morphisms. The pushout functor from $\nabla\mathscr{C}^*$ to $\mathscr{C}^*$ that to each triple $(A \xleftarrow{\alpha} C \xrightarrow{\beta} B)$ assigns the amalgamated free product $A \bigstar_C B$ has a right adjoint, viz. the constant functor that to each A assigns the triple $(A \leftarrow A \to A)$.

2.12. *Amalgamated Free Products of Banach Spaces.* If we are given Banach spaces $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$, with bounded linear operators $\alpha : \mathfrak{Z} \to \mathfrak{X}$ and $\beta : \mathfrak{Z} \to \mathfrak{Y}$, we define the *amalgamated free product* as the quotient space

$$\mathfrak{X} \bigstar_{\mathfrak{Z}} \mathfrak{Y} = (\mathfrak{X} \oplus \mathfrak{Y})/\mathfrak{L},$$

where we use the 1-norm on $\mathfrak{X} \oplus \mathfrak{Y}$ and set

$$\mathfrak{L} = \{(\alpha(z), \beta(-z)) \in \mathfrak{X} \oplus \mathfrak{Y} \mid z \in \mathfrak{Z}\}^{=}.$$

This gives the commutative diagram below, to the left, where γ and δ are the obvious coordinate embeddings, followed by the quotient map. If we have another commutative diagram of Banach spaces as the one below, to the right,

$$\begin{tikzcd} \mathfrak{Z} \arrow[r, "\beta"'] \arrow[d, "\alpha"] & \mathfrak{Y} \arrow[d, "\gamma"] \\ \mathfrak{X} \arrow[r, "\delta"] & \mathfrak{X} \bigstar_{\mathfrak{Z}} \mathfrak{Y} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{Z} \arrow[r, "\beta"'] \arrow[d, "\alpha"] & \mathfrak{Y} \arrow[d, "\psi"] \\ \mathfrak{X} \arrow[r, "\varphi"] & \mathfrak{W} \end{tikzcd}$$

with φ and ψ bounded (respectively contractive) linear operators, there is a unique bounded (respectively contractive) linear operator $\sigma : \mathfrak{X} \bigstar_{\mathfrak{Z}} \mathfrak{Y} \to \mathfrak{W}$, such that $\varphi = \sigma \circ \delta$ and $\psi = \sigma \circ \gamma$. Thus $\mathfrak{X} \bigstar_{\mathfrak{Z}} \mathfrak{Y}$ is the universal solution that defines a pushout diagram for Banach spaces. As a frequently overlooked application, pointed out to the author by Vern Paulsen, we see that if $\mathfrak{X} = \mathfrak{Y}$ as linear spaces, and if we take $\mathfrak{Z}$ as $\mathfrak{X}(=\mathfrak{Y})$ equipped with the sup or the sum norm, then $\mathfrak{X} \star_{\mathfrak{Z}} \mathfrak{Y}$ becomes $\mathfrak{X}$ equipped with the largest norm dominated by both the $\mathfrak{X}$-norm and the $\mathfrak{Y}$-norm.

Note that, due to the absence of multiplicative structure, the amalgamated free product of Banach spaces is a much simpler construction than for C^*-algebras. By contrast, the definition and construction of the *restricted direct sum* of Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$, relative to bounded (respectively contractive) linear operators $\alpha : \mathfrak{X} \to \mathfrak{Z}$ and $\beta : \mathfrak{Y} \to \mathfrak{Z}$ into some Banach space $\mathfrak{Z}$, is exactly the same as before (with the ∞-norm on $\mathfrak{X} \oplus \mathfrak{Y}$):

$$\mathfrak{X} \oplus_{\mathfrak{Z}} \mathfrak{Y} = \{(x, y) \in \mathfrak{X} \oplus \mathfrak{Y} \mid \alpha(x) = \beta(y)\}.$$

2.13. Proposition. *Given a pullback or a pushout diagram of Banach spaces*

$$\begin{tikzcd} \mathfrak{W} \arrow[r, "\gamma"'] \arrow[d, "\delta"] & \mathfrak{Y} \arrow[d, "\beta"] \\ \mathfrak{X} \arrow[r, "\alpha"] & \mathfrak{Z} \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} \mathfrak{Z} \arrow[r, "\beta"'] \arrow[d, "\alpha"] & \mathfrak{Y} \arrow[d, "\gamma"] \\ \mathfrak{X} \arrow[r, "\delta"] & \mathfrak{W} \end{tikzcd}$$

we obtain by transposition a diagram which is a pushout or a pullback, respectively,

$$\begin{tikzcd} \mathfrak{W}^* & \mathfrak{Y}^* \arrow[l, "\gamma^*"] \\ \mathfrak{X}^* \arrow[u, "\delta^*"] & \mathfrak{Z}^* \arrow[l, "\alpha^*"'] \arrow[u, "\beta^*"'] \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} \mathfrak{Z}^* & \mathfrak{Y}^* \arrow[l, "\beta^*"] \\ \mathfrak{X}^* \arrow[u, "\alpha^*"] & \mathfrak{W}^* \arrow[l, "\delta^*"'] \arrow[u, "\gamma^*"'] \end{tikzcd}$$

Proof. Corresponding to the category $\mathscr{B}$ of Banach spaces with bounded linear operators as morphisms we have the category $\mathscr{B}^*$ of dual Banach spaces, i.e., Banach spaces $\mathfrak{Y}$ of the form $\mathfrak{Y} = \mathfrak{X}^*$ for some $\mathfrak{X}$ in $\mathscr{B}$. Thus each $\mathfrak{Y}$ comes equipped with a weak* topology such that the closed subspace $\mathfrak{X}$ of $\mathfrak{Y}^*$ consisting of the weak* continuous functionals is the predual of $\mathfrak{Y}$. The morphisms in $\mathscr{B}^*$ are the weak* continuous linear operators. Since these are precisely the operators that are transposed of bounded linear operators between the preduals, we obtain (passing to the opposite category for convenience) *covariant* functors $\mathfrak{X} \to \mathfrak{X}^*$ and $\mathfrak{Y} \to \mathfrak{Y}_*$ between $\mathscr{B}$ and $(\mathscr{B}^*)^{opp}$ by taking dual or predual spaces. These two functors are both left and right adjoints to one another and therefore preserve both pullbacks and pushouts, cf. 2.10. But pullbacks in $(\mathscr{B}^*)^{opp}$ are pushouts in $\mathscr{B}^*$ and conversely. ∎

2.14. COROLLARY. *If we have a restricted direct sum of C^*-algebras $X = A \oplus_C B$, then X^* is isometrically $*$-isomorphic to $A^* \star_{C^*} B^*$. In particular, the state space $S(X)$ of X is the image in X^* of $S(A) \oplus S(B)$.*

2.15. *Tensor Products.* In most situations the natural tensor product between C^*-algebras A and B to consider is the *minimal* (or *spatial*) tensor product $A \otimes_{\min} B$, obtained by choosing faithful representations of A and B on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively, and defining $A \otimes_{\min} B$ as the completion of the algebraic tensor product $A \odot B$ on $\mathfrak{H} \otimes \mathfrak{K}$, cf. [26, Chapt. 12] or [49, Sect. 1].

The minimal tensor product behaves well under inclusions: If $A_1 \subset A$ and $B_1 \subset B$, then $A_1 \otimes_{\min} B_1 \subset A \otimes_{\min} B$. It behaves less satisfactory under extensions: If $X \in \operatorname{ext}(A, B)$ and Y is another C^*-algebra, then although $Y \otimes_{\min} A$ is a closed ideal of $Y \otimes_{\min} X$ and $Y \otimes_{\min} B$ is a quotient of $Y \otimes_{\min} X$, the kernel of the quotient map may not equal $Y \otimes_{\min} A$. (However, $Y \otimes_{\min} X / Y \otimes_{\min} A = Y \otimes_{\alpha} B$ for some larger cross norm α.) C^*-algebras Y for which we *always* obtain an extension

$$0 \to Y \otimes_{\min} A \to Y \otimes_{\min} X \to Y \otimes_{\min} B \to 0$$

are called *exact*, [27, 48, 49].

For our purposes the *maximal* tensor product $A \otimes_{\max} B$ will be more useful, especially in dealing with pushout diagrams. The (largest) cross norm defining this completion is namely given by a universal condition: If $(\pi, \mathfrak{H})$ and $(\rho, \mathfrak{H})$ is a pair of *commuting* representations of A and B (i.e., $\pi(A) \subset \rho(B)'$ in $\mathbb{B}(\mathfrak{H})$), then there is a unique norm decreasing representation $(\pi \otimes \rho, \mathfrak{H})$ of $A \otimes_{\max} B$ such that

$$(\pi \otimes \rho)(a \otimes b) = \pi(a)\, \rho(b), \qquad a \in A, \quad b \in B.$$

The maximal tensor product is less wellbehaved under inclusions: If $A_1 \subset A$, then the natural morphism $A_1 \otimes_{\max} B \to A \otimes_{\max} B$ need not be injective. However, if A_1 is a closed ideal in A we do have an embedding $A_1 \otimes_{\max} B$. Thus, $A \otimes_{\max} B \subset \tilde{A} \otimes_{\max} \tilde{B}$ and $A \otimes_{\max} B \subset M(A) \otimes_{\max} B$. More importantly, if $X \in \operatorname{ext}(A, B)$, then for any C^*-algebra Y we obtain another extension

$$0 \to Y \otimes_{\max} A \to Y \otimes_{\max} X \to Y \otimes_{\max} B \to 0,$$

cf. [49, 1.9].

The class of *nuclear* C^*-algebras, i.e. C^*-algebras A such that $A \otimes_{\max} B = A \otimes_{\min} B$ for any other C^*-algebra B, is designed to make all tensor product difficulties vanish. The class is pleasantly large. It includes all C^*-algebras of type I and is closed under direct sums, tensor products and inductive limits, and it behaves well under extensions and hereditary subalgebras. However, a C^*-subalgebra of a nuclear C^*-algebra need not be nuclear. On the other hand such an algebra is always exact, and as shown by Kirchberg, [29], the separable, exact C^*-algebras are precisely the subalgebras of nuclear, separable C^*-algebras.

Passing to an arbitrary number of factors (A_i) we see the relationship between the direct sum $\bigoplus A_i$, the (maximal) tensor product $\bigotimes A_i$, and the free product $\bigstar A_i$: The first is the universal solution for families of morphisms $\varphi_i : A_i \to B$ with *orthogonal* images, the second for families with *commuting* images, and the third for *arbitrary* families.

2.16. *Joint Free Products.* For two unital C^*-algebras A and B the free product $A \star_{\mathbb{C}} B$, amalgamated over the common unit, would seem to be the proper noncommutative analogue of the tensor product $A \otimes_{\max} B$. For nonunital algebras A and B the free product $A \star B$ is the only possible analogue (and the unitizations behave nicely, as $(A \star B)^{\sim} = \tilde{A} \star_{\mathbb{C}} \tilde{B}$), but now the deviations from the tensor product construction begin to show. True, if C is a C^*-subalgebra of A, then $C \star B$ is naturally embedded as a C^*-subalgebra of $A \star B$; and if $\pi : A \to D$ is a quotient map, it induces a quotient map $\tilde{\pi} : A \star B \to D \star B$. But if I is an ideal of A, then $I \star B$ is not an ideal of $A \star B$, because $I \star B$ contains a copy of B by construction. Thus, the free product does not preserve extensions.

A possible way out of this dilemma would be to define the *joint free product* $A \mathbin{\hat{\star}} B$ as the completion in $A \star B$ of those words that contain elements from *both* A and B. Evidently $A \mathbin{\hat{\star}} B$ will be a closed ideal in $A \star B$ giving rise to the (nonsplit) extension:

$$0 \to A \mathbin{\hat{\star}} B \to A \star B \to A \oplus B \to 0.$$

Thus we now only have embeddings of the algebras A and B into the multiplier algebra $M(A \mathbin{\hat{\star}} B)$. Note that when both A and B are unital, then $A \star_{\mathbb{C}} B$ will be the quotient of $A \mathbin{\hat{\star}} B$ by the ideal generated by the two multipliers $\mathbf{1} - \mathbf{1}_A$ and $\mathbf{1} - \mathbf{1}_B$.

With this new product we see that if $A \to X \to B$ is an extension, then we again have an extension

$$0 \to Y \mathbin{\hat{\star}} A \to Y \mathbin{\hat{\star}} X \to Y \mathbin{\hat{\star}} B \to 0.$$

One could now go ahead and prove the analogues of Theorems 3.8, 4.7, and 5.7, replacing the maximal tensor product with the joint free product.

We shall not pursue the theory of free products here. Instead we shall consider C^*-algebras $X = A \star_C B$ where the amalgamation C is "large," relative to A and B. In this way even very civilized C^*-algebras X, such as subhomogeneous algebras over CW complexes (cf. Sect. 11), can appear as amalgamated free products, see Theorem 11.16.

3. PULLBACKS AND TENSOR PRODUCTS

3.1. PROPOSITION. *A commutative diagram of C^*-algebras*

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\alpha} & C \end{array}$$

is a pullback if and only if the following conditions hold:

(i) $\ker \gamma \cap \ker \delta = \{0\}$,

(ii) $\beta^{-1}(\alpha(A)) = \gamma(X)$,

(iii) $\delta(\ker \gamma) = \ker \alpha$.

Proof. The two coherent morphisms γ and δ define a unique morphism $\sigma : X \to A \oplus_C B$. If the diagram is a pullback, σ is an isomorphism, so (i) and (ii) are clearly satisfied. To prove (iii) take a in $\ker \alpha$, and consider the element $(a, 0)$ in $A \oplus_C B$. By assumption $(a, 0) = \sigma(x)$ for some x in X, which means that $a = \delta(x)$ and $0 = \gamma(x)$. Thus $\ker \alpha \subset \delta(\ker \gamma)$, and the reverse inclusion is automatic.

If the three conditions are satisfied, then σ is injective by (i). To prove surjectivity take (a, b) in $A \oplus_C B$. Then $b \in \beta^{-1}(\alpha(A))$, since $\alpha(a) = \beta(b)$, so $b = \gamma(x)$ for some x in X by (ii). Now

$$\alpha(a - \delta(x)) = \alpha(a) - \alpha(\delta(x)) = \beta(b) - \beta(\gamma(x)) = 0,$$

so $a - \delta(x) = \delta(y)$ for some y in $\ker \gamma$ by (iii). Consequently

$$\sigma(x + y) = (\delta(x + y), \gamma(x)) = (a, b),$$

so that σ is surjective. ∎

Reversing rows and columns in diagram above we see that also the conditions $\alpha^{-1}(\beta(B)) = \delta(X)$ and $\gamma(\ker \delta) = \ker \beta$ are satisfied in a pullback diagram.

3.2. *Remark.* It follows from the preceding result that without changing X one may replace B and C with $\beta^{-1}(\alpha(A)) = \gamma(X)$ and $\alpha(A)$ in a pullback diagram. This means that essentially every pullback diagram fits into a morphism between two extensions in a "Diagram I" situation, cf. [20, Sect. 1]:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, equal] & X \arrow[r, "\gamma"'] \arrow[d, "\delta"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & I \arrow[r, hook] & A \arrow[r, "\alpha"] & C \arrow[r] & 0 \end{tikzcd}$$

Here we have identified $\ker \alpha$ and $\ker \gamma$, since δ is an isomorphism between them by (i) and (iii). Note that this diagram is also a pushout by Theorem 2.5.

Going further, we can replace A and C with $\delta(X)$ and $\beta(B)$, still without changing X. Now all morphisms are surjective, and the diagram fits into a commutative 3×3 diagram in which all rows and columns are extensions:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & J \arrow[r, equal] \arrow[d] & J \arrow[d] \\ I \arrow[r] \arrow[d, equal] & X \arrow[r, "\gamma"] \arrow[d, "\delta"] & B \arrow[d, "\beta"] \\ I \arrow[r] & A \arrow[r, "\alpha"] & C \end{tikzcd}$$

3.3. EXAMPLES. **A.** For arbitrary C^*-algebras X and Y consider the three trivial diagrams

$$\begin{tikzcd} X \oplus Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \star Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\alpha"'] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "\alpha"] & Y \end{tikzcd}$$

These are examples of diagrams that are both pullbacks and pushouts.

B. Given morphisms $\alpha_i : X \to Y$ for $i = 1, 2$ and automorphisms σ and τ of X and Y, respectively, such that $\alpha_2 \circ \sigma = \tau \circ \alpha_1$, we can form the semi-trivial diagram below, to the left, which is also both a pullback and a pushout. Further, if we have an extension X in $\operatorname{ext}(A, B)$, then the other diagram below, to the right, is both a pullback and a pushout

$$\begin{tikzcd} X \arrow[r, "\alpha_1"'] \arrow[d, "\sigma"'] & Y \arrow[d, "\tau"'] \\ X \arrow[r, "\alpha_2"] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[d] \\ X \arrow[r, "\beta"] & B \end{tikzcd}$$

It should be noted that a commutative diagram as above, to the right, can be a pullback or a pushout (but not both!) without X being an extension of A by B. A moments reflection (cf. Proposition 3.1 and Theorem 2.5) reveals that such a diagram is a pullback if α is injective with $\alpha(A) = \ker \beta$. It is a pushout if β is surjective and $\ker \beta$ is generated as an ideal by $\alpha(A)$.

C. The reader may have wondered how often one can find diagrams of C^*-algebras that are simultaneously pullbacks and pushouts. Actually this happens as often as we please: If we consider an arbitrary pushout diagram, to the left, below and form $C_0 = A \oplus_X B$, then (α, β) is a coherent pair and thus defines a morphism $\sigma : C \to C_0$. (Assuming, as we may, that $\ker \alpha \cap \ker \beta = \{0\}$, this is even an injection.) The new diagram, to the right,

$$
\begin{tikzcd}
C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\
A \arrow[r, "\delta"] & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
C_0 \arrow[r, "\beta_0"'] \arrow[d, "\alpha_0"] & B \arrow[d, "\gamma"] \\
A \arrow[r, "\delta"] & X
\end{tikzcd}
$$

is of course a pullback; but for any pair of morphisms $\varphi\colon A \to Y$ and $\psi : B \to Y$ we have $\varphi \circ \alpha = \psi \circ \beta$ if and only if $\varphi \circ \alpha_0 = \psi \circ \beta_0$, and thus the diagram is also a pushout.

Similarly, if we have an arbitrary pullback diagram to the left, below, we can form $C_0 = A \star_X B$. Viewing (α, β) as a coherent pair we obtain a morphism $\sigma : C_0 \to C$. (Assuming, as we may, that C is generated by $\alpha(A) \cup \beta(B)$, this is even a surjection.) The new diagram to the right

$$
\begin{tikzcd}
C \arrow[r, "\gamma"'] \arrow[d, "\delta"] & B \arrow[d, "\beta"] \\
A \arrow[r, "\alpha"] & C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "\gamma"'] \arrow[d, "\delta"] & B \arrow[d, "\beta_0"] \\
A \arrow[r, "\alpha_0"] & C_0
\end{tikzcd}
$$

is a pushout by construction; but for any pair of morphisms $\varphi\colon Y \to A$ and $\psi : Y \to B$ we have $\alpha \circ \varphi = \beta \circ \psi$ if and only if $\alpha_0 \circ \varphi = \beta_0 \circ \psi$, so the diagram is also a pullback.

D. If both A and B are C^*-subalgebras of a larger algebra C, and α and β denote the inclusion morphisms, then we simply get $A \oplus_C B = A \cap B$.

In a similar vein, if A and B are quotients of a larger C^*-algebra C, and α and β denote the quotient morphisms, then $A \star_C B = C/(\ker \alpha + \ker \beta)$.

E. Finally we wish to mention that several constructions familiar from K-theory relate directly to pullback diagrams, although the formal identification is not always stressed. If $\alpha : B \to A$ and $\beta : B \to A$ are morphisms between C^*-algebras A and B, and $\mathbb{I}A = C([0, 1], A)$ denotes the cylinder algebra over A (more about this in Sect. 11), we define a pullback diagram

$$\begin{tikzcd} X \arrow[r, "\gamma"'] \arrow[d, "\delta"] & B \arrow[d, "\varepsilon"] \\ \mathbb{I}A \arrow[r, "\partial"] & A \oplus A \end{tikzcd}$$

Here ∂ is the boundary map $\partial f = (f(0), f(1))$ and $\varepsilon(b) = (\alpha(b), \beta(b))$. In the case where $\alpha = 0$ the C^*-algebra X is known as the *mapping cone*, cf. [3, 15.3.1] or [50, 6.4.5]. In the case where $B = A$ and $\alpha = \text{id}$ the algebra X is called the *mapping torus* or *mapping cylinder*, cf. [3, 10.3.1] or [50, 9K].

3.4. PROPOSITION. *Let $\mathscr{C}$ be a class of C^*-algebras which is closed under formation of ideals, quotients and extensions. Then $\mathscr{C}$ is also closed under formation of pullbacks.*

Proof. Consider a pullback diagram of C^*-algebras

$$\begin{tikzcd} X \arrow[r, "\gamma"'] \arrow[d, "\delta"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & C \end{tikzcd}$$

with A, B and C in $\mathscr{C}$. By (ii) in Proposition 3.1 we have $\gamma(X) = \beta^{-1}(\alpha(A))$, and thus an extension

$$0 \to \ker \beta \to \gamma(X) \to \alpha(A) \to 0.$$

By assumption $\alpha(A)$ belongs to $\mathscr{C}$, and so does $\ker \beta$, so $\gamma(X) \in \mathscr{C}$.

By (i) in Proposition 3.1 it follows that δ is an isomorphism of $\ker \gamma$ onto $\ker \alpha$, so that we again have an extension

$$0 \to \ker \alpha \to X \to \gamma(X) \to 0.$$

By assumption $\ker \alpha \in \mathscr{C}$, and we just proved the same for $\gamma(X)$, so we conclude that $X \in \mathscr{C}$, as desired. ∎

3.5. *Remarks.* Proposition 3.4 applies to show that the four main categories of C^*-algebras: separable, exact, nuclear and of type I, are all closed under pullbacks. It also shows that the category of finitely generated C^*-algebras is closed under pullbacks, as claimed (without tangible evidence) in the proof of [21, Lemma 2.4.3]. Evidently that class is closed under the formation of quotients and extensions. To show that it is also closed under formation of ideals, let I be a closed ideal in a (necessarily separable) C^*-algebra A with generators $\{a_1, ..., a_n\}$. If h is a strictly positive contraction in I, then the set $\{ha_1, ..., ha_n, h\}$ will generate I. The straightforward argument for this claim uses a quasicentral approximate unit chosen from the algebra $\{f(h) \mid f \in C_0(]0, 1])\}$, cf. [38, 3.12.14].

Even for categories that are not closed under arbitrary extensions one may obtain stability results for pullbacks with surjective rows as in 3.2. Thus it is proved in [12, Theorem 5.7] that if we have a pullback diagram of C^*-algebras

$$\begin{tikzcd} X \arrow[r, "\gamma"'] \arrow[d, "\delta"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & C \end{tikzcd}$$

in which α (hence also γ) is surjective, then if A, B and C have topological stable rank one, or have real rank zero, or are extremally rich, the same is true for X—extremal richness, though, only if in addition β is extreme-point-preserving. Simple examples, cf. [12, Example 5.9], show that the surjection condition can not be deleted, and that the condition that β be extreme-point-preserving in necessary in the case of extremal richness.

3.6. Proposition. *Given two extensions of C^*-algebras*

$$0 \to A_i \to X \to B_i \to 0,$$

where $i = 1, 2$, we obtain, taking $C = X/(A_1 + A_2)$, a third extension

$$0 \to A_1 \cap A_2 \to X \to B_1 \oplus_C B_2 \to 0.$$

Proof. Identifying as usual $A_i/(A_1 \cap A_2)$ with $(A_1 + A_2)/A_i$, $i = 1, 2$, we obtain a commutative diagram

$$\begin{tikzcd} A_1 \cap A_2 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & (A_1+A_2)/A_2 \arrow[d] \\ A_2 \arrow[r] \arrow[d] & X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B_2 \arrow[d, "\beta"] \\ (A_1+A_2)/A_1 \arrow[r] & B_1 \arrow[r, "\alpha"] & C \end{tikzcd}$$

in which all rows and columns are extensions. Evidently the coherent pair of morphisms (π_1, π_2) gives rise to a morphism

$$\sigma : X \to B_1 \oplus_C B_2$$

such that $\gamma\circ\sigma=\pi_1$ and $\delta\circ\sigma=\pi_2$, where γ and δ are the projections on first and second coordinates in $B_1\oplus_C B_2$, respectively. Moreover,

$$\ker\sigma=\ker\pi_1\cap\ker\pi_2=A_1\cap A_2.$$

To show that σ is surjective take $b=(b_1,b_2)$ in $B_1\oplus_C B_2$. Choose x such that $\pi_1(x)=b_1$ and note that

$$\beta(\pi_2(x)-b_2)=\alpha(\pi_1(x))-\beta(b_2)=\alpha(b_1)-\beta(b_2)=0,$$

so that $\pi_2(x)-b_2\in\ker\beta$. Now observe that $\ker\beta=\pi_2(A_1)$, so we can find a_1 in A_1 such that $b_2=\pi_2(x+a_1)$. We will have $b_1=\pi_1(x+a_1)$, so $\sigma(x+a_1)=b$, as desired. ∎

3.7. LEMMA. *If $X_i\in\operatorname{ext}(A_i,B_i)$, $i=1,2$, are two extensions of C^*-algebras and both are exact, then with $\otimes$ denoting the minimal tensor product we obtain two new extensions*:

$$0\to A_1\otimes A_2\to X_1\otimes X_2\to(B_1\otimes X_2)\bigoplus_{B_1\otimes B_2}(X_1\otimes B_2)\to 0;\qquad (*)$$

$$0\to A_1\otimes X_2+X_1\otimes A_2\to X_1\otimes X_2\to B_1\otimes B_2\to 0.\qquad (**)$$

If instead $\otimes=\otimes_{\max}$, the formulae $(*)$ and $(**)$ are valid for all C^*-algebras X_i in $\operatorname{ext}(A_i,B_i)$, $i=1,2$.

Proof. With $\otimes=\otimes_{\min}$ we consider the commutative diagram

$$\begin{tikzcd}
A_1\otimes A_2 \arrow[r] \arrow[d] & A_1\otimes X_2 \arrow[r, "\rho_2"] \arrow[d] & A_1\otimes B_2 \arrow[d] \\
X_1\otimes A_2 \arrow[r] \arrow[d, "\rho_1"] & X_1\otimes X_2 \arrow[r, "\sigma_2"] \arrow[d, "\sigma_1"] & X_1\otimes B_2 \arrow[d, "\tau_1"] \\
B_1\otimes A_2 \arrow[r] & B_1\otimes X_2 \arrow[r, "\tau_2"] & B_1\otimes B_2
\end{tikzcd}$$

in which $\rho_1=\pi_1\otimes\iota_2$ and similarly for all the other quotient morphisms. Since X_1 and X_2 (hence also A_1, A_2, B_1 and B_2) are exact, all rows and columns in the diagram are extensions. Evidently

$$X_1\otimes A_2\cap A_1\otimes X_2=A_1\otimes A_2,$$

so the extension $(*)$ follows from Proposition 3.6.

The extension $(**)$ also follows from Proposition 3.6, identifying $B_1\otimes B_2$ with $X_1\otimes X_2/(A_1\otimes X_2+X_1\otimes A_2)$.

Since the maximal tensor product preserves extensions and embeddings of ideals, the same proof applies (for arbitrary X_1 and X_2) when $\otimes$ is taken as $\otimes_{\max}$, cf. [5, Proposition 3.15]. ∎

3.8. THEOREM. *If we are given two pullback diagrams*

$$\begin{tikzcd} X_1 \arrow[r, "\gamma_1"'] \arrow[d, "\delta_1"] & B_1 \arrow[d, "\beta_1"] \\ A_1 \arrow[r, "\alpha_1"] & C_1 \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} X_2 \arrow[r, "\gamma_2"'] \arrow[d, "\delta_2"] & B_2 \arrow[d, "\beta_2"] \\ A_2 \arrow[r, "\alpha_2"] & C_2 \end{tikzcd}$$

in which α_1 and α_2 are both surjective, we obtain a new pullback diagram with $\otimes$ denoting the maximal tensor product:

$$\begin{tikzcd} X_1 \otimes X_2 \arrow[r, "\bar{\gamma}"'] \arrow[d, "\bar{\delta}"] & (X_1 \otimes B_2) \underset{B_1 \otimes B_2}{\oplus} (B_1 \otimes X_2) \arrow[d, "\bar{\beta}"] \\ A_1 \otimes A_2 \arrow[r, "\bar{\alpha}"] & (A_1 \otimes C_2) \underset{C_1 \otimes C_2}{\oplus} (C_1 \otimes A_2) \end{tikzcd}$$

Here $\bar{\alpha}$ and $\bar{\gamma}$ are the quotient morphisms with kernels $\ker \alpha_1 \otimes \ker \alpha_2$ *and* $\ker \gamma_1 \otimes \ker \gamma_2$, *respectively. Moreover,* $\bar{\delta} = \delta_1 \otimes \delta_2$, *whereas* $\bar{\beta} = (\delta_1 \otimes \beta_2) \oplus (\beta_1 \otimes \delta_2)$.

The same formula prevails if instead $\otimes$ denotes the minimal tensor product and both X_1 and X_2 are exact,

Proof. Assume first that $\otimes$ is the minimal tensor product, and let $I_i = \ker \gamma_i$ and $J_i = \ker \alpha_i$, $i = 1, 2$. It follows from Proposition 3.1 that $J_i = \delta_i(I_i)$ for $i = 1, 2$ (and δ_i is an isomorphism, cf. Remark 3.2). From $(*)$ in Lemma 3.7 we conclude that

$$\ker \bar{\gamma} = I_1 \otimes I_2, \qquad \ker \bar{\alpha} = J_1 \otimes J_2.$$

Consequently,

$$\bar{\delta}(\ker \bar{\gamma}) = (\delta_1 \otimes \delta_2)(I_1 \otimes I_2) = J_1 \otimes J_2 = \ker \bar{\alpha},$$

and the new diagram is a pullback by Proposition 3.1, since evidently

$$\ker \bar{\delta} \cap \ker \bar{\gamma} = (\ker \delta_1 \otimes X_2 + X_1 \otimes \ker \delta_2) \cap (I_1 \otimes I_2) = \{0\}.$$

As the maximal tensor product preserves extensions and embeddings of ideals, the same proof applies (for arbitrary X_1 and X_2) when $\otimes$ is taken as $\otimes_{\max}$. ∎

The proof of a slightly weakened version of the next Theorem (both X and Y are exact, and α is surjective) can be obtained from Theorem 3.8 by substituting

$$\begin{array}{ccc} X_2 & \longrightarrow & B_2 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & C_2 \end{array} \quad \text{with} \quad \begin{array}{ccc} Y & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & 0 \end{array}$$

However, by a direct argument one discovers that it suffices to demand only that the algebra Y is exact. We leave this as an exercise for the reader.

3.9. THEOREM (Cf. [49, 1.11]). *Consider the two commutative diagrams of C^*-algebras*

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & B \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & C \end{array} \quad \textit{and} \quad \begin{array}{ccc} Y \otimes_{\min} X & \xrightarrow{\bar{\gamma}} & Y \otimes_{\min} B \\ \downarrow{\scriptstyle \bar{\delta}} & & \downarrow{\scriptstyle \bar{\beta}} \\ Y \otimes_{\min} A & \xrightarrow{\bar{\alpha}} & Y \otimes_{\min} C \end{array}$$

where $\bar{\alpha} = \iota \otimes \alpha$, and similarly for $\bar{\beta}$, $\bar{\gamma}$, and $\bar{\delta}$. If Y is an exact C^*-algebra and the first diagram is a pullback, then so is the second.

3.10. *Remark.* The condition that Y be exact is necessary for the preceeding result. For if $X \in \operatorname{ext}(A, B)$ we can form the two commuting diagrams

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array} \quad \text{and} \quad \begin{array}{ccc} Y \otimes_{\min} A & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ Y \otimes_{\min} X & \longrightarrow & Y \otimes_{\min} C \end{array}$$

The first is a pullback, cf. Example 3.3.B; but the second is only a pullback if $Y \otimes_{\min} A$ is the kernel of the quotient morphism of $Y \otimes_{\min} X$ onto $Y \otimes_{\min} C$, and that requires Y to be exact.

Replacing $\otimes_{\min}$ with $\otimes_{\max}$ in Theorem 3.9 saddles us with the same problems already encountered in Theorem 3.8, namely that, say $\bar{\alpha}$ as a morphism between $Y \otimes_{\max} A$ and $Y \otimes_{\max} C$ does not, necessarily, have image $Y \otimes_{\max} \alpha(A)$ (cf. 2.15). The embedding $\alpha(A) \subset C$ gives a morphism $Y \otimes_{\max} \alpha(A) \to Y \otimes_{\max} C$, but not necessarily an injective one. Thus, even though α and β are injective, we can not assert the same for $\bar{\alpha}$ and $\bar{\beta}$. We can, however, control the kernel if the original morphism is surjective, and that is not unusual, cf. Remark 3.2.

In any case the direct argument for Theorem 3.9 mentioned above applies to any pullback diagram of C^*-algebras and any C^*-algebra Y to produce a new pullback

$$\begin{array}{ccc} Y \otimes_{\max} X & \xrightarrow[\bar{\gamma}]{} & Y \otimes_{\max} B \\ \Big\downarrow{\scriptstyle \bar{\delta}} & & \Big\downarrow{\scriptstyle \bar{\beta}} \\ Y \otimes_{\max} A & \xrightarrow{\bar{\alpha}} & Y \otimes_{\max} C \end{array}$$

3.11. *Multirestricted Direct Sums.* The results in Theorem 3.8 are not symmetric in A and B (but probably more useful as stated). To formulate a symmetric version we need to expand the notion of restricted direct sums to include more summands and more targets, thus abandoning the lush world of diagrams in favour of algebraic austerity.

Given a family $\{A_i \mid i \in I\}$ of C^*-algebras, and for each nondiagonal pair (i, j) in $I \times I$ a morphism $\alpha_{ij}: A_i \to C_{ij}$ into some C^*-algebras C_{ij}, where $C_{ij} = C_{ji}$ (and $i \neq j$), we define the *multirestricted direct sum*

$$\bigoplus_{C_{ij}} A_i = \{(a_i) \in \bigoplus A_i \mid \alpha_{ij}(a_i) = \alpha_{ji}(a_j) \; \forall i, j\}.$$

Evidently this C^*-algebra is the universal solution to the problem of finding a C^*-algebra A with morphisms $\delta_i: A \to A_i$, such that $\alpha_{ij} \circ \delta_i = \alpha_{ji} \circ \delta_j$ for all i and j; in the sense that any other solution must factor through $\bigoplus_{C_{ij}} A_i$.

We may assume that all C_{ij} are identical ($= C$), which greatly simplifies the notation. Either this reduction is given at the outset, or we force it by taking $C = \bigoplus_{i<j} C_{ij}$ and defining $\alpha_k : A_k \to C$ by $(\alpha_k(a_k))_{ij} = \alpha_{kj}(a_k)$ if $i = k$, and $= \alpha_{ij}(a_k)$ if $j = k$; zero elsewhere.

We may also consider the "dual" definition of multiamalgamated free products, and in the case of a single amalgamation algebra (i.e. $C_{ij} = C$) this was done already in [1, Sect. 3].

It is straightforward to generalize Proposition 3.1 to show that if I is finite and $\bigoplus_C A_i$ is determined by morphisms $\alpha_i : A_i \to C$, $i \in I$, and if we have a family of morphisms $\delta_i : X \to A_i$ for some C^*-algebra X, such that $\alpha_i \circ \delta_i = \alpha_j \circ \delta_j$ for all i and j, then $X = \bigoplus_C A_i$ if and only if

(i) $\bigcap \ker \delta_i = \{0\}$,

(ii) $\alpha_1^{-1}(\bigcap \alpha_i(A_i)) = \delta_1(X)$,

(iii) $\delta_j(\bigcap_{i \neq j} \ker \delta_i) = \ker \alpha_j$

for all $j > 1$.

Using the result above we can generalize Theorem 3.9 and show that if Y is an exact C^*-algebra then

$$\left(\bigoplus_C A_i\right)\otimes Y = \bigoplus_{C\otimes Y} A_i \otimes Y, \tag{$*$}$$

where $\otimes$ denotes the minimal tensor product. If instead we use the maximal tensor product the formula $(*)$ holds for every Y.

3.12. *Tensor Products of Restricted Direct Sums.* Assume now that $A = \bigoplus_C A_i$ and $B = \bigoplus_D B_j$ are restricted direct sums of finite families of C^*-algebras $\{A_i \mid i \in I\}$ and $\{B_j \mid j \in J\}$, respectively, determined by morphisms $\alpha_i : A_i \to C$ and $\beta_j : B_j \to D$ for some C^*-algebras C and D. Then with $\otimes$ denoting the maximal tensor product we obtain by iterated use of $(*)$ in 3.11 that

$$A \otimes B = \bigoplus_{C\otimes B} A_i \otimes B = \bigoplus_{C\otimes B}\left(\bigoplus_{A_i\otimes D} A_i \otimes B_j\right).$$

Observing that $C \otimes B = \bigoplus_{C\otimes D} C \otimes B_j$ we see that elements in $A \otimes B$ can be described as those (z_{ij}) in $\bigoplus A_i \otimes B_i$ such that

$$\begin{aligned}
\iota_i \otimes \beta_j(z_{ij}) &= \iota_i \otimes \beta_\ell(z_{i\ell}), && (i, j, \ell) \in I \times J \times J,\\
\alpha_i \otimes \iota_j(z_{ij}) &= \alpha_k \otimes \iota_\ell(z_{k\ell}), && (i, k, j) \in I \times I \times J,\\
\alpha_i \otimes \beta_j(z_{ij}) &= \alpha_k \otimes \beta_\ell(z_{k\ell}), && (i, k, j, \ell) \in I \times I \times J \times J.
\end{aligned}$$

Thus if for (i, j, k, ℓ) in $I \times J \times I \times J$, where $(i, j) < (k, \ell)$ (in the lexicographic order), we define $E = \bigoplus E_{ijk\ell}$, where

$$E_{ijk\ell} = \left(A_i \otimes D \bigoplus_{C\otimes D} C \otimes B_j\right) \bigoplus_{C\otimes D} \left(A_k \otimes D \bigoplus_{C\otimes D} C \otimes B_\ell\right),$$

we can write the tensor product as a multirestricted direct sum:

$$\left(\bigoplus_C A_i\right) \otimes \left(\bigoplus_D B_j\right) = \bigoplus_E A_i \otimes B_j.$$

If the C^*-algebras involved are all exact, the formula above holds for the minimal tensor product.

4. STRUCTURE IN PUSHOUTS

4.1. *Universal Embeddings.* If we consider a pushout diagram written as in 2.3 there are some nontrivial relations between the kernels of the morphisms α, β, γ and δ. In [1, Theorem 3.1] Blackadar proved that if both α and β are injective, then so are γ and δ. The proof (he complains) is curiously nonconstructive and uses the universal representations of the algebras involved. For convenience we shall here refer to any representation $(\pi, \mathfrak{H})$ of a C^*-algebra A as *universal*, if it is nondegenerate, and if every functional φ in A^* can be represented as a vector functional $\varphi(x) = (\pi(x)\,\xi \mid \eta)$, $x \in A$, for some ξ and η in $\mathfrak{H}$.

Using the same approach, we obtain an extension of Blackadar's result.

4.2. THEOREM. *Assume that we have embeddings of C^*-algebras $C \subset A_1 \subset A_2$ and $C \subset B_1 \subset B_2$. Then also the natural morphisms $\delta_i \colon A_i \to A_i \star_C B_i$ and $\gamma_i \colon B_i \to A_i \star_C B_i$ are injective for $i = 1, 2$. Moreover, the natural morphism $\sigma \colon A_1 \star_C B_1 \to A_2 \star_C B_2$ is injective.*

Proof. Put $X_i = A_i \star_C B_i$ for $i = 1, 2$ and consider the commutative diagram

$$\begin{tikzcd}
 & C \arrow[rr, "\beta_2"] \arrow[dd, "\alpha_2"' near start] & & B_2 \arrow[dd, "\gamma_2"] \\
C \arrow[ur, "\iota"] \arrow[rr, "\beta_1" near end] \arrow[dd, "\alpha_1"'] & & B_1 \arrow[ur, "\iota"] \arrow[dd, "\gamma_1" near start] & \\
 & A_2 \arrow[rr, "\delta_2" near start] & & X_2 \\
A_1 \arrow[ur, "\iota"] \arrow[rr, "\delta_1"] & & X_1 \arrow[ur, "\sigma"] &
\end{tikzcd}$$

Here all morphisms ι, α_i and β_i, $i = 1, 2$, are injections. To prove that σ is injective, assume first that $A_1 = A_2 (= A)$ (so that $\alpha_1 = \alpha_2 = \alpha$), and assume moreover that γ_1 is injective. Consider universal representations $(\pi, \mathfrak{H})$ and $(\rho, \mathfrak{K})$ of X_1 and B_2, respectively. Identifying the subalgebras $\gamma_1(B_1)$ and B_1 it follows that $\pi(\gamma_1(B_1))''$ and $\rho(B_1)''$ are both isomorphic to the enveloping von Neumann algebra of B_1, cf. [38, 3.7.9]. After suitable amplifications of π and ρ we may therefore assume that $\mathfrak{K} = \mathfrak{H}$ and that $\pi(\gamma_1(B_1)) = u\rho(B_1)\,u^*$ for some unitary u on $\mathfrak{H}$, cf. [38, 3.8.7]. Then π and $\rho_0 = \operatorname{Ad} u \circ \rho$ are universal representations of X_1 and B_2, respectively, such that $\rho_0 \mid B_1 = \pi \circ \gamma_1$. Since $\delta_1(A) \subset X_1$, there is a morphism τ of X_2 into the C^*-algebra generated by $\pi(X_1) \cup \rho_0(B_2)$ such that $\pi \circ \delta_1 = \tau \circ \delta_2$ and $\rho_0 = \tau \circ \gamma_2$, whence $\pi \circ \gamma_1 = \rho_0 \mid B_1 = \tau \circ \gamma_2 \mid B_1$. The natural morphism σ: $X_1 \to X_2$ is obtained from the morphisms δ_2 and $\gamma_2 \mid B_1$ of A and B_1 into X_2, so $\delta_2 = \sigma \circ \delta_1$ and $\gamma_2 \mid B_1 = \sigma \circ \gamma_1$. It follows that $\pi = \pi \circ \sigma$ both on $\delta(A)$ and on $\gamma_1(B_1)$, whence $\pi = \tau \circ \sigma$ on X_1. Since π is injective, so is σ.

Applying this preliminary result to the case where $B_1 = C$ and $B_2 = B$, we have $X_1 = A$, cf. Example 3.3.B, so the assumption that γ_1 be injective is fulfilled. It follows that $\sigma = \delta : A \to A \bigstar_C B$ is injective. By symmetry also $\gamma : B \to A \bigstar_C B$ is injective, so we have established Blackadar's result. Thus, δ_i and γ_i are injective in general (for $i = 1, 2$), and consequently also $\sigma : X_1 \to X_2$ is injective (when $A_1 = A_2$).

The general case follows by applying the above argument twice, first holding A_1 fixed and passing from B_1 to B_2, then with B_2 fixed passing from A_1 to A_2. ∎

4.3. *Remarks.* The general problem of embedding one amalgamated free product into another is quite tricky. Thus the result in Theorem 4.2 may fail if, say, the morphisms β_i: $C \to B_i$, $i = 1, 2$, are not injective. It suffices to consider the box diagram below, modeled on the diagram in the proof of Theorem 4.2, where the front and the hind squares are pushouts.

$$\begin{tikzcd}
& \mathbb{C}^2 \arrow[rr] \arrow[dd] & & \mathbb{C} \arrow[dd] \\
\mathbb{C}^2 \arrow[ur, "\iota"] \arrow[rr, crossing over] \arrow[dd] & & \mathbb{C} \arrow[ur, "\iota"] & \\
& \mathbb{M}_2 \arrow[rr] & & 0 \\
\mathbb{C}^2 \arrow[ur, "\iota"] \arrow[rr] & & \mathbb{C} \arrow[ur, "\sigma"] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

Be warned also that the representation $\pi \bigstar \rho_0$ of X_2 in Theorem 4.2, its large apparent "size" notwithstanding, is, in general, much smaller than the universal representation of X_2, and need not even be faithful. This even applies to the case where $B_1 = C$ and $B_2 = B$. Take, e.g., $C = 0$ and $A = B = \mathbb{C}$.

Our next result shows how to determine the kernels of the morphisms γ and δ without reference to the amalgamated product X. In particular it proves that every amalgamated free product can be obtained from an injective pushout diagram.

4.4. Theorem. *Consider a pushout diagram of* C^**-algebras*

$$\begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\ A \arrow[r, "\delta"] & X \end{tikzcd}$$

If $I = \ker(\delta \circ \alpha) = \ker(\gamma \circ \beta)$, *then* I *is the smallest closed ideal of* C *containing* $\ker \alpha + \ker \beta$ *such that*

$$\mathrm{Id}(\alpha(I)) \cap \alpha(C) = \alpha(I) \qquad \textit{and} \qquad \mathrm{Id}(\beta(I)) \cap \beta(C) = \beta(I), \tag{$*$}$$

where $\mathrm{Id}(E)$ *denotes the smallest closed ideal generated by a set* E. *Moreover,*

$$\ker \gamma = \mathrm{Id}(\beta(I)) \qquad \textit{and} \qquad \ker \delta = \mathrm{Id}(\alpha(I)).$$

Proof. Since $\alpha(I) \subset \ker \delta$ we also have $\mathrm{Id}(\alpha(I)) \subset \ker \delta$. Similarly $\mathrm{Id}(\beta(I)) \subset \ker \gamma$, so we can form the commutative diagram

$$\begin{tikzcd} C \arrow[rr, "\beta"'] \arrow[dr] \arrow[dd, "\alpha"] & & B \arrow[d] \\ & C/I \arrow[r, "\tilde{\beta}"] \arrow[d, "\tilde{\alpha}"] & B/\mathrm{Id}(\beta(I)) \arrow[d, "\tilde{\gamma}"] \\ A \arrow[r] & A/\mathrm{Id}(\alpha(I)) \arrow[r, "\tilde{\delta}"] & X \end{tikzcd}$$

Since the quotient morphisms of A and B are surjective it follows easily that the smaller SE square is a pushout.

We now estimate

$$\alpha(I) \subset \mathrm{Id}(\alpha(I)) \cap \alpha(C) \subset \ker \delta \cap \alpha(C) = \alpha(I).$$

This means that

$$\ker \tilde{\alpha} = \alpha^{-1}(\mathrm{Id}(\alpha(I)))/I = \{0\},$$

so that $\tilde{\alpha}$—and by symmetry also $\tilde{\beta}$—are both injective. By Blackadar's theorem proved in Theorem 4.2, also $\tilde{\gamma}$ and $\tilde{\delta}$ are injective; and it follows that

$$\ker \gamma = \mathrm{Id}(\beta(I)) \qquad \text{and} \qquad \ker \delta = \mathrm{Id}(\alpha(I)),$$

as desired.

If J is another closed ideal of C containing $\ker\alpha + \ker\beta$ such that

$$\mathrm{Id}(\alpha(J)) \cap \alpha(C) = \alpha(J) \qquad \text{and} \qquad \mathrm{Id}(\beta(J)) \cap \beta(C) = \beta(J), \tag{$*$}$$

then we can form a commutative diagram exactly as above with I replaced by J and X replaced by the C^*-algebra

$$X_0 = A/\mathrm{Id}(\alpha(J)) \star_{C/J} B/\mathrm{Id}(\beta(J)).$$

The two quotient morphisms $\pi : A \to A/\mathrm{Id}(\alpha(J))$ and $\rho : B \to B/\mathrm{Id}(\beta(J))$, followed by the embeddings $\tilde{\delta}$ and $\tilde{\gamma}$ into X_0, form a coherent pair, so that we obtain a (quotient) morphism $\sigma : X \to X_0$ such that

$$\sigma \circ \delta = \tilde{\delta} \circ \pi \qquad \text{and} \qquad \sigma \circ \gamma = \tilde{\gamma} \circ \rho.$$

In particular, $\sigma \circ \delta \circ \alpha = \tilde{\delta} \circ \pi \circ \alpha$, so

$$I = \ker(\delta \circ \alpha) \subset \ker(\sigma \circ \delta \circ \alpha) = \ker(\tilde{\delta} \circ \pi \circ \alpha) = \ker(\pi \circ \alpha) = J.$$

Here the argument for the last equality sign uses the special properties $(*)$ of J, and the injectivity of $\tilde{\delta}$ follows from Blackadar's result, applied to the embeddings $\tilde{\alpha}$ and $\tilde{\beta}$ of C/J into $A/\mathrm{Id}(\alpha(J))$ and $B/\mathrm{Id}(\beta(J))$, respectively.

It follows that I is indeed characterized as being the smallest closed ideal in C satisfying the relations $(*)$. ∎

4.5. Proposition. *Consider a commutative diagram of C^*-algebras*

$$\begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\ A \arrow[r, "\delta"] & X \end{tikzcd}$$

in which $\ker\beta \subset \ker\alpha$ *and* $\beta(C)$ *is a hereditary* C^*-*subalgebra of* B. *Then* $\mathrm{Id}(\beta(\ker\alpha)) \cap \beta(C) = \beta(\ker\alpha)$. *If the diagram is a pushout we therefore have that*

(i) *X is generated (as a C^*-algebra) by $\delta(A) \cup \gamma(B)$,*

(ii) *δ is injective,*

(iii) $\ker\gamma = \mathrm{Id}(\beta(\ker\alpha))$.

Conversely, if these three conditions are satisfied and α is a proper morphism then the diagram is a pushout.

Proof. Put $I = \ker\alpha$. If $x \in \mathrm{Id}(\beta(I)) \cap \beta(C)$, there is for each $\varepsilon > 0$ a finite sum of the form $y = \sum b_n x_n b_n'$, with b_n and b_n' in B and x_n in $\beta(I)$, such that $\|x - y\| < \varepsilon$. If now (u_λ) is an approximate unit for $\beta(C)$, then evidently $u_\lambda x u_\lambda \to x$ and $u_\lambda x_n u_\lambda \to x_n$ for all n. Since $\|u_\lambda(x-y)\,u_\lambda\| < \varepsilon$ we may therefore assume that $\|x - y_\lambda\| < 2\varepsilon$, where $y_\lambda = \sum u_\lambda b_n u_\lambda x_n u_\lambda b'_n u_\lambda$. Note now that $u_\lambda b_n u_\lambda \in \beta(C)\, B\beta(C) = \beta(C)$ and $x_n \in \beta(I)$, which is an ideal in $\beta(C)$. It follows that $y_\lambda \in \beta(I)$; whence in the limit $x \in \beta(I)$, as desired. A rather short version of this argument is found in [37, 3.2.7].

If the diagram is a pushout we see that $I = \ker \alpha$ satisfies the requirements ($*$) in Theorem 4.4, and thus the three conditions follow.

That the three conditions suffice to make the diagram a pushout when α is proper requires considerably more work. As we shall see, it is a special case of Theorem 5.11, and we defer the proof until then. ∎

4.6. Lemma. *Consider a commutative diagram of C^*-algebras*

$$\begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\ A \arrow[r, "\delta"] & D \end{tikzcd}$$

If α is a proper morphism, i.e., $\alpha(C)$ generates A as a hereditary C^*-algebra, then $\delta(A)$ is contained in the hereditary C^*-subalgebra of D generated by $\gamma(B)$. In particular, if D is hereditarily generated by $\delta(A) \cup \gamma(B)$, then γ is a proper morphism.

If A is only generated by $\alpha(C)$ as a closed ideal, i.e. $A = \mathrm{Id}(\alpha(C))$, then $\delta(A) \subset \mathrm{Id}(\gamma(B))$ in D. In particular, if $D = \mathrm{Id}(\gamma(B) \cup \delta(A))$, then already $D = \mathrm{Id}(\gamma(B))$, and if also $B = \mathrm{Id}(\beta(C))$ then actually $D = \mathrm{Id}(\delta(\alpha(C)))$.

Proof. Let E denote the hereditary C^*-subalgebra of D generated by $\gamma(B)$. Since α is proper we have $A = \alpha(C)\, A\alpha(C)$, cf. [40, Sect. 4], whence

$$\delta(A) = \delta(\alpha(C)\, A\alpha(C)) = \gamma(\beta(C))\, \delta(A)\, \gamma(\beta(C)) \subset \gamma(B)\, D\gamma(B) \subset E.$$

In particular, if D is the smallest hereditary algebra containing $\gamma(B) \cup \delta(A)$ then $D = E$, which means that γ is a proper morphism, cf. [20, 2.1] or [39, 2.1].

To prove the second half of the lemma let $I = \mathrm{Id}(\gamma(B))$ in D. Then the diagram

$$\begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "0"] \\ A \arrow[r, "\tilde{\delta}"] & D/I \end{tikzcd}$$

is still commutative. Evidently $\tilde{\delta} \circ \alpha = 0$, so $\alpha(C) \subset \ker \tilde{\delta} = \delta^{-1}(I)$. By assumption this means that $\delta^{-1}(I) = A$, whence $\delta(A) \subset I$.

Finally, if $B = \mathrm{Id}(\beta(C))$, let $J = \mathrm{Id}(\gamma(\beta(C)))$. Then $\beta(C) \subset \gamma^{-1}(J)$, so $B = \gamma^{-1}(J)$ and $\gamma(B) \subset J$. Symmetrically, $\delta(A) \subset J (= \mathrm{Id}(\delta(\alpha(C))))$, so $D = J$. ∎

4.7. Proposition. *Consider a sequence of pushout diagrams as below, to the left, and assume that each α_n is a proper morphism. Then we obtain the new pushout diagram below, to the right:*

$$\begin{array}{ccc} C_n & \xrightarrow{\beta_n} & B_n \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \gamma_n} \\ A_n & \xrightarrow{\delta_n} & X_n \end{array} \qquad \textit{gives} \qquad \begin{array}{ccc} \bigoplus C_n & \xrightarrow{\oplus \beta_n} & \bigoplus B_n \\ \downarrow{\scriptstyle \oplus \alpha_n} & & \downarrow{\scriptstyle \oplus \gamma_n} \\ \bigoplus A_n & \xrightarrow{\oplus \delta_n} & \bigoplus X_n \end{array}$$

Proof. For ease of notation put $X = \bigoplus X_n$, and likewise for A, B and C. Given now a coherent pair of morphisms $\varphi : A \to Y$ and $\psi : B \to Y$ into some C^*-algebra Y (i.e. $\varphi \circ \bigoplus \alpha_n = \psi \circ \bigoplus \beta_n$), we may assume that Y is generated by $\varphi(A) \cup \psi(B)$. Since each α_n is proper, $\bigoplus \alpha_n$ is proper, and by Lemma 4.6 so is the morphism ψ.

For each n let $\varphi_n = \varphi \mid A_n$ and $\psi_n = \psi \mid B_n$, and let Y_n denote the hereditary C^*-subalgebra generated by $\psi_n(B_n)$. Since $A_n = \alpha_n(C_n) A_n \alpha_n(C_n)$ by properness, and $\varphi_n(\alpha_n(C_n)) = \psi_n(\beta_n(C_n)) \subset Y_n$, it follows that $\varphi_n(A_n) \subset Y_n$. Each pair (φ_n, ψ_n) is therefore coherent for the nth diagram and defines there a unique morphism $\sigma_n : X_n \to Y_n$ such that $\varphi_n = \sigma_n \circ \delta_n$ and $\psi_n = \sigma_n \circ \gamma_n$. Since the Y_n's are pairwise orthogonal C^*-algebras we can unambiguously define the morphism

$$\bigoplus \sigma_n : X \to \bigoplus Y_n \subset Y,$$

and evidently $\varphi = \bigoplus \sigma_n \circ \bigoplus \delta_n$ and $\psi = \bigoplus \sigma_n \circ \bigoplus \gamma_n$. ∎

4.8. Proposition. *If we have a sequence of pullback diagrams as below, to the left, then we obtain the new pullback diagram below, to the right:*

$$\begin{array}{ccc} X_n & \xrightarrow{\gamma_n} & B_n \\ \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \beta_n} \\ A_n & \xrightarrow{\alpha_n} & C_n \end{array} \qquad \textit{gives} \qquad \begin{array}{ccc} \bigoplus X_n & \xrightarrow{\oplus \gamma_n} & \bigoplus B_n \\ \downarrow{\scriptstyle \oplus \delta_n} & & \downarrow{\scriptstyle \oplus \beta_n} \\ \bigoplus A_n & \xrightarrow{\oplus \alpha_n} & \bigoplus C_n \end{array}$$

If desired, one may even replace direct sums with direct products.

Proof. The relations defining the restricted direct sums carry over to the sequence spaces, so the proof is almost trivial. The result for products is true in any category with pullbacks, since the product preserves pullbacks, cf. Example 2.11.A. ∎

4.9. PROPOSITION. *Let (A_n), (B_n) and (C_n) be increasing sequences of C^*-algebras. Assuming that for each n we have morphisms $\alpha_n : A_n \to C_n$ and $\beta_n : B_n \to C_n$, compatible with the embeddings, there is a natural embedding of the C^*-algebra $\varinjlim(A_n \oplus_{C_n} B_n)$ into $\varinjlim A_n \oplus_{\varinjlim C_n} \varinjlim B_n$. If either α_n or β_n are surjective for infinitely many n this embedding is an isomorphism.*

Proof. For each n we get $A_n \oplus_{C_n} B_n \subset A_{n+1} \oplus_{C_{n+1}} B_{n+1}$, and thus, with $A = \varinjlim A_n$, $B = \varinjlim B_n$ and $C = \varinjlim C_n$, an embedding $\varinjlim(A_n \oplus_{C_n} B_n) \subset A \oplus_C B$.

To prove surjectivity, consider $x = (a, b)$ in $A \oplus_C B$. For each $\varepsilon > 0$ there is then an n, and elements a_n in A_n and b_n in B_n, such that $\|a - a_n\| < \varepsilon$ and $\|b - b_n\| < \varepsilon$; whence

$$\begin{aligned}\|\alpha_n(a_n) - \beta_n(b_n)\| &= \|\alpha(a_n) - \beta(b_n)\| \\ &\leqslant \|\alpha(a_n - a)\| + \|\beta(b - b_n)\| < 2\varepsilon.\end{aligned}$$

Choosing n suitably large we may assume that α_n is a surjective morphism. There is then an a'_n in A_n such that $\alpha_n(a'_n) = \beta_n(b_n)$ and $\|a_n - a'_n\| < 2\varepsilon$. But then $x_n = (a'_n, b_n) \in A_n \oplus_{C_n} B_n$, and $\|x - x_n\| \leqslant \max\{\|a - a'_n\|, \|b - b_n\|\} < 3\varepsilon$. It follows that $\varinjlim(A_n \oplus_{C_n} B_n) = A \oplus_C B$, as desired. ∎

4.10. EXAMPLE. The condition above, that either α_n or β_n be surjective infinitely often, is necessary. To see this, let Δ denote the closed unit disk and put $C_n = C(\Delta)$ for all n. With $r_n = 1 - (1/n)$ let A_n be the closed C^*-subalgebra of $C(\Delta)$ consisting of functions f such that for all θ in $[0, 2\pi]$

$$r \geqslant r_n \Rightarrow f(re^{i\theta}) = f(r_n e^{i\theta}).$$

Let B_n denote the C^*-subalgebra of functions g in $C(\Delta)$ such that

$$r \geqslant r_n \Rightarrow g(re^{i\theta}) = g(r_n e^{i(\theta + r - r_n)}).$$

Thus, the functions in A_n are continued constantly from $r_n\Delta$ to Δ along radial lines, whereas those in B_n are continued along curved lines. Evidently $A_n \subset A_{n+1}$ for all n, but a moments reflection shows that also $B_n \subset B_{n+1}$. In both cases we have isomorphisms $A_n = B_n = C(r_n\Delta)$. It follows that $\varinjlim A_n = \varinjlim B_n = C(\Delta)$, and therefore

$$\varinjlim A_n \oplus_{\varinjlim C_n} \varinjlim B_n = C(\Delta) \cap C(\Delta) = C(\Delta),$$

cf. Example 3.3.D. However, $A_n \oplus_{C_n} B_n = A_n \cap B_n$, which consists of functions f in $C(\Delta)$ such that

$$f(e^{i\theta}) = f(r_n e^{i\theta}) = f(r_n e^{i(\theta+1-r_n)}) = f(r_n e^{i(\theta+(1/n))})$$

for all θ. Since 2π is irrational, this condition implies that f is constant on the annulus $\{re^{i\theta} \mid r_n \leqslant r \leqslant 1 \ \&\ 0 \leqslant \theta \leqslant 2\pi\}$. It follows that

$$\varinjlim(A_n \oplus_{C_n} B_n) \neq C(\Delta),$$

since every function in the direct limit must be constant on the unit circle.

4.11. *Generalized Direct Limits.* A result for pushouts, analogous to Proposition 4.9, is difficult to formulate, because it is hard to find conditions on a pair of triples of algebras that will ensure that their amalgamated free products embed in one another, cf. Example 4.3. Suppose as in [45, Sect. 5] we renounce on embeddings, and define the *generalized direct limit* for any sequence of C^*-algebras (A_n) with morphisms $\varphi_n : A_n \to A_{n+1}$ (not necessarily injective) to be the completion of the *-algebra of equivalence classes of sequences (x_n), $x_n \in A_n$, such that $\varphi(x_n) = x_{n+1}$ for all $n \geqslant n_0$. Here equivalence of sequences means equality from a certain step, and the completion is made with respect to the C^*-seminorm $\|(x_n)\| = \lim \|x_n\|$ (the norm sequence being decreasing). This C^*-algebra (still denoted by $\varinjlim A_n$) has the universal property that for each coherent sequence of morphisms $\tau_n : A_n \to B$, i.e., $\tau_n = \tau_{n+1} \circ \varphi_n$ for each n, there is a unique morphism $\tau : \varinjlim A_n \to B$, such that $\tau_n = \tau \circ \bar{\varphi}_n$ for every n. Here $\bar{\varphi}_m : A_m \to \varinjlim A_n$ is the morphism that takes an element x in A_m to the sequence $(x, \varphi_m(x), \varphi_{m+1}(\varphi_m(x)), \ldots)$. Note that we have $\varinjlim A_n = \varinjlim \bar{\varphi}_n(A_n)$, where the second term is an oldfashioned direct limit of embedded algebras. Of course, this wellknown construction also has an analogue for generalized sequences (nets).

4.12. Proposition. *Let (A_n), (B_n) and (C_n) be sequences of C^*-algebras such that for each n we have morphisms $\varphi_n : A_n \to A_{n+1}$, $\psi_n = B_n \to B_{n+1}$ and $\eta_n : C_n \to C_{n+1}$. Assume furthermore that we have morphisms $\alpha_n : C_n \to A_n$ and $\beta_n : C_n \to B_n$, such that $\varphi_n \circ \alpha_n = \alpha_{n+1} \circ \eta_n$ and $\psi_n \circ \beta_n = \psi_{n+1} \circ \eta_n$ for each n. There is then a natural isomorphism between the generalized direct limit of the amalgamated free products and the amalgamated free product of the generalized direct limits, i.e.*

$$\varinjlim(A_n \star_{C_n} B_n) = \varinjlim A_n \star_{\varinjlim C_n} \varinjlim B_n.$$

Proof. The category $\vec{\mathbb{N}}\mathscr{C}^*$ of directed sequence of C^*-algebras defined in Example 2.11.C has pushouts. These are simply obtained from $(A_n) \leftarrow (C_n) \rightarrow (B_n)$ as the directed sequence $(A_n \star_{C_n} B_n)$. Since the function $\varinjlim$ has a right adjoint, cf. Example 2.11.C, it preserves pushouts from $\vec{\mathbb{N}}\mathscr{C}^*$ to $\mathscr{C}^*$, and that is the content of the proposition. ∎

4.13. COROLLARY. *If (A_n) and (B_n) are increasing sequences of C^*-algebras, all of which contain a common C^*-subalgebra C, then there is a natural isomorphism*

$$\varinjlim(A_n \star_C B_n) = \varinjlim A_n \star_C \varinjlim B_n,$$

where $\varinjlim$ now denotes the ordinary direct limit.

Proof. By Theorem 4.2 we have injections $A_n \star_C B_n \subset A_{n+1} \star_C B_{n+1}$ for every n, so that Proposition 4.12 applies with ordinary direct limits. ∎

4.14. PROPOSITION. *Let (A_n), (B_n) and (C_n) be increasing sequences of C^*-algebras. Assume that for each n we have embeddings $C_n \subset A_n$ and $C_n \subset B_n$, such that $A_{n+1} = A_n \star_{C_n} C_{n+1}$. Then $A_n \star_{C_n} B_n \subset A_{n+1} \star_{C_{n+1}} B_{n+1}$ for every n, so that we have natural isomorphisms*

$$\varinjlim(A_n \star_{C_n} B_n) = \varinjlim A_n \star_{\varinjlim C_n} \varinjlim B_n = A_1 \star_{C_1} \varinjlim B_n.$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & C_{n+1} \arrow[rr] \arrow[dd] & & B_{n+1} \arrow[dd] \\
C_n \arrow[ur] \arrow[rr] \arrow[dd] & & B_n \arrow[ur] \arrow[dd] & \\
 & A_{n+1} \arrow[rr] & & A_{n+1} \star_{C_{n+1}} B_{n+1} \\
A_n \arrow[ur] \arrow[rr] & & A_n \star_{C_n} B_n \arrow[ur] &
\end{tikzcd}$$

Since $A_{n+1} = A_n \star_{C_n} C_{n+1}$ by assumption, we can concatenate the left and hind pushout sides of the box diagram, cf. Proposition 2.7. (More about this in Theorem 8.3.) It follows that

$$A_{n+1} \star_{C_{n+1}} B_{n+1} = A_n \star_{C_n} B_{n+1} \supset A_n \star_{C_n} B_n,$$

where the last inclusion results from Theorem 4.2. Thus, Proposition 4.12 applies with ordinary direct limits.

Iterating the equality above we see that $A_n \star_{C_n} B_n = A_1 \star_{C_1} B_n$ for every n, whence the second equality in $(*)$ follows. ∎

4.15. *Inverse Limits.* If (A_n) is a sequence of C^*-algebras, and if for each n we have a morphism $\varphi_n : A_{n+1} \to A_n$, we define the *inverse limit* as the C^*-algebra $\varprojlim A_n$ of bounded sequences $x = (x_n)$ in $\prod A_n$, such that $\varphi_n(x_{n+1}) = x_n$ for all n.

For every m define the morphism $\bar{\varphi}_m : \varprojlim A_n \to A_m$ by evaluating an element $x = (x_n)$ in $\varprojlim A_n$ at m. Note that $\varphi_n \circ \bar{\varphi}_{n+1} = \bar{\varphi}_n$ for every n. Then $\varprojlim A_n$ has the universal property that for each coherent sequence of morphisms $\sigma_n : X \to A_n$ (i.e. $\sigma_n = \varphi_n \circ \sigma_{n+1}$) there is a unique morphism $\bar{\sigma} : X \to \varprojlim A_n$ such that $\sigma_n = \bar{\sigma} \circ \bar{\varphi}_n$ for all n.

For each m let $B_m = \bar{\varphi}_m(\varprojlim A_n) \subset A_m$. Then $\varphi_n(B_{n+1}) = B_n$ and $\varprojlim B_n = \varprojlim A_n$. This shows that the natural assumption that each morphism φ_n be surjective can always be realized by a slight change of target algebras.

In stark contrast to direct limits, the inverse limit of C^*-algebras is practically absent from the general theory. The primary reason is that the resulting algebras tend to be unmanageably large. However, in [12, Sect. 3] we show that in many instances the multiplier algebra $M(A)$ of a C^*-algebra A can be described as an inverse limits of quotients of A. With this in mind the size of the inverse limit becomes understandable, and the construction suddenly seems much more interesting. Especially since some of its universal properties are good. Thus, real rank zero, stable rank one and extremal richness are all preserved under inverse limits (with surjective morphisms φ_n), cf. [12, Theorem 3.8].

4.16. Proposition. *Let (A_n), (B_n) and (C_n) be sequences of C^*-algebras such that for each n we have morphisms $\varphi_n : A_{n+1} \to A_n$, $\psi_n : B_{n+1} \to B_n$ and $\eta_n : C_{n+1} \to C_n$. Assume furthermore that we have morphisms $\alpha_n : A_n \to C_n$ and $\beta_n : B_n \to C_n$, such that $\alpha_n \circ \varphi_n = \eta_n \circ \alpha_{n+1}$ and $\beta_n \circ \psi_n = \eta_n \circ \psi_{n+1}$ for each n. There is then a natural isomorphism between the inverse limit of the restricted direct sums and the restricted direct sum of the inverse limits, i.e.,*

$$\varprojlim (A_n \oplus_{C_n} B_n) = \varprojlim A_n \oplus_{\varprojlim C_n} \varprojlim B_n.$$

Proof. The category $\overleftarrow{\mathbb{N}}\mathscr{C}^*$ of inversely directed sequences of $\mathscr{C}^*$-algebras defined in Example 2.11.D has pullbacks. These are obtained from $(A_n) \to (C_n) \leftarrow (B_n)$ as the inversely directed sequence $(A_n \oplus_{C_n} B_n)$. Since the functor $\varprojlim$ has a left adjoint, cf. Example 2.11.D, it preserves pullbacks from $\overleftarrow{\mathbb{N}}\mathscr{C}^*$ to $\mathscr{C}^*$, and that is the content of the proposition. ∎

4.17. Example. Consider sequences of C^*-algebras (A_n), (B_n) and (C_n) such that we have morphisms $\varphi_n : A_{n+1} \to A_n$, $\psi_n : B_{n+1} \to B_n$ and $\eta_n : C_{n+1} \to C_n$ for each n. Assume furthermore that we have morphisms $\alpha_n : C_n \to A_n$ and $\beta_n : C_n \to B_n$, such that $\varphi_n \circ \alpha_n = \alpha_{n+1} \circ \eta_n$ and $\psi_n \circ \beta_n = \psi_{n+1} \circ \eta_n$ for each n. There is than a natural morphism ρ from the amalgamated free product of the inverse limits into inverse limit of the amalgamated free products, i.e.,

$$\varprojlim A_n \bigstar_{\varprojlim C_n} \varprojlim B_n \xrightarrow{\rho} \varprojlim (A_n \bigstar_{C_n} B_n).$$

This follows by straightforward manipulations with the definitions, as in 4.9, 4.12, and 4.16. However, the morphism ρ shows no inclination to be an isomorphism. And category theory is of no help here, since the functor $\varprojlim$ need not preserve pushouts, cf. Example 2.11.D. For the same reasons the results in Propositions 4.7 and 4.9 are the more interesting, since they rely on specific C^*-algebra properties.

A simple counterexample to the injectivity of ρ is obtained by taking $A_n = C([0, n])$, $B_n = 0$ and $C_n = C_0(\mathbb{R}_+)$, with $\alpha_n(f) = f \mid [0, n]$ for each f in C_n. It follows that $X_n = A_n \bigstar_{C_n} B_n = 0$ for every n. Taking inverse limits we get $\varprojlim A_n = C_b([0, \infty]) = C(\beta\mathbb{R}_+)$, $\varprojlim B_n = 0$ and $\varprojlim C_n = C_0(\mathbb{R}_+)$. Moreover, the connecting morphism $\alpha : C_0(\mathbb{R}_+) \to C(\beta\mathbb{R}_+)$ is simply the embedding map. Consequently, the amalgamated free product of the inverse limits equals $C(\beta\mathbb{R}_+ \backslash \mathbb{R}_+)$, cf. Example 3.3.B, whereas the inverse limit of the amalgamated free products X_n equals 0.

5. IDEAL AND HEREDITARY PUSHOUTS

5.1. Proposition. *If we have a pushout diagram of C^*-algebras in concatenated from*

$$\begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B_0 \arrow[r, hook] \arrow[d, "\gamma"] & B \arrow[d, "\gamma"] \\ A \arrow[r, "\delta"] & X_0 \arrow[r, hook] & X \end{tikzcd}$$

in which B_0 and X_0 are hereditary C^*-subalgebras of B and X, respectively, such that $\beta(C) \subset B_0$, $\gamma(B_0) \subset X_0$, and $\delta(A) \subset X_0$, then both the left and the right square in the diagram are pushouts, provided that X_0 is generated by $\delta(A) \cup \gamma(B_0)$. The last condition is satisfied if α is proper, and X_0 is the hereditary C^*-subalgebra of X generated by $\delta(A) \cup \gamma(B_0)$.

Proof. To prove that the left square is a pushout, consider a coherent pair of morphisms $\varphi : A \to Y$ and $\psi : B_0 \to Y$ (i.e. $\varphi \circ \alpha = \psi \circ \beta$). Assuming that $Y \subset \mathbb{B}(\mathfrak{H})$ we can find a representation $\tilde{\psi} : B \to \mathbb{B}(\mathfrak{H} \oplus \mathfrak{K})$ such that $\tilde{\psi} \mid B_0 = \psi \oplus 0$. Defining $\tilde{\varphi} = \varphi \oplus 0$ it follows that $(\tilde{\varphi}, \tilde{\psi})$ is a coherent pair of morphisms of A and B into some C^*-algebra $Z \subset \mathbb{B}(\mathfrak{H} \oplus \mathfrak{K})$. By assumption there is therefore a morphism $\tilde{\sigma} : X \to Z$ such that $\tilde{\sigma} \circ \delta = \tilde{\varphi}$ and $\tilde{\sigma} \circ \gamma = \tilde{\psi}$. Put $\sigma = \tilde{\sigma} \mid X_0$ and note that $\sigma \circ \delta = \varphi$ and $\sigma \circ \gamma \mid B_0 = \psi$. Since X_0 is generated by $\delta(A) \cup \gamma(B_0)$ it follows that $X_0 = A \bigstar_C B_0$.

By Proposition 2.9 we see that also the right square in the diagram is a pushout.

If α is proper and X_0 is hereditarily generated by $\delta(A) \cup \gamma(B_0)$ then $\gamma \mid B_0$ is proper by Lemma 4.6. Each element x_0 in X_0 therefore has the form $\gamma(b_0)\, x \gamma(b_0)$ for some x in X and b_0 in B_0. However, x can be approximated by sums of products $\gamma(b_1)\, \delta(a_1) \cdots \gamma(b_n)\, \delta(a_n)$. Writing each a_k as $\alpha(c_k)\, a'_k \alpha(c_k)$ for some c_k in C, and using that B_0 is hereditary in B and contains $\beta(C)$, it follows that x_0 lies in the C^*-algebra generated by $\delta(A) \cup \gamma(B_0)$, as desired. ∎

5.2. Lemma (Cf. [33, 5.1.2]). *Consider a commutative diagram of C^*-algebras*

$$\begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\ A \arrow[r, "\delta"] & D \end{tikzcd}$$

Assume that α is a proper morphism and that D is generated by $\delta(A) \cup \gamma(B)$. Then $\delta(A)$ is a hereditary C^*-subalgebra of D, provided that $\beta(C)$ is hereditary in B. Moreover, $\delta(A)$ is an ideal in D, provided that $\beta(C)$ is an ideal in B. In any of these cases $\delta(A) \cap \gamma(B) = \gamma(\beta(C))$ $(= \delta(\alpha(C)))$ and $\mathrm{Id}(\delta(A)) \cap \gamma(B) = \gamma(\mathrm{Id}(\beta(C)))$.

Proof. Since α is proper we have $A = \alpha(C)\, A \alpha(C)$, cf. [40, Sect. 4], and thus

$$\gamma(B)\, \delta(A) = \gamma(B)\, \delta(\alpha(C)\, A) = \gamma(B\beta(C))\, \delta(A). \tag{$*$}$$

Assuming that $\beta(C)$ is hereditary in B it follows that

$$\begin{aligned} \delta(A)\, \delta(B)\, \delta(A) &= \delta(A)\, \gamma(\beta(C)\, B\beta(C))\, \delta(A) \subset \delta(A)\, \gamma(\beta(C))\, \delta(A) \\ &= \delta(A\alpha(C)\, A) = \delta(A). \end{aligned}$$

Consequently, $\delta(A)\,x\delta(A)\subset\delta(A)$ for every x in the C^*-subalgebra of D generated by $\delta(A)\cup\gamma(B)$. That algebra being D by assumption, $\delta(A)$ is hereditary in D.

Assuming now that $\beta(C)$ is an ideal in B we have by $(*)$

$$\gamma(B)\,\delta(A)=\delta(B\beta(C))\,\delta(A)=\gamma(\beta(C))\,\delta(A)\,\delta(\alpha(C)\,A)=\delta(A).$$

It follows that $\delta(A)\,x+x\delta(A)\subset\delta(A)$ for every x in the C^*-subalgebra of D generated by $\delta(A)\cup\gamma(B)$. That algebra being D by assumption, $\delta(A)$ is an ideal in D.

If $\delta(A)$ is hereditary in D and $\gamma(b)$ is an element in $\delta(A)\cap\gamma(B)$, then

$$\gamma(\beta(u_\lambda)\,b\beta(u_\lambda))=\delta(\alpha(u_\lambda))\,\gamma(b)(\delta(\alpha(u_\lambda)))\to\gamma(b)$$

for any approximate unit (u_λ) for C, since α is proper. Since $\beta(C)$ is hereditary in B it follows that $\operatorname{dist}(\gamma(b),\gamma(\beta(C))$ can be made arbitrarily small, whence $\gamma(b)\in\gamma(\beta(C))$, as claimed.

To prove the last assertion put $I=\operatorname{Id}(\beta(C))$ and $J=\operatorname{Id}(\delta(B))$. From $(*)$ we see that $\gamma(B)\,\delta(A)\subset\gamma(I)\,\delta(A)$, which shows that J is hereditarily generated by $\gamma(I)$. Since moreover $\gamma(B\beta(C)\,B)\subset D\delta(A)\,D$, we conclude that $\gamma(I)\subset J\cap\gamma(B)$ and that $J=C^*(\gamma(I)\cup\delta(A))$. Consider now an element x in $\gamma(B)$ such that $x\in J$. If (u_λ) is an approximate unit for I, then $\gamma((u_\lambda))$ is an approximate unit for $\gamma(I)$ and for $\delta(A)=\gamma(\beta(C))\,\delta(A)$, hence also for J—the algebra they generate. Thus $x-\gamma(u_\lambda)\,x\to 0$, which implies that $x\in\gamma(I)$, as desired. ∎

Note that in the case where α is proper, D is C^*-generated by $\delta(A)\cup\gamma(B)$ and $\beta(C)$ is an ideal in B, the net result is that $D=\delta(A)+\gamma(B)$, so that the natural morphism

$$B/\beta(C)\to D/\delta(A)$$

is a surjective morphism whose kernel is $(\ker\gamma+\beta(C))/\beta(C)$.

5.3. Theorem. *Consider a commutative diagram of C^*-algebras*

$$\begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\ A \arrow[r, "\delta"] & X \end{tikzcd}$$

in which α is a proper morphism and $\beta(C)$ is an ideal in B. Such a diagram is a pushout if and only if

(i) *X is generated (as a C^*-algebra) by $\delta(A) \cup \gamma(B)$,*

(ii) $\ker \delta$ *is generated as an ideal by* $\alpha(\ker \beta)$,

(iii) $\ker \gamma \subset \beta(C)$.

In that case $\delta(A)$ is an ideal in X and $\ker \gamma = \beta(\alpha^{-1}(\ker \delta))$. *Moreover, we have a commutative diagram in which $\tilde{\gamma}$ is an isomorphism*:

$$\begin{tikzcd}
\ker \beta \arrow[r, hook, "\iota"'] \arrow[d, "\alpha"] & C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & \beta(C) \arrow[r, hook, "\iota"'] \arrow[d, "\gamma"] & B \arrow[r] \arrow[d, "\gamma"] & B/\beta(C) \arrow[d, "\tilde{\gamma}"] \\
\ker \delta \arrow[r, hook, "\iota"] & A \arrow[r, "\delta"] & \delta(A) \arrow[r, hook, "\iota"] & X \arrow[r] & X/\delta(A)
\end{tikzcd}$$

Proof. If the three conditions are satisfied, the existence and commutativity of the larger diagram follow from the second part of Lemma 5.2, with $\tilde{\gamma}$ a surjection. To prove that $\tilde{\gamma}$ is also injective, assume that $\tilde{\gamma}(b + \beta(C)) = 0$ for some b in B. Thus $\gamma(b) \in \delta(A)$. By the third part of Lemma 5.2 this implies that $\gamma(b) = \gamma(\beta(c))$ for some c in C, so $b \in \beta(C) + \ker \gamma = \beta(C)$ by condition (iii), and thus $b + \beta(C) = 0$ in $B/\beta(C)$, as desired. Moreover,

$$\ker \gamma = \beta(\ker (\gamma \circ \beta)) = \beta(\ker (\delta \circ \alpha)) = \beta(\alpha^{-1}(\ker \delta)).$$

Since both morphisms $\gamma : \beta(C) \to \delta(A)$ and $\gamma : B \to X$ are proper, we can apply Theorem 2.4 to the right half of the large diagram to conclude that the third square (from left) is a pushout. However, we can also apply Theorem 2.5 to the left half of the diagram (courtesy of condition (ii)) to conclude that the second square is a pushout. An easy diagram chase (Proposition 2.7) shows that the concatenation of two pushout diagrams is again a pushout, giving us the desired conclusion.

Conversely, if the original diagram is a pushout, then evidently condition (i) is satisfied. Thus Lemma 5.2 applies to give the larger diagram with $X = \delta(A) + \gamma(B)$ and $\tilde{\gamma}$ a surjective morphism. Observe now that the zero morphism on A, coupled with the quotient map $B \to B/\beta(C)$, is a coherent pair, and so, by assumption, must factor through X. Consequently $\ker \gamma \subset \beta(C)$, so (iii) is satisfied and $\tilde{\gamma}$ is an isomorphism.

By Proposition 5.1 both of the two middle squares in the diagram are pushouts. Applying Theorem 2.5 to the left one it follows that $\ker \delta$ is generated as an ideal by $\alpha(\ker \beta)$. Thus also condition (ii) is satisfied. ∎

5.4. Example. The preceding result is rather satisfying, in that it characterizes those "ideal" pushout diagrams that can be built from morphisms $\alpha : C \to A$ and $\beta : C \to B$, where α is proper and $\beta(C)$ is an ideal in B. On the other hand it also shows that we get nothing essentially new: Each such diagram is a concatenation of two pushout diagrams, one where β is surjective (covered by Theorem 2.5) and one where β is injective (covered by Theorem 2.4).

The condition that α be a proper morphism can not be deleted in Theorem 5.3. This is seen from the following commutative diagram, in which the upper left corner is assumed to be a pushout:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\beta"'] \arrow[d, "\alpha"] & \mathbb{C}^2 \arrow[r, "\pi"'] \arrow[d, "\gamma"] & \mathbb{C} \arrow[d, equal] \\
\mathbb{M}_2 \arrow[r, "\delta"] \arrow[d] & X \arrow[r, "\rho"] \arrow[d] & \mathbb{C} \arrow[d, equal] \\
\ker\rho \arrow[r] & X \arrow[r, "\rho"] & \mathbb{C}
\end{tikzcd}$$

Here $\alpha(\mathbf{1}) = e_{11}$ and $\beta(\mathbf{1}) = (\mathbf{1}, 0)$. It follows that X is the universal C^*-algebra generated by three elements p, q and x, such that p and q are projections orthogonal to one another, and $x^*x = p$ with $x^2 = 0$. The algebra X has irreducible representations of arbitrary high order, since in $\mathbb{M}_n$ we can choose the generators $p = e_{11}$, $x = e_{21}$ and $q = (1/n-1)\sum_{i,\, j=2}^{n} e_{ij}$. The coherent pair $(0, \pi)$ on $\mathbb{M}_2$ and $\mathbb{C}^2$ defines the morphism $\rho \colon X \to \mathbb{C}$, and we see that $\mathbb{M}_2$ is properly contained in $\ker\rho$, since $X \neq \mathbb{M}_2 \oplus \mathbb{C}$.

In the language of Theorem 5.3 we have constructed a pushout diagram in which $\beta(C)$ is an ideal in B without $\delta(A)$ being an ideal in X. Replacing $\mathbb{M}_2$ wit $\ker\rho$ (i.e. deleting the middle row in the diagram above) we have an example of a commutative diagram, where the algebras satisfy conditions (i), (ii) and (iii) without the diagram being a pushout. Note that although the morphism α is not proper, it does satisfy that $\alpha(C)$ generates A as an ideal. This condition is therefore not enough to establish Theorem 5.3 (or Theorem 2.4).

5.5. THEOREM. *Consider the two commutative diagrams of C^*-algebras*

$$\begin{tikzcd}
C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\
A \arrow[r, "\delta"] & X
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
Y \otimes_{\max} C \arrow[r, "\bar\beta"'] \arrow[d, "\bar\alpha"] & Y \otimes_{\max} B \arrow[d, "\bar\gamma"] \\
Y \otimes_{\max} A \arrow[r, "\bar\delta"] & Y \otimes_{\max} X
\end{tikzcd}$$

where $\bar\alpha = \iota \otimes \alpha$, and similarly for $\bar\beta$, $\bar\gamma$, and $\bar\delta$. If α is a proper morphism and the first diagram is a pushout, so is the second.

Proof. Given a coherent pair of morphisms $\varphi: Y \otimes_{\max} A \to Z$ and $\psi: Y \otimes_{\max} B \to Z$ (i.e. $\varphi \circ \bar{\alpha} = \psi \circ \bar{\beta}$) we may assume that Z is generated by the images of φ and ψ. It follows from Lemma 4.6 that ψ is proper (since $\bar{\alpha}$ is proper because α is). Thus ψ extends uniquely to a morphism $\bar{\psi}: M(Y \otimes_{\max} B) \to M(Z)$, cf. 7.1. Identifying Y and B with $Y \otimes \mathbf{1}$ and $\mathbf{1} \otimes B$ in $M(Y \otimes_{\max} B)$ we define morphisms $\rho: Y \to M(Z)$ and $\psi_0: B \to M(Z)$ by restriction. Thus, $\psi = \rho \otimes \psi_0$. Assuming that $Z \subset \mathbb{B}(\mathfrak{H})$ is a faithful and nondegenerate representation it follows from the properness of ψ that both ρ and ψ_0 are nondegenerate representations.

By definition of the maximal tensor product there is a pair of commuting representations $(\eta, \mathfrak{H})$ and $(\varphi_0, \mathfrak{H})$ of Y and A, respectively, such that $\varphi = \eta \otimes \varphi_0$. For each a in A, b in B, c in C and y in Y we therefore have

$$\eta(y)\, \varphi_0(\alpha(c)) = \varphi(y \otimes \alpha(c)) = \psi(y \otimes \beta(c)) = \rho(y)\, \psi_0(\alpha(c)).$$

Let now (u_λ) be an approximate unit for Y. Then strong-lim $\eta(u_\lambda)$ is a projection p in the commutant of $\varphi_0(A)$, and strong-lim $\rho(u_\lambda) = \mathbf{1}$, since ρ is nondegenerate. It follows from above that $p\varphi_0(\alpha(c)) = \psi_0(\alpha(c))$ for every c in C. This means that $p\varphi_0$ and ψ_0 is a coherent pair of morphisms of A and B. Consequently there is a morphism $\sigma: X \to \mathbb{B}(\mathfrak{H})$ such that $\sigma \circ \delta = p\varphi_0$ and $\sigma \circ \gamma = \psi_0$.

Evidently,

$$\rho(y)\, \sigma(\gamma(b)) = \rho(y)\, \psi_0(b) = \psi(y \otimes b),$$

which shows that $\rho(y)$ commutes with $\sigma(\gamma(B))$. Moreover,

$$\begin{aligned}\rho(y)\, \sigma(\delta(\alpha(c)\, a)) &= \rho(y)\, p\varphi_0(\alpha(c)\, a) = \rho(y)\, p\varphi_0(\alpha(c))\, \varphi_0(a)\\ &= \rho(y)\, \psi_0(\beta(c))\, \varphi_0(a) = \psi(y \otimes \beta(c))\, \varphi_0(a)\\ &= \varphi(y \otimes \alpha(c))\, \varphi_0(a) = \varphi(y \otimes \alpha(c)\, a).\end{aligned}$$

From this we see that $\rho(y)$ also commutes with $\sigma(\delta(\alpha(C)\, A\alpha(C)))$. But since α is proper this set equals $\sigma(\delta(A))$. As $\delta(A) \cup \gamma(B)$ generates X it follows that $\rho(Y)$ commutes with $\sigma(X)$. By definition of the maximal tensor product there is therefore a unique morphism $\bar{\sigma}: Y \otimes_{\max} X \to \mathbb{B}(\mathfrak{H})$ determined by $\bar{\sigma} = \rho \otimes \sigma$.

To show that $\bar{\sigma}$ factors correctly we compute

$$\begin{aligned}\bar{\sigma}(\bar{\gamma}(y \otimes b)) &= \bar{\sigma}(y \otimes \gamma(b)) = \rho(y) \otimes \sigma(\gamma(b))\\ &= \rho(y)\, \psi_0(b) = \psi(y \otimes b),\end{aligned}$$

which shows that $\bar{\sigma} \circ \bar{\gamma} = \psi$. Moreover, as above,

$$\begin{aligned} \bar{\sigma}(\bar{\delta}(y \otimes \alpha(c)\, a)) &= \bar{\sigma}(y \otimes \delta(\alpha(c)\, a)) = \rho(y)\, \sigma(\delta(\alpha(c)\, a)) \\ &= \varphi(y \otimes \alpha(c)\, a), \end{aligned}$$

and since $A = \alpha(C)\, A$ this means that also $\bar{\sigma} \circ \bar{\delta} = \varphi$, as desired. ∎

5.6. *Remarks.* Note that the formula

$$Y \otimes_{\max} (A \star_C B) = (Y \otimes_{\max} A) \star_{Y \otimes_{\max} C} (Y \otimes_{\max} B)$$

established in Theorem 5.5 does not hold unrestrictedly. For example, it is evidently false if we take $C = 0$. (But then none of the morphisms α and β are proper.) The same restrictions apply to Proposition 4.7 and Theorem 6.3.

As a neat application of Theorem 5.5 consider the unital pushout diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C}^2 \arrow[d] \\ \mathbb{C}^2 \arrow[r] & \mathbb{C}^2 \star_{\mathbb{C}} \mathbb{C}^2 \end{tikzcd}$$

It is well known that

$$\mathbb{C}^2 \star_{\mathbb{C}} \mathbb{C}^2 = \{ f \in C([0, 1], \mathbb{M}_2) \mid f(0) \in \mathbb{C}^2,\ f(1) \in \mathbb{C}^2 \},$$

which describes the universal C^*-algebra generated by two symmetries. Here we have identified $\mathbb{C}^2$ with the diagonal matrices in $\mathbb{M}_2$. Tensoring the diagram with $C_0(]\,0, 1])$, i.e. taking the cones over the algebras involved, cf. 10.1, we obtain a new pushout diagram. Evidently $\mathbf{C}(\mathbb{C}^2 \star_{\mathbb{C}} \mathbb{C}^2)$ consists of those functions f in $C([0, 1]^2, \mathbb{M}^2)$ such that

$$f(x, 0) = 0, \qquad f(0, t) \in \mathbb{C}^2 \qquad \text{and} \qquad f(1, t) \in \mathbb{C}^2$$

for all s, t in $[0, 1]$. But since

$$\mathbf{C}(\mathbb{C}^2 \star_{\mathbb{C}} \mathbb{C}^2) = \mathbf{C}\mathbb{C}^2 \star_{\mathbf{C}\mathbb{C}} \mathbf{C}\mathbb{C}^2,$$

we see that this algebra of matrix functions describes the universal C^*-algebra generated by two selfadjoint contractions x and y satisfying the relation $x^2 = y^2$.

This algebra is a prime example of an *NCCW* complex of topological dimension 2, cf. 11.2. Being a cone its K-theory vanishes, so the algebra can be added to the list in [21, Sect. 8.2] of algebras with weakly stable relations.

5.7. LEMMA. *Consider C^*-algebras A and B with C^*-subalgebras $A_0 \subset A$ and $B_0 \subset B$, and assume that A_0 and B_0 generate A and B as closed ideals. Then $A_0 \odot B_0$ generates $A \otimes_\alpha B$ as a closed ideal for any C^*-tensor product $\otimes_\alpha$.*

Proof. Suppose that

$$A_0 \odot B_0 \subset I \subset A \otimes_\alpha B$$

for some closed ideal I, and take any representation $(\pi, \mathfrak{H})$ of $A \otimes_\alpha B$ such that $\pi(I) = 0$. There is a unique pair $(\rho, \mathfrak{H})$ and $(\sigma, \mathfrak{H})$ of commuting representations of A and B, respectively, such that

$$\pi(a \otimes b) = \rho(a)\, \sigma(b), \qquad a \in A, \quad b \in B.$$

Since A_0 generates A as an ideal, the linear span of products in AA_0A is dense in A. Likewise for B. But

$$\begin{aligned} \rho(AA_0A)\, \sigma(BB_0B) &= \rho(A)\, \rho(A_0)\, \rho(A)\, \sigma(B)\, \sigma(B_0)\, \sigma(B) \\ &= \rho(A)\, \sigma(B)\, \rho(A_0)\, \sigma(B_0)\, \rho(A)\, \sigma(B) \\ &= \pi(A \odot B)\, \pi(A_0 \odot B_0)\, \pi(A \odot B) = 0. \end{aligned}$$

Consequently $\pi(A \odot B) = \rho(A)\, \sigma(B) = 0$, so that $\pi = 0$. Since $(\pi, \mathfrak{H})$ is arbitrary it follows that $I = A \otimes_\alpha B$, as claimed. ∎

5.8. THEOREM. *If we have two pushout diagrams*

$$\begin{array}{ccc} C_1 & \xrightarrow{\beta_1} & B_1 \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \gamma_1} \\ A_1 & \xrightarrow{\delta_1} & X_1 \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} C_2 & \xrightarrow{\beta_2} & B_2 \\ {\scriptstyle \alpha_2}\downarrow & & \downarrow{\scriptstyle \gamma_2} \\ A_2 & \xrightarrow{\delta_2} & X_2 \end{array}$$

in which α_i is proper and β_i is surjective for $i = 1, 2$, then with $\bar{\alpha} = \alpha_1 \otimes \alpha_2$, and likewise for $\bar{\beta}$, $\bar{\gamma}$, and $\bar{\delta}$, we obtain a new pushout diagram of the same type,

$$\begin{array}{ccc} C_1 \otimes C_2 & \xrightarrow{\bar{\beta}} & B_1 \otimes B_2 \\ {\scriptstyle \bar{\alpha}}\downarrow & & \downarrow{\scriptstyle \bar{\gamma}} \\ A_1 \otimes A_2 & \xrightarrow{\bar{\delta}} & X_1 \otimes X_2 \end{array}$$

Here either $\otimes$ is the maximal tensor product, or $\otimes$ denotes the minimal tensor product, in which case we must further assume that C_i and A_i are exact for $i = 1, 2$.

Proof. Consider the diagram

$$\begin{tikzcd}
\ker \beta_1 \otimes C_2 + C_1 \otimes \ker \beta_2 \arrow[r] \arrow[d, "\alpha"] & C_1 \otimes C_2 \arrow[r, "\bar{\beta}"'] \arrow[d, "\bar{\alpha}"] & B_1 \otimes B_2 \arrow[d, "\bar{\gamma}"] \\
\ker \delta_1 \otimes A_2 + A_1 \otimes \ker \delta_2 \arrow[r] & A_1 \otimes A_2 \arrow[r, "\bar{\delta}"] & X_1 \otimes X_2
\end{tikzcd}$$

By $(**)$ in Lemma 3.7 both rows are extensions. This uses the fact that δ_i is surjective for $i=1, 2$ (as $X_i/\delta_i(A_i) = B_i/\beta_i(C_i)$ by Theorem 5.3).

Evidently $\bar{\delta} \circ \bar{\alpha} = \bar{\gamma} \circ \bar{\beta}$, and we define α as the restriction of $\bar{\alpha}$ to the kernels of $\bar{\beta}$ and $\bar{\delta}$ to obtain a commutative diagram.

By condition (ii) in Theorem 5.3 the algebra $\ker \delta_1$ is generated as an ideal by $\alpha_1(\ker \beta_1)$, and thus by Lemma 5.7 (using that $\alpha_2(C_2)$ generates A_2 as a hereditary C^*-algebra) it follows that the C^*-algebra

$$\bar{\alpha}(\ker \beta_1 \otimes C_2) = (\alpha_1(\ker \beta_1) \odot \alpha_2(C_2))^{=}$$

generates $\ker \delta_1 \otimes A_2$ as an ideal. By symmetry $\bar{\alpha}(C_1 \otimes \ker \beta_1)$ generates $A_1 \otimes \ker \delta_2$ as an ideal, and by addition we see that $\bar{\alpha}(\ker \bar{\beta})$ generates $\ker \bar{\delta}$ as an ideal. But then the right square in the diagram is a pushout by Theorem 2.5 (or Theorem 5.3), as desired. ∎

5.9. Theorem. *If we have two pushout diagrams*

$$\begin{tikzcd}
C_1 \arrow[r, "\beta_1"'] \arrow[d, "\alpha_1"] & B_1 \arrow[d, "\gamma_1"] \\
A_1 \arrow[r, "\delta_1"] & X_1
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
C_2 \arrow[r, "\beta_2"'] \arrow[d, "\alpha_2"] & B_2 \arrow[d, "\gamma_2"] \\
A_2 \arrow[r, "\delta_2"] & X_2
\end{tikzcd}$$

in which α_i is proper and β_i is an injection of C_i as an ideal in B_i for $i=1, 2$, then we obtain a new pushout diagram of the same type

$$\begin{tikzcd}
\beta_1(C_1) \otimes B_2 + B_1 \otimes \beta_2(C_2) \arrow[r, "\bar{\beta}"'] \arrow[d, "\bar{\alpha}"] & B_1 \otimes B_2 \arrow[d, "\bar{\gamma}"] \\
\delta_1(A_1) \otimes X_2 + X_1 \otimes \delta_2(A_2) \arrow[r, "\bar{\delta}"] & X_1 \otimes X_2
\end{tikzcd}$$

Here either $\otimes$ denotes the maximal tensor product, or $\otimes$ denotes the minimal tensor product, in which case we must further assume that B_i and X_i are exact for $i=1, 2$.

Proof. It follows from Theorem 5.3 that $\delta_i(A_i)$ is an ideal in X_i and that $\ker \delta_i = 0$ (because it is generated by $\alpha_i(\ker \beta_i)$). Suppressing β_i and δ_i, i.e., regarding C_i and A_i as ideals in B_i and X_i, respectively, we see, again from Theorem 5.3, that

$$B_i/C_i = X_i/A_i = Q_i, \qquad i = 1, 2.$$

From $(**)$ in Lemma 3.7 we obtain a diagram

$$\begin{tikzcd}
0 \arrow[r] & C_1 \otimes B_2 + B_1 \otimes C_2 \arrow[r] \arrow[d, "\bar{\alpha}"] & B_1 \otimes B_2 \arrow[r] \arrow[d, "\bar{\gamma}"] & Q_1 \otimes Q_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_1 \otimes X_2 + X_1 \otimes A_2 \arrow[r] & X_1 \otimes X_2 \arrow[r] & Q_1 \otimes Q_2 \arrow[r] & 0
\end{tikzcd}$$

in which both rows are extensions. Here $\bar{\gamma} = \gamma_1 \otimes \gamma_2$ and $\bar{\alpha}$ is the restriction of $\bar{\gamma}$. Since $\alpha_i(C_i) \subset A_i$ we actually have

$$\bar{\alpha}(C_1 \otimes B_2 + B_1 \otimes C_2) \subset A_1 \otimes X_2 + X_1 \otimes A_2,$$

and thus the diagram commutes, the morphism between the quotients $Q_1 \otimes Q_2$ being an isomorphism.

Since α_i is proper, so is γ_i by Lemma 4.6. Let (u^i_λ) and (v^i_λ) be approximate units for C_i and B_i, respectively, and put

$$x_\lambda = u^1_\lambda \otimes v^2_\lambda + v^1_\lambda \otimes u^2_\lambda.$$

Choose a net (ε_λ) in $\mathbb{R}_+$ converging to zero and define

$$w_\lambda = \bar{\alpha}(x_\lambda(\varepsilon_\lambda + x_\lambda)^{-1}).$$

To prove that (w_λ) is an approximate unit for $A_1 \otimes X_2 + X_1 \otimes A_2$ it suffices to check this on simple tensors of the form $a_1 \otimes x_2$ or $x_1 \otimes a_2$. However, since the function $t \to t(\varepsilon + t)^{-1}$ is operator monotone on $\mathbb{R}_+$ and $1 - t(\varepsilon + t)^{-1} \leqslant 1 - t + \varepsilon$ for $0 \leqslant t \leqslant 1$, we can estimate

$$\begin{aligned}
&\|(1 - w_\lambda)(a_1 \otimes x_2)\|^2 \\
&\quad \leqslant \|(a_1^* \otimes x_2^*)(1 - w_\lambda)(a_1 \otimes x_2)\| \\
&\quad \leqslant \|(a_1^* \otimes x_2^*)(1 - \bar{\alpha}(u^1_\lambda \otimes v^2_\lambda(\varepsilon_\lambda + u^1_\lambda \otimes v^2_\lambda)^{-1}))(a_1 \otimes x_2)\| \\
&\quad \leqslant \|(a_1^* \otimes x_2^*)(1 - \bar{\alpha}(u^1_\lambda \otimes v^2_\lambda) + \varepsilon_\lambda)(a_1 \otimes x_2)\| \\
&\quad \leqslant \varepsilon_\lambda \, \|a_1 \otimes x_2\|^2 + \|a_1 \otimes x_2\| \; \|a_1 \otimes x_2 - \alpha_1(u^1_\lambda)\, a_1 \otimes \gamma_2(v^2_\lambda)\, x_2\|;
\end{aligned}$$

and this tends to zero as $\lambda \to \infty$.

We have shown that $\bar{\alpha}$ is a proper morphism, and it follows from Theorem 2.4 (or Theorem 5.3) that the left square in the large diagram is a pushout, as desired. ∎

5.10. *Remark*. Both Theorem 5.5 and 5.8 may be seen as special cases of a general process of taking two pushout diagrams and tensoring them together in the four corners. Unfortunately this process will not in general produce a pushout diagram, not even with diagrams of the ideal type described in Theorem 5.3. A particularly glaring counterexample is obtained by taking arbitrary unital, nuclear C^*-algebras X and Y and consider the three commutative diagrams

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \otimes X \end{array}$$

The first two are pushouts, cf. Example 3.3.B, but the third—their "tensor product"—is not. The algebra $Y \otimes X$ is the universal solution to pairs of *commuting* morphisms of X and Y, and this is a much smaller object than $X \bigstar_{\mathbb{C}} Y$.

5.11. THEOREM. *Consider a commutative diagram of C^*-algebras*

$$\begin{array}{ccc} C & \xrightarrow[\beta]{} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ A & \xrightarrow{\delta} & X \end{array}$$

in which α is a proper morphism and $\beta(C)$ is hereditary in B. Such a diagram is a pushout if and only if

(i) *X is generated (as a C^*-algebra) by $\delta(A) \cup \gamma(B)$,*

(ii) *$\ker \delta$ is generated as an ideal by $\alpha(\ker \beta)$,*

(iii) *$\ker \gamma \subset \mathrm{Id}(\beta(C))$.*

In that case $\delta(A)$ is hereditary in X and we have the commutative diagram below, in which all three middle squares are pushouts. Moreover, γ and both of its restrictions are proper morphisms and $\tilde{\gamma}$ is an isomorphism:

$$\begin{array}{ccccccccccc} \ker \beta & \hookrightarrow & C & \xrightarrow[\beta]{} & \beta(C) & \hookrightarrow & \mathrm{Id}(\beta(C)) & \hookrightarrow & B & \longrightarrow & B/\mathrm{Id}(\beta(C)) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \tilde{\gamma}} \\ \ker \delta & \hookrightarrow & A & \xrightarrow{\delta} & \delta(A) & \hookrightarrow & \mathrm{Id}(\delta(A)) & \hookrightarrow & X & \longrightarrow & X/\mathrm{Id}(\delta(A)) \end{array}$$

Proof. If the three conditions are satisfied, the existence and commutativity of the larger diagram follow from Lemma 5.2, with $\delta(A)$ hereditary in X. Put $I = \mathrm{Id}(\beta(C))$ and $J = \mathrm{Id}(\delta(A))$. Since α is proper, we know from Lemma 5.2 that $\gamma(I) = J \cap \gamma(B)$ and that $J = C^*(\gamma(I) \cup \delta(A))$. Also, $\gamma : \beta(C) \to \delta(A)$ and $\gamma : I \to J$ are both proper morphisms by Lemma 4.6.

Since $X = J + \gamma(B)$ it follows that $\tilde{\gamma}$ is a welldefined surjective morphism. To prove that it is injective consider an element $b + I$ in B/I such that $\tilde{\gamma}(b + I) = 0$ in X/J. This means that $\gamma(b) \in J$, whence $b \in \ker \gamma + I \subset I$ by Lemma 5.2 in conjunction with condition (iii). Consequently $\tilde{\gamma}$ is injective, as desired.

We claim that the three middle squares—hence also the original concatenated diagram—are all pushouts. For the first this follows from condition (ii) and Theorem 2.5. For the third it follows from Theorem 2.4, since $\tilde{\gamma}$ is an isomorphism. For the middle square we let $Y = \delta(A) \bigstar_{\beta(C)} I$, and consider the morphism $\sigma : Y \to J$ induced by the coherent pair (ι, γ) (where ι denotes the embedding of $\delta(A)$ in J. We proved above that J was generated by $\delta(A) \cup \gamma(I)$, which shows that σ is surjective. Moreover, since ι is injective we must have $\ker \sigma \cap \delta(A) = \{0\}$. As $\delta(A)$ is a full, hereditary C^*-subalgebra of J (cf. [6,1.2]), this implies that $\ker \sigma \cap J = \{0\}$; so σ is injective and thus an isomorphism, as claimed.

Conversely, if we have a pushout diagram, the existence of the large commutative diagram is a straightforward computation, and it follows from Lemma 5.2 that $\delta(A)$ is a hereditary algebra in X, whence by Proposition 5.1 the second and the concatenated third and fourth squares are pushout diagrams. Moreover, $\ker \delta = \mathrm{Id}(\alpha(\ker \beta))$ by Theorem 2.5. Since $I = \mathrm{Id}(\beta(C))$ and $J = \mathrm{Id}(\delta(A))$ are hereditary in B and X, respectively, and α is proper, we find as above that J is C^*-generated by $\delta(A) \cup \gamma(I)$. By Proposition 5.1 this implies that both the third and the fourth square are pushout diagrams. Moreover, $\gamma \mid \beta(C)$ and $\gamma \mid \mathrm{Id}(\beta(C))$ are both proper morphisms and $\tilde{\gamma}$ is an isomorphism by Theorem 5.3. ∎

5.12. *Remarks.* Using Theorem 5.3 we can now characterize each amalgamated free product arising from a hereditary pushout as in Theorem 5.11 in terms of ideals and quotients of A, B, and C, except for the middle square, which describes a pushout diagram

$$\begin{array}{ccc} C & \xrightarrow{\beta} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} \\ A & \xrightarrow{\delta} & X \end{array}$$

in which α and γ are proper, β and δ are injective, and C and A are full, hereditary C^*-subalgebras of B and X, respectively, cf. [6, 1.2]. However, if B and C, hence also A and X, are σ-unital this implies that X is stably isomorphic to A (and B to C) by [6, 2.8]. In particular, X is Rieffel–Morita equivalent to A by [9]. Inserting this information in the large diagram from 5.11 one may now characterize X up to Rieffel–Morita equivalence using only ideals and quotients of A, B and C.

We shall characterize diagrams as above directly (without passing to Rieffel–Morita equivalence) in the case of corner embeddings, where $B \subset C \otimes \mathbb{K}$ and $\beta = \iota \otimes e_{11}$; see Theorems 10.4 and 10.12 and Corollary 10.13.

Evidently the results in Theorem 5.11 can be used to generalize Theorem 5.9 to the case where the $\beta(C_i)$ are only hereditary subalgebras of the B_i's. The usual problems with addition of hereditary C^*-subalgebras disappear in this case, since the algebras $\beta(C_1) \otimes B_2$ and $B_1 \otimes \beta(C_2)$ have commuting open support projections of the form $p_1 \otimes 1_{B_2}$ and $1_{B_1} \otimes p_2$. The algebras are *q-commuting* in Akemann's sense. The sum is therefore supported by the open projection

$$1 - (1_{B_1} - p_1) \otimes (1_{b_2} - p_2).$$

6. CROSSED PRODUCTS

6.1. *Crossed Products.* A C^*-dynamical system (A, G) consists of a C^*-algebra A together with a strongly continuous action of a locally compact group G as automorphisms of A. The universal object for covariant representations $(\pi, u, \mathfrak{H})$ of (A, G), i.e. representations of A and of G such that

$$u_g \pi(a)\, u_g^* = \pi(g(a)), \qquad a \in A, \quad g \in G,$$

is the crossed product $G \ltimes A$, which may be regarded as a skew tensor product between $C^*(G)$ and A, see [38, Chap. 7]. Indeed, if the action is trivial $G \ltimes A = C^*(G) \otimes_{\max} A$.

In accordance with its universal construction the crossed product preserves exact sequences. Thus if I is a G-invariant ideal in A, so that we obtain C^*-dynamical systems (I, G) and $(A/I, G)$ by restriction and quotient actions, then we have canonically a new extension

$$0 \to G \ltimes I \to G \ltimes A \to G \ltimes (A/I) \to 0,$$

cf. [41, 2.8.2] or [46, Theorem 2.6].

6.2. PROPOSITION. *If we have a pullback or a pushout diagram of C^*-algebras*

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\alpha} & C \end{array} \qquad or \qquad \begin{array}{ccc} C & \xrightarrow{\beta} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} \\ A & \xrightarrow{\delta} & X \end{array}$$

and (A, G), (B, G), and (C, G) are C^*-dynamical systems such that α and β are G-equivariant morphisms, then there is a unique C^*-dynamical system (X, G) for which the morphisms γ and δ are G-equivariant.

Proof. For the pullback diagram this is almost a triviality. If

$$x = (a, b) \in X = A \oplus_C B$$

we define $g(x) = (g(a), g(b))$ to obtain a strongly continuous action of G on X, and evidently this is the only action for which the coordinate projections γ and δ are G-equivariant.

In the pushout situation we use the universal properties: For each g in G consider the coherent pair of morphisms

$$\delta \circ g : A \to X \qquad \text{and} \qquad \gamma \circ g : B \to X,$$

which determines a unique morphism $\sigma_g : X \to X$ such that

$$\delta \circ g = \sigma_g \circ \delta \qquad \text{and} \qquad \gamma \circ g = \sigma_g \circ \gamma.$$

By uniqueness it follows that σ is a representation of G as automorphisms of X, and γ and δ are G-equivariant by construction. Since the functions $g \to \sigma_g(\delta(a))$ and $g \to \sigma_g(\gamma(b))$ are continuous, and X is generated by $\delta(A) \cup \gamma(B)$, it follows that σ is a strongly continuous action on X. ∎

6.3. THEOREM. *If we have a pullback or a pushout diagram of C^*-algebras*

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & B \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\alpha} & C \end{array} \qquad or \qquad \begin{array}{ccc} C & \xrightarrow{\beta} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} \\ A & \xrightarrow{\delta} & X \end{array}$$

and (A, G), (B, G), and (C, G) are C^*-dynamical systems such that α and β are G-equivariant morphisms, then with the actions defined in 6.2 we obtain *new pullback or pushout diagrams, assuming in the last case that* α *is a proper morphism*:

$$\begin{tikzcd} G\ltimes X \arrow[r, "\tilde\gamma"'] \arrow[d, "\tilde\delta"] & G\ltimes B \arrow[d, "\tilde\beta"] \\ G\ltimes A \arrow[r, "\tilde\alpha"] & G\ltimes C \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} G\ltimes C \arrow[r, "\tilde\beta"'] \arrow[d, "\tilde\alpha"] & G\ltimes B \arrow[d, "\tilde\gamma"] \\ G\ltimes A \arrow[r, "\tilde\delta"] & G\ltimes X \end{tikzcd}$$

Proof. The existence and commutativity of the diagrams for the crossed products follow from simple covariance principles. However, just as for the maximal tensor product construction, cf. 2.15, if A_1 is a G-invariant C^*-subalgebra of A, then although we get a morphism $G\ltimes A_1 \to G\ltimes A$ this needs not be injective, except when A_1 is also an ideal in A.

In the pullback situation it is clear that as linear spaces we have an isomorphism

$$C_c(G, X) = (C_c(G, A)) \bigoplus_{C_c(G, C)} (C_c(G, B)).$$

This becomes an algebraic *-isomorphism for the convolution product defined, say, on $C_c(G, A)$ by

$$(x\times y)(g) = \int x(h)\, h(y(h^{-1}g))\, dh,$$

cf. [38, 7.6]. To show that it extends to an isometric *-isomorphism between the two C^*-algebras

$$G\ltimes (A\oplus_C B) \qquad \text{and} \qquad (G\ltimes A)\oplus_{G\ltimes C} G\ltimes B)$$

we check that the conditions in Proposition 3.1 are satisfied. Note first that

$$\ker\tilde\alpha = G\ltimes \ker\alpha,$$

and similarly for the kernels of $\tilde\beta$, $\tilde\gamma$ and $\tilde\delta$. Consequently,

$$\begin{aligned}\ker\tilde\gamma\cap\ker\tilde\delta &= (G\ltimes\ker\gamma)(G\ltimes\ker\delta)\\ &= ((C_c(G, \ker\gamma))(C_c(G, \ker\delta)))^{=} = \{0\}.\end{aligned}$$

Moreover,

$$\begin{aligned}\tilde\beta^{-1}(\tilde\alpha(G\ltimes A)) &= (\tilde\beta^{-1}(\tilde\alpha(C_c(G, A))))^{=}\\ &= (C_c(G, \beta^{-1}(\alpha(A))))^{=} = (C_c(G, \gamma(X)))^{=} = \tilde\gamma(G\ltimes X).\end{aligned}$$

Finally,

$$\tilde{\delta}(\ker \tilde{\gamma}) = \tilde{\delta}(G \ltimes \ker \gamma) = (\tilde{\delta}(C_c(G, \ker \gamma)))^{=}$$
$$= (C_c(G, \delta(\ker \gamma)))^{=} = (C_c(G, \ker \alpha))^{=} = G \ltimes \ker \alpha = \ker \tilde{\alpha}.$$

In the case of pushout diagrams we assume that α is a proper morphism and consider a coherent pair of morphisms $\varphi : G \ltimes A \to Y$ and $\psi : G \ltimes B \to Y$ (i.e., $\varphi \circ \tilde{\alpha} = \psi \circ \tilde{\beta}$). With $Y \subset \mathbb{B}(\mathfrak{H})$ it follows that there are unique covariant representations $(\varphi_1, u, \mathfrak{H})$ and $(\psi_1, v, \mathfrak{H})$ of (A, G) and (B, G), respectively, such that $\varphi = \varphi_1 \times u$ and $\psi = \psi_1 \times v$, cf. [38, 7.6.6].

For each c in C and f in $C_c(G)$ we compute

$$\varphi_1(\alpha(c))\, u_f = \varphi(\alpha(c) \otimes f) = \varphi(\tilde{\alpha}(c \otimes f))$$
$$= \psi(\tilde{\beta}(c \otimes f)) = \psi(\beta(c) \otimes f) = \psi_1(\beta(c))\, v_f.$$

Letting f range over an approximate unit for $L^1(G)$ it follows that $\varphi_1(\alpha(c)) = \psi_1(\beta(c))$, so that the pair (φ_1, ψ_1) is coherent. Consequently there is a unique morphism $\sigma_1 : X \to \mathbb{B}(\mathfrak{H})$ such that $\varphi_1 = \sigma_1 \circ \delta$ and $\psi_1 = \sigma_1 \circ \gamma$. Inserting this in the equation above, and setting

$$y = \sigma_1(\delta(\alpha(c))) = \sigma_1(\gamma(\beta(c))) = \varphi_1(\alpha(c)) = \psi_1(\beta(c)),$$

we find that

$$y u_f = y v_f, \qquad f \in C_c(G).$$

Consequently,

$$u_g\, y u_f = \varphi_1(g(\alpha(c)))\, u_g u_f = \psi_1(g(\beta(c)))\, v_g v_f = v_g\, y v_f.$$

Again letting f range over an approximate unit it follows that

$$u_g\, y = v_g\, y, \qquad g \in G.$$

We claim that (σ_1, v) is a covariant representation for (X, G). Indeed, if $b \in B$ we have

$$v_g \sigma_1(\gamma(b))\, v_g^* = v_g \psi_1(b)\, v_g^* = \psi_1(g(b)) = \sigma_1(\gamma(g(b))) = \sigma_1(g(\gamma(b))).$$

If $a \in A$ it has the form $\alpha(c)\, a_1 \alpha(c)$ for some c in C and a_1 in A, cf. [40, Sect. 4], so

$$v_g \sigma_1(\delta(a))\, v_g^* = v_g \varphi_1(a)\, v_g^* = v_g\, y \varphi_1(a_1)\; y^* v_g^*$$
$$= u_g\, y \varphi_1(a_1)\; y^* u_g^* = u_g \varphi_1(a)\, u_g^*$$
$$= \varphi_1(g(a)) = \sigma_1(g(\delta(a))).$$

Since $\delta(A) \cup \gamma(B)$ generates X the claim is established.

Let $\sigma = \sigma_1 \times v$ be the integrated representation of $G \ltimes X$. Then for b in B and f in $C_c(G)$ we have

$$\sigma(\tilde{\gamma}(b \otimes f)) = \sigma(\gamma(b) \otimes f) = \sigma_1(\gamma(b))\, v_f = \psi_1(b)\, v_f = \psi(b \otimes f).$$

Similarly, for a in A, written in the form $a = a_1 \alpha(c)$, we compute

$$\begin{aligned} \sigma(\tilde{\delta}(a \otimes f)) &= \sigma(\delta(a) \otimes f) = \sigma_1(\delta(a))\, v_f \\ &= \varphi(a_1)\, y v_f = \varphi_1(a_1)\, y u_f = \varphi_1(a)\, u_f = \varphi(a \otimes f). \end{aligned}$$

It follows that $\sigma \circ \tilde{\gamma} = \psi$ and $\sigma \circ \tilde{\delta} = \varphi$, so that we have found a factorization. Since φ, ψ was an arbitrary pair we have shown that $G \ltimes X$ is the amalgamated free product, i.e.

$$G \ltimes (A \star_C B) = (G \ltimes A) \star_{G \ltimes C} (G \ltimes B). \quad \blacksquare$$

6.4. *Exact Groups.* If (A, G) is a C^*-dynamical system and $(\pi, \mathfrak{H})$ is a faithful representation of A we can define the *left regular representation* $\lambda \times \pi$ of $G \ltimes A$ on $L^2(G) \otimes \mathfrak{H}$. The image algebra is known as the *reduced crossed product*,

$$G \ltimes_r A = (\lambda \times \pi)(G \ltimes A),$$

and does not depend on $(\pi, \mathfrak{H})$, cf. [38, 7.7.5]. In many cases, notably when G is amenable, $\lambda \times \pi$ is faithful, but in general it has a nonzero kernel. Note that if the action is trivial, then $G \ltimes_r A = C_r^*(G) \otimes_{\min} A$, where $C_r^*(G) = \lambda(C^*(G))$ is the reduced group C^*-algebra for G.

Following Kirchberg and Wassermann, cf. [28, 48], we say that a locally compact group G is *exact* if the operation of taking the reduced crossed product preserves short exact sequences. Thus, whenever we have an extension

$$0 \to A \to X \to B \to 0,$$

and a strongly continuous action of G on X for which A is a G-invariant ideal (and thus induces actions of G on A and on B), we obtain canonically a new extension

$$0 \to G \ltimes_r A \to G \ltimes_r X \to G \ltimes_r B \to 0.$$

Evidently this implies that $C_r^*(G)$ is an exact C^*-algebra, and in many cases, e.g., when G is discrete, this condition also suffices to show that G is exact, cf. [32, Theorem 5.2]. On the other hand, there are no known examples of discrete, nonexact groups, cf. the discussion in [31, Sect. 6].

6.5. THEOREM. *If we have a pullback diagram of C^*-algebras as below, to the left, and (A, G), (B, G), and (C, G) are C^*-dynamical systems for which α and β are G-equivariant morphisms, then if G is exact and the actions are defined as in* 6.2, *we obtain a new pullback diagram as below, to the right*:

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & B \\ \Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\alpha} & C \end{array} \qquad \textit{gives} \qquad \begin{array}{ccc} G \ltimes_r A & \xrightarrow{\tilde{\gamma}} & G \ltimes_r B \\ \Big\downarrow{\scriptstyle\tilde{\delta}} & & \Big\downarrow{\scriptstyle\tilde{\beta}} \\ G \ltimes_r A & \xrightarrow{\tilde{\alpha}} & G \ltimes_r C \end{array}$$

Proof. It follows from the definition of the reduced crossed product that whenever (A, G) is a C^*-dynamical system and A_1 is a G-invariant C^*-subalgebra of A we obtain a canonical embedding

$$G \ltimes_r A_1 \subset G \ltimes_r A.$$

This fact will be used repeatedly and without further comment below.

Consider the extension

$$0 \to \ker \alpha \to A \to \alpha(A) \to 0.$$

Since both $\ker \alpha$ and $\alpha(A)$ are G-invariant and G is exact, this results in a new extension

$$0 \to G \ltimes_r \ker \alpha \to G \ltimes_r A \xrightarrow{\tilde{\alpha}} G \ltimes_r \alpha(A) \to 0.$$

Since $\alpha(A) \subset C$ we actually have the morphism $\tilde{\alpha} : G \ltimes_r A \to G \ltimes_r C$, with $\ker \tilde{\alpha} = G \ltimes_r \ker \alpha$ and $\tilde{\alpha}(G \ltimes_r A) = G \ltimes_r \alpha(A)$. Similarly we define $\tilde{\beta}$, $\tilde{\gamma}$, and $\tilde{\delta}$, and identify their kernels and their ranges as reduced crossed products.

To verify the three conditions in Proposition 3.1 for the new diagram we first compute

$$\ker \tilde{\gamma} \cap \ker \tilde{\delta} = (G \ltimes_r \ker \gamma)(G \ltimes_r \ker \delta) = \{0\}.$$

For condition (ii) we note that its validity in the first diagram gives an extension

$$0 \to \ker \beta \to \gamma(X) \xrightarrow{\beta} \alpha(A) \to 0,$$

and since G is exact, this results in a new extension

$$0 \to \ker \tilde{\beta} \to \tilde{\gamma}(G \ltimes_r A) \xrightarrow{\tilde{\beta}} \tilde{\alpha}(G \ltimes_r A) \to 0,$$

which neatly shows that

$$\tilde{\gamma}(G \ltimes_r X) = \tilde{\beta}^{-1}(\tilde{\alpha}(G \ltimes_r A)).$$

Finally, since δ is an isomorphism of $\ker \gamma$ onto $\ker \alpha$, we see that $\tilde{\delta}$ is an isomorphism of $G \ltimes_r \ker \gamma$ onto $G \ltimes_r \ker \alpha$, i.e. $\tilde{\delta}(\ker \tilde{\gamma}) = \ker \tilde{\alpha}$. ∎

6.6. *Remarks.* One may view the process of forming the crossed product with a fixed group G as a functor $G \ltimes$ from the category $\mathscr{C}^*(G)$ of C^*-dynamical G-systems with G-equivariant morphisms to $\mathscr{C}^*$. Since $G \ltimes$ transforms a proper G-equivariant morphism $\alpha : A \to B$ into a proper morphism $\tilde{\alpha} : G \ltimes A \to G \ltimes B$ we may also regard it as a functor from $\mathscr{C}_p^*(G)$ to $\mathscr{C}_p^*$—the categories where only proper morphisms are allowed.

It follows from the preceding results that $B \ltimes$ preserves finite products, pullbacks, kernels, and cokernels. The restriction to $\mathscr{C}_p^*(G)$ even preserves pushouts. The functor does neither preserve free products nor infinite products, since already the tensor functor $A \to Y \otimes_{\max} A$ fails to do so.

When G is exact the functor $G \ltimes_r$ preserves finite products, pullbacks, kernels and cokernels.

7. MULTIPLIER DIAGRAMS

7.1. This section is concerned with some elementary looking results about pullbacks and pushouts of multiplier algebras. The important fact here is that every proper morphism between C^*-algebras

$$\varphi : A \to B$$

extends uniquely to a unital (and strictly continuous) morphism

$$\bar{\varphi} : M(A) \to M(B) \qquad (*)$$

such that $\varphi(ma) = \bar{\varphi}(m)\,\varphi(a)$ for every a in A and m in $M(A)$; cf. [20, 2.1] or [40, Theorem 5.10]. If φ is surjective and A is σ-unital the extension is also surjective; see [38, 3.12.10].

7.2. Proposition. *If we have a pullback diagram in which all morphisms are proper, then the extended diagram on the multiplier algebras is also a pullback*:

$$\begin{tikzcd} X \arrow[r, "\gamma"'] \arrow[d, "\delta"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & C \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} M(X) \arrow[r, "\bar{\gamma}"'] \arrow[d, "\bar{\delta}"] & M(B) \arrow[d, "\bar{\beta}"] \\ M(A) \arrow[r, "\bar{\alpha}"] & M(C) \end{tikzcd}$$

Proof. Put $M = M(A) \oplus_{M(C)} M(B)$ and observe that in the commutative diagram to the right the two coherent morphisms $\bar{\gamma}$ and $\bar{\delta}$ define a morphism

$$\sigma : M(X) \to M.$$

Since

$$\ker \bar{\gamma} = \{m \in M(X) \mid mX + Xm \subset \ker \gamma\},$$

cf. $(*)$ in 7.1, it follows that

$$\begin{aligned}\ker \sigma &= \ker \bar{\gamma} \cap \ker \bar{\delta} \\ &= \{m \in M(X) \mid mX + Xm \subset \ker \gamma \cap \ker \delta\} = \{0\}.\end{aligned}$$

Thus σ is injective and may be regarded as an embedding

$$X \subset M(X) \subset M.$$

For each $m = (y, z)$ in M and $x = (a, b)$ in X we have

$$mx = (ya, zb) \in A \oplus B.$$

Moreover,

$$\alpha(ya) = \bar{\alpha}(y)\, \alpha(a) = \bar{\beta}(z)\, \beta(b) = \beta(zb),$$

which shows that actually $mx \in X$. Consequently, X is an ideal in M. If $m = (y, z) \in M$ and $mX = 0$, take an approximate unit $(u_\lambda) = (a_\lambda, b_\lambda)$ for X. Since γ and δ are proper morphisms, (a_λ) and (b_λ) are approximate units for A and B, respectively. As $ya_\lambda = 0 = zb_\lambda$ this implies that $y = 0 = z$, whence $m = 0$. Thus X is an essential ideal in M, and therefore $M \subset M(X)$, whence $M = M(X)$. ∎

Examples where the conditions of Proposition 7.2 are met are not hard to find: If α is surjective (cf. 3.2) and β is proper, then also δ is proper (and γ is surjective).

7.3. Example. If we have a pushout diagram in which all the morphisms are proper, then by extension we obtain a commutative diagram on the multiplier algebras:

$$\begin{array}{ccc} C & \xrightarrow{\beta} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ A & \xrightarrow{\delta} & X \end{array} \quad \text{gives} \quad \begin{array}{ccc} M(C) & \xrightarrow{\bar{\beta}} & M(B) \\ \downarrow{\scriptstyle\bar{\alpha}} & & \downarrow{\scriptstyle\bar{\gamma}} \\ M(A) & \xrightarrow{\bar{\delta}} & M(X) \end{array}$$

However, this diagram need not be a pushout. The following counterexample arose in conversation with R. Nest, F. Radulescu, and D. Shlyakhtenko. (The author supplied the wine.)

We take $A = B = C = C_0(\mathbb{R}_+)$ and let $\alpha = \beta$ be the transposed of the proper, continuous map $\theta : \mathbb{R}_+ \to \mathbb{R}_+)$ defined by

$$\theta(t) = \begin{cases} n + t - 4^{n-1} & \text{for} \quad 4^{n-1} - 1 \leqslant t \leqslant 4^{n-1} \\ n & \text{for} \quad 4^{n-1} \leqslant t \leqslant 4^n - 1 \end{cases}$$

for n in $\mathbb{N}$. If we consider the filtered space

$$\Omega = \{(s, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \theta(s) = \theta(t)\},$$

which looks like a sequence of exponentially increasing squares connected by short diagonal lines, then $C_0(\Omega)$ is isomorphic to the balanced tensor product $C_0(\mathbb{R}_+) \otimes_{C_0(\mathbb{R}_+)} C_0(\mathbb{R}_+)$ (cf. 7.5), and thus a (maximal commutative) quotient of the amalgamated free product. The idea behind our example is that the Stone–Čech compactification of Ω is not the filtration of $\beta\mathbb{R}_+ \times \beta\mathbb{R}_+$ with respect to $\bar{\theta}$.

Assume, to obtain a contradiction, that we had a pushout diagram

$$\begin{array}{ccc} C_b(\mathbb{R}_+) & \xrightarrow[\bar{\beta}]{} & C_b(\mathbb{R}_+) \\ \Big\downarrow{\scriptstyle \bar{\alpha}} & & \Big\downarrow{\scriptstyle \bar{\gamma}} \\ C_b(\mathbb{R}_+) & \xrightarrow{\bar{\delta}} & M(X) \end{array}$$

where $X = C_0(\mathbb{R}_+) \star_{C_0(\mathbb{R}_+)} C_0(\mathbb{R}_+)$, and let $\sigma : X \to C_0(\Omega)$ denote the natural quotient map induced by the coherent morphisms of $C_0(\mathbb{R}_+)$ into $C_0(\Omega)$. Moreover, let $\bar{\sigma} : M(X) \to C_b(\Omega)$ be its canonical extension. Note that by construction the morphisms $\bar{\delta}$, $\bar{\gamma}$ and $\bar{\sigma}$ are strictly continuous.

Define a strictly continuous morphism $\varphi_k : C_b(\mathbb{R}_+) \to \ell^\infty$ for each k in $\mathbb{N}$ by

$$\varphi_k(f)(n) = \begin{cases} f(4^n) & \text{if} \quad 4^n < k \\ f(4^n + k) & \text{if} \quad k \leqslant 4^n. \end{cases}$$

By computation we find that

$$\varphi_k(\bar{\alpha}(f))(n) = \begin{cases} f(\theta(4^n)) = f(n+1) & \text{if} \quad 4^n < k \\ f(\theta(4^n + k)) = f(n+1) & \text{if} \quad k \leqslant 4^n. \end{cases}$$

If we therefore define $\psi_k = \varphi_k$, but regard ψ_k as a morphism of the other copy of $C_b(\mathbb{R}_+)$, then $\varphi_k \circ \bar{\alpha} = \psi_\ell \circ \bar{\beta}$, so that we have a coherent pair of morphisms (φ_k, ψ_ℓ) for all (k, ℓ) in $\mathbb{N} \times \mathbb{N}$.

Define two sequences (x_m) and (y_m) in $C_0(\mathbb{R}_+)$ of positive, norm one, pairwise orthogonal functions given by

$$x_m(t) = y_m(t) = (1 - 2\,|t - m|) \vee 0.$$

Then let

$$z = \sum \delta(x_m)\,\gamma(y_m) \in M(X),$$

the sum being strictly convergent. Note that $\bar{\sigma}(z)$ is the function on Ω given by

$$\bar{\sigma}(z)(s, t) = \sum x_m(s)\; y_m(t).$$

For each k, ℓ the morphism $\varphi_k \star \psi_\ell \colon M(X) \to \ell^\infty$ factors through $\bar{\sigma}$ (since ℓ^∞ is commutative) and we have

$$((\varphi_k \star \psi_\ell)(z))(n) = \sum \varphi_k(x_m)\,\psi_\ell(y_m)(n).$$

If $k \leqslant 4^n$ and $\ell \leqslant 4^n$ then with δ the Kronecker symbol

$$\begin{aligned}
\varphi_k(x_m)\,\psi_\ell(y_m)(n) &= x_m(4^n + k)\; y_m(4^n + k)\\
&= \delta(m, 4^n + k)\,\delta(m, 4^n + \ell)\\
&= \delta(k, \ell)\,\delta(m, 4^n + k).
\end{aligned}$$

If $k \leqslant 4^n < \ell$ then

$$\varphi_k(x_m)\,\psi_\ell(y_m)(n) = x_m(4^n + k)\; y_m(4^n) = 0.$$

If $4^n < k$ and $4^n < \ell$ then

$$\varphi_k(x_m)\,\psi_\ell(y_m)(n) = x_m(4^n)\; y_m(4^n) = \delta(m, 4^n).$$

It follows that

$$((\varphi_k \star \psi_\ell)(z))(n) = \begin{cases} \delta(k, \ell) & \text{if } \max(k, \ell) \leqslant 4^n\\ 0 & \text{if } k \leqslant 4^n < \ell\\ 0 & \text{if } \ell \leqslant 4^n < k\\ 1 & \text{if } 4^n < \min(k, \ell)\end{cases}$$

By assumption $M(X)$ is generated as a C^*-algebra by $\bar{\delta}(C_b(\mathbb{R}_+)) \cup \bar{\gamma}(C_b(\mathbb{R}_+))$. There is therefore a polynomial p with noncommuting variables from this set, such that $\|z - p\| < \frac{1}{2}$. We can simplify matters by considering

$$\bar{\sigma}(p) = \sum_{j=1}^{m} f_j \otimes g_j \in C_b(\Omega),$$

where $f_j \in C_b(\mathbb{R}_+)$ and $g_j \in C_b(\mathbb{R}_+)$ (but $f_j \otimes g_j$ is restricted to Ω), since then we can estimate

$$((\varphi_k \bigstar \psi_\ell)(p))(n) = \sum_{j=1}^{m} \varphi_k(f_j)\, \psi_\ell(g_j)(n). \tag{$*$}$$

Choose now a character ω on ℓ^∞/c_0 (corresponding to a nontrivial universal net in $\mathbb{N}$) and define $\tilde{\varphi}_k = \omega \circ \varphi_k$ and $\tilde{\psi}_\ell = \omega \circ \psi_\ell$. Note that the pairs $(\tilde{\varphi}_K, \tilde{\psi}_\ell)$ are coherent morphisms into $\mathbb{C}$, and that $\tilde{\varphi}_k \bigstar \tilde{\psi}_\ell = \omega \circ (\varphi_k \bigstar \psi_\ell)$. From our earlier computations it follows that

$$\tilde{\varphi}_k \bigstar \tilde{\psi}_\ell(z) = \delta(k, \ell).$$

However, it follows from $(*)$ that

$$\tilde{\varphi}_k \bigstar \tilde{\psi}_\ell(p) = \sum_{j=1}^{m} \omega(\varphi_k(f_j))\, \omega(\psi_\ell(g_j)) = \sum_{j=1}^{m} s_{jk} t_{j\ell}.$$

Evidently $|\delta(k, \ell) - \sum_{j=1}^{m} s_{jk} t_{j\ell}| < \frac{1}{2}$ for all k and ℓ is impossible, because the sequences $(\mathbf{s}_k)$ and $(\mathbf{t}_\ell)$ of vectors in $\mathbb{C}^m$ have limit points.

7.4. Example. The preceding counterexample was essentially commutative. Here is one which exploit the proper embedding $\alpha\colon c_0 \to \mathbb{K}$, identifying c_0 with the diagonal compact operators on ℓ^2 in a given orthonormal basis. If we let $\beta : c_0 \to c_0$ denote the quotient map with kernel $\mathbb{C} = \mathbb{C}e_{11}$, then we obtain the two commutative diagrams

$$\begin{tikzcd} c_0 \arrow[r, "\beta"'] \arrow[d, "\alpha"] & c_0 \arrow[d, "\gamma"] \\ \mathbb{K} \arrow[r, "\delta"] & 0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \ell^\infty \arrow[r, "\bar{\beta}"'] \arrow[d, "\bar{\alpha}"] & \ell^\infty \arrow[d, "\bar{\gamma}"] \\ \mathbb{B}(\ell^2) \arrow[r, "\bar{\delta}"] & 0 \end{tikzcd}$$

The first is a pushout by Theorem 2.5, since $\alpha(\ker \beta)$ generates $\ker \delta$ $(= \mathbb{K})$ as an ideal. But in the diagram on the multiplier algebras $\bar{\alpha}(\ker \bar{\delta})$ fails to generate $\ker \bar{\delta}$ as an ideal, so this is not a pushout. (Evidently he amalgamated free product between $\mathbb{B}(\ell^2)$ and ℓ^∞ is the Calkin algebra in this case.)

Note that the counterexample above even occurred for one of our "ideal" pushouts, where one of the morphisms is surjective, cf. Remark 5.4. However, as T. G. Houghton-Larsen proved to the author, if in a pushout diagram *both* α and β (hence also γ and δ) are surjective, and the C^*-algebras are σ-unital, *then* the extended diagram on the multiplier algebras is again a pushout, cf. Example 3.3.D.

7.5. *Remark.* One may view the process of forming the multiplier algebra as a functor M from the category $\mathscr{C}_p^*$ of C^*-algebras with proper morphisms to the category $\mathscr{C}_1^*$ of unital C^*-algebras with unital morphisms. We see from the previous results that M preserves pullbacks. It also trivially preserves cokernels, since all cokernels in these two categories are zero. The fact that M preserves products is easily established, but not nearly as important as the formula $M(\bigoplus A_n) = \prod M(A_n)$. However, M does neither preserve free products nor pushouts. The question of kernels does not arise, since the two categories only have trivial kernel morphisms. (The zero object in $\mathscr{C}_1^*$ is the unital C^*-algebra 0 by convention.)

7.6. *Multiplier Amalgamations.* If C is a unital C^*-algebra and $\alpha : C \to M(A)$ and $\beta : C \to M(B)$ are unital morphisms into the multiplier algebras of A and B, respectively, we define the *multiplier amalgamated free product* as the C^*-algebra

$$A \star_C B = (A \star B)/I,$$

where I is the closed ideal in the free product generated by elements of the form

$$a(\alpha(c) - \beta(c))\, b, \qquad a \in A, \quad b \in B, \quad c \in C.$$

This algebra, together with the canonical morphisms $\delta : A \to A \star_C B$ and $\gamma : B \to A \star_C B$, is the universal solution to the following problem: Given nondegenerate representations $\varphi : A \to \mathbb{B}(\mathfrak{H})$ and $\psi : B \to \mathbb{B}(\mathfrak{H})$ such that the extensions $\bar{\varphi} : M(A) \to \mathbb{B}(\mathfrak{H})$ and $\bar{\psi} : M(B) \to \mathbb{B}(\mathfrak{H})$ satisfy the coherence relation $\bar{\varphi} \circ \alpha = \bar{\psi} \circ \beta$, there is a unique morphism $\sigma : A \star_C B \to \mathbb{B}(\mathfrak{H})$ such that $\varphi = \sigma \circ \delta$ and $\psi = \sigma \circ \gamma$.

If we are given an ordinary pushout construction as in the diagram below, to the left, with the morphisms α and β proper, we have extensions $\bar{\alpha}$ and $\bar{\beta}$ and can form the multiplier amalgamated free product below, to the right:

$$\begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\ A \arrow[r, "\delta"] & A \star_C B \end{tikzcd} \qquad \begin{tikzcd} M(C) \arrow[r, "\bar{\beta}"'] \arrow[d, "\bar{\alpha}"] & M(B) & & B \arrow[ll] \arrow[d, "\gamma"] \\ M(A) & & A \arrow[ll] \arrow[r, "\delta"] & A \star_{M(C)} B \end{tikzcd}$$

It follows from the definitions that $A \star_C B = A \star_{M(C)} B$, so that the construction in this case yields nothing new—as expected.

Also in the case where we form the free product $A \star B$ for some nonunital C^*-algebras, and afterwards amalgamate over the common unit in the multiplier algebras $M(A)$ and $M(B)$ we just get $A \star_{\mathbb{C}} B = A \star B$. However, Theorem 5.5 generalizes to the setting of multiplier amalgamations, and we derive the formula $Y \otimes (A \star B) = (Y \otimes A) \star_Y (Y \otimes B)$ for any C^*-algebra Y and $\otimes = \otimes_{\max}$, using the embedding $y \to y \otimes \mathbf{1}$ of Y into $M(Y \otimes A)$ and similarly for B. More generally, with C^+ the forced unitization of a C^*-algebra C, we obtain the formula

$$Y \otimes (A \star_C B) = (Y \otimes A) \star_{Y \otimes C^+} (Y \otimes B).$$

This vindicates the example in 5.10, and shows that the concept of multiplier amalgamated free products has useful applications.

Replacing free products with tensor products in the construction above is known as the *C-balanced tensor product* of A and B. By necessity the "balancing" must be taken over a common subalgebra in the centers of the multiplier algebras of the C^*-algebras involved. Consider for example two C^*-algebras A and B such that $Z(M(A)) = Z(M(B))$. By the Dauns–Hofmann theorem, [38, 4.4.8], this means that $C_b(\hat{A}) = C_b(\hat{B})$, where $\hat{A}$ denotes the spectrum of (equivalence classes of) irreducible representations of A equipped with the Jacobson topology. So in the most elementary cases the two algebras are Rieffel-Morita equivalent. Note that if the C^*-algebras A and B are nonunital we have to pass to the multiplier algebras in order to recover algebraically the similarities between their irreducible representations. The C-balanced tensor product is studied in [4, 5, 16, 17].

8. BOXES AND PRISMS

The first results in this section about commutative three-dimensional diagrams of C^*-algebras can presumably be found—in a slightly different language—in textbooks on axiomatic category theory. Certainly the proofs require no more than standard diagram chases. However, the results have immediate applications, and their potential usefulness should not be underestimated.

8.1. THEOREM. *Consider a commutative diagram of C^*-algebras in box form*

$$\begin{tikzcd}
& X_1 \arrow[rr] \arrow[dd] & & B_1 \arrow[dd] \\
X_2 \arrow[ur] \arrow[rr] \arrow[dd] & & B_2 \arrow[ur] \arrow[dd] & \\
& A_1 \arrow[rr] & & C_1 \\
A_2 \arrow[ur] \arrow[rr] & & C_2 \arrow[ur] &
\end{tikzcd}$$

If $X_1 = A_1 \oplus_{C_1} B_1$, $X_2 = X_1 \oplus_{A_1} A_2$ *and* $B_2 = B_1 \oplus_{C_1} C_2$, *then* $X_2 = A_2 \oplus_{C_2} B_2$.

Proof. The contents of the theorem (already used in the proof of [20, Proposition 4.1]) is that when the back and the two sides of the box are pullbacks, then so is the front.

By concatenating the left side and the back (cf. Proposition 2.7) we see that the diagram below, to the left, is a pullback. But (using the front and the right side of the box) this diagram also has the concatenated form below, to the right, and decatenating the right square, using Proposition 2.9, we obtain the desired result:

$$\begin{tikzcd}
X_2 \arrow[r] \arrow[d] & B_1 \arrow[d] \\
A_2 \arrow[r] & C_1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_1 \arrow[d] \\
A_2 \arrow[r] & C_2 \arrow[r] & C_1
\end{tikzcd} \quad \blacksquare$$

8.2. Corollary. *Consider a commutative diagram of* C^**-algebras in prismatic form*

$$\begin{tikzcd}
X_2 \arrow[rr] \arrow[dr] \arrow[dd] & & B_2 \arrow[dr] \arrow[dd] & \\
& C_1 \arrow[rr] & & B_1 \\
A_2 \arrow[ur] \arrow[rr] & & C_2 \arrow[ur] &
\end{tikzcd}$$

If $X_2 = B_2 \oplus_{B_1} C_1$ *and* $A_2 = C_1 \oplus_{B_1} C_2$, *then* $X_2 = A_2 \oplus_{C_2} B_2$.

Proof. Replace the edge $C_1 \to B_1$ with the pullback diagram

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d, equal] & B_1 \arrow[d, equal] \\ C_1 \arrow[r] & B_1 \end{tikzcd}$$

(cf. Example 3.3.B) and apply Theorem 8.1. ∎

8.3. THEOREM. *Consider a commutative diagram of C^*-algebras in box form*

$$\begin{tikzcd} & C_2 \arrow[rr] \arrow[dd] & & B_2 \arrow[dd] \\ C_1 \arrow[ur] \arrow[rr] \arrow[dd] & & B_1 \arrow[ur] \arrow[dd] & \\ & A_2 \arrow[rr] & & X_2 \\ A_1 \arrow[ur] \arrow[rr] & & X_1 \arrow[ur] & \end{tikzcd}$$

If $X_1 = A_1 \star_{C_1} B_1$, $A_2 = A_1 \star_{C_1} C_2$ *and* $X_2 = X_1 \star_{B_1} B_2$, *then* $X_2 = A_2 \star_{C_2} B_2$.

Proof. We know that the front and both sides of the box are pushouts and wish to conclude that the back is a pushout.

By concatenating the front and the right side we see that the diagram below, to the left, is a pushout. But this diagram also has the concatenated form below, to the right, and decatenating the left square, using Proposition 2.9, we obtain the desired result.

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] & B_2 \arrow[d] \\ A_1 \arrow[r] & X_2 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & B_2 \arrow[d] \\ A_1 \arrow[r] & A_2 \arrow[r] & X_2 \end{tikzcd}$$ ∎

8.4. COROLLARY. *Consider a commutative diagram of C^*-algebras in prismatic form*

$$\begin{tikzcd} & C_2 \arrow[rr] \arrow[dd] & & B_2 \arrow[dd] \\ C_1 \arrow[ur] \arrow[rr] \arrow[dr] & & B_1 \arrow[ur] \arrow[dr] & \\ & A_2 \arrow[rr] & & X_2 \end{tikzcd}$$

If $B_2 = B_1 \star_{C_1} C_2$ *and* $X_2 = A_2 \star_{C_1} B_1$, *then* $X_2 = A_2 \star_{C_2} B_2$.

Proof. Replace the edge $C_1 \to B_1$ with the pushout diagram

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d, equal] & B_1 \arrow[d, equal] \\ C_1 \arrow[r] & B_1 \end{tikzcd}$$

(cf. Example 3.3.B) and apply Theorem 8.3. ▮

8.5. PROPORTION. *Given a commutative diagram of C^*-algebras*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & A_0 \arrow[d] & A_2 \arrow[l] \arrow[d] \\ C_1 \arrow[r] & C_0 & C_2 \arrow[l] \\ B_1 \arrow[r] \arrow[u] & B_0 \arrow[u] & B_2 \arrow[l] \arrow[u] \end{tikzcd}$$

there is a natural isomorphism:

$$(A_1 \oplus_{C_1} B_1) \underset{A_0 \oplus_{C_0} B_0}{\oplus} (A_2 \oplus_{C_2} B_2) = (A_1 \oplus_{A_0} A_2) \underset{C_1 \oplus_{C_0} C_2}{\oplus} (B_1 \oplus_{B_0} B_2).$$

Proof. The category $\Delta\mathscr{C}^*$ of inward directed triples of C^*-algebras defined in Example 2.11.E has pullbacks, obtained as the inward directed triples of pullbacks. Since the pullback functor from $\Delta\mathscr{C}^*$ to $\mathscr{C}^*$ has a left adjoint it preserves pullbacks, cf. 2.10, and that is precisely the content of the proposition. ▮

8.6. COROLLARY. *If we have a commutative diagram of C^*-algebras*

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & B_1 \arrow[d] & & B_2 \arrow[d] & X_2 \arrow[l] \arrow[d] \\ A_1 \arrow[r] & C_1 \arrow[r] & D & C_2 \arrow[l] & A_2 \arrow[l] \end{tikzcd}$$

in which the two squares above are pullbacks, we obtain a new pullback diagram using the induced morphisms:

$$\begin{tikzcd} X_1 \oplus_D X_2 \arrow[r] \arrow[d] & B_1 \oplus_D B_2 \arrow[d] \\ A_1 \oplus_D A_2 \arrow[r] & C_1 \oplus_D C_2 \end{tikzcd}$$

Proof. Take $A_0 = B_0 = C_0 = D$ in Proposition 8.5. ▮

8.7. PROPOSITION. *Given a commutative diagram of C^*-algebras*

$$\begin{tikzcd}
A_1 & A_0 \arrow[l] \arrow[r] & A_2 \\
C_1 \arrow[u] \arrow[d] & C_0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & C_2 \arrow[u] \arrow[d] \\
B_1 & B_0 \arrow[l] \arrow[r] & B_2
\end{tikzcd}$$

there is a natural isomorphism:

$$(A_1 \bigstar_{C_1} B_1) \bigstar_{A_0 \star_{C_0} B_0} (A_2 \bigstar_{C_2} B_2) = (A_1 \bigstar_{A_0} A_2) \bigstar_{C_1 \star_{C_0} C_2} (B_1 \bigstar_{B_0} B_2).$$

Proof. The category $\nabla\mathscr{C}^*$ of outward directed triples of C^*-algebras defined in Example 2.11.F has pushouts, obtained as the outward directed triples of pushouts. Since the pushout functor from $\nabla\mathscr{C}^*$ has a right adjoint it preserves pushouts, and that is just the content of the proposition. ▮

8.8. COROLLARY. *If we have a commutative diagram of C^*-algebras*

$$\begin{tikzcd}
B_1 \arrow[d] & C_1 \arrow[l] \arrow[d] & D \arrow[l] \arrow[r] & C_2 \arrow[r] \arrow[d] & B_2 \arrow[d] \\
X_1 & A_1 \arrow[l] & & A_2 \arrow[r] & X_2
\end{tikzcd}$$

in which the two squares above are pushouts, we obtain a new pushout diagram using the induced morphisms:

$$\begin{tikzcd}
C_1 \bigstar_D C_2 \arrow[r] \arrow[d] & B_1 \bigstar_D B_2 \arrow[d] \\
A_1 \bigstar_D A_2 \arrow[r] & X_1 \bigstar_D X_2
\end{tikzcd}$$

Proof. Take $A_0 = B_0 = C_0 = D$ in Proposition 8.7. ▮

8.9. PROPOSITION. *If we have a diagram of C^*-algebras*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & X \arrow[r] & B \arrow[r] \arrow[d] & 0 \\
& & Y \arrow[r, "\alpha"] & D &
\end{tikzcd}$$

in which the upper row is an extension, we obtain canonically a new extension

$$0 \to A \to Y \bigoplus_D X \to Y \bigoplus_D B \to 0.$$

Proof. Applying Corollary 8.6 to the first diagram below we see that the second diagram is a pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] & Y \arrow[d, "\alpha"] \\ X \arrow[r] & B \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} Y \oplus_D A \arrow[r] \arrow[d] & Y \oplus_D 0 \arrow[d] \\ Y \oplus_D X \arrow[r] & Y \oplus_D B \end{tikzcd}$$

Evidently $Y \oplus_D A = \ker \alpha \oplus A$ and $Y \oplus_D 0 = \ker \alpha$. By concatenation with the pullback diagram to the left, below, we obtain the pullback diagram to the right:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ \ker \alpha \oplus A \arrow[r] & \ker \alpha \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ Y \oplus_D X \arrow[r] & Y \oplus_D B \end{tikzcd}$$

Since $Y \oplus_D X$ surjects onto $Y \oplus_D B$ this describes an extension, cf. Example 3.3.B. ▌

8.10. Proposition. *If we have a diagram of C^*-algebras*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & X \arrow[r] & B \arrow[r] & 0 \\ & D \arrow[u, "\alpha"] \arrow[r, "\beta"] & Y \end{tikzcd}$$

in which the upper row is an extension and β is a proper morphism, we obtain canonically a new extension

$$0 \to Y \bigstar_D A \to Y \bigstar_D X \to B \to 0.$$

Proof. Applying Corollary 8.8 to the first diagram below we see that the second is a pushout

$$\begin{tikzcd} D \arrow[r, "\alpha"'] \arrow[d, "\beta"] & A \arrow[r] \arrow[d] & 0 \arrow[d] \\ Y & X \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} Y \bigstar_D A \arrow[r] \arrow[d, "\sigma"] & Y \bigstar_D 0 \arrow[d] \\ Y \bigstar_D X \arrow[r, "\pi"] & Y \bigstar_D B \end{tikzcd}$$

However, since D maps to zero in both $Y\star_D 0$ and $Y\star_D B$ we see that $\beta(D)$ is contained in the kernel of the canonical morphisms $Y\to Y\star_D 0$ and $Y\star_D B$. Since $\beta(D)$ contains an approximate unit for Y it follows that $Y\star_D 0=0$ and $Y\star_D B=B$. We already know that π is surjective and that $\sigma(Y\star_D A)$ generates $\ker\pi$ as an ideal (cf. Example 3.3.B), so all we need to show to obtain an extension is that σ embeds $Y\star_D A$ as an ideal in $Y\star_D X$.

Towards this end consider the diagram

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[r] \arrow[d, "\delta"] & X \arrow[d] \\ Y \arrow[r] & Y\star_D A \arrow[r, "\sigma"] & Y\star_D X \end{tikzcd}$$

By construction the left square and the big, concatenated square are both pushouts. By decatenation, cf. Proposition 2.9, also the right square is a pushout. Since β, hence also δ are proper morphisms and A is embedded as an ideal in X it follows from Theorem 5.3 that σ is an embedding of $Y\star_D A$ as an ideal in $Y\star_D X$ (and that the quotient is $X/A=B$). ∎

8.11. *Conditionally Projective Diagrams.* Recall from [21, 5.1.1] that a commutative diagram of C^*-algebras as below, to the left, is *conditionally projective* if, whenever we have an extended commutative diagram as below, to the right, there is a morphism $\sigma\colon A\to M$ such that $\sigma\circ\alpha=\psi\circ\beta$ and $\pi\circ\sigma=\varphi\circ\delta$.

$$\begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\ A \arrow[r, "\delta"] & D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[r, "\psi"'] \arrow[d, "\gamma"] & M \arrow[d, "\pi"] \\ A \arrow[r, "\delta"] & D \arrow[r, "\varphi"] & Q \end{tikzcd}$$

Here $\pi: M\to Q$ is any quotient morphism, but just as for projective algebras it suffices by Busby theory to consider the case where $M=M(E)$ and $Q=Q(E)$ for an arbitrary nonunital C^*-algebra E. The notion of a conditionally projective diagram grew out of ad hoc methods used in [35] and [20]. The philosophy behind it is that the morphism α encodes the obstructions for lifting morphisms out of A, whereas the right column in the diagram serves as a filter for the class of morphisms from A and C that we want to consider. Thus in the case with no filter, where $B=C$ and $C=A$, we simply say that $\alpha: C\to A$ is a *conditionally projective* morphism, cf. [21, 5.3]; and when every obstacle collapses, so that $D=A$ and $B=C=0$, we recover the definition of a projective C^*-algebra, cf. [33, 34].

An important generalization occurs if we consider only situations where $\ker\pi$ is the closure of an increasing sequence of ideals I_n, so that we have a sequence of morphisms $\pi_n: M/I_n\to Q$. We then say that the diagram is *conditionally semiprojective* if for each $\psi_n = B \to M/I_n$ and φ as before we can find $m \geqslant n$ and $\sigma : A \to M/I_m$ such that $\sigma \circ \alpha = \psi_m \circ \beta$ and $\pi_m \circ \sigma = \varphi \circ \delta$ (with ψ_m the composition of ψ_n with the quotient morphism $M/I_n \to M/I_m$). Conditionally semiprojective diagrams and morphisms play a key rôle in the proof that every one-dimensional *NCCW* complex is semiprojective, [19, 6.2.2].

8.12. Proposition. *Consider a commutative diagram of C^*-algebras in box form*

$$\begin{tikzcd}
 & C_2 \arrow[rr, "\beta_2"] \arrow[dd, "\alpha_2"] & & B_2 \arrow[dd, "\gamma_2"] \\
C_1 \arrow[ur, "\gamma_{12}"] \arrow[rr, "\beta_1"] \arrow[dd, "\alpha_1"] & & B_1 \arrow[ur, "\beta_{12}"] \arrow[dd, "\gamma_1"] & \\
 & A_2 \arrow[rr, "\delta_2"] & & D_2 \\
A_1 \arrow[ur, "\alpha_{12}"] \arrow[rr, "\delta_1"] & & D_1 \arrow[ur, "\delta_{12}"] &
\end{tikzcd}$$

If the left vertical side of the box is a pushout and the front side is conditionally (*semi*)*projective, then also the hind side is conditionally* (*semi*)*projective.*

Proof. We only treat the projective case, as the semiprojective is quite analogous.

Assume therefore that we have a quotient morphism $\pi : M \to Q$ and coherent morphisms $\psi : B_2 \to M$ and $\varphi : D_2 \to Q$. By assumption we can then find a morphism $\sigma_1 : A_1 \to M$ such that

$$\sigma_1 \circ \alpha_1 = \psi \circ \beta_2 \circ \gamma_{12} \qquad \text{and} \qquad \pi \circ \sigma_1 = \varphi \circ \delta_2 \circ \alpha_{12}.$$

Since $A_2 = A_1 \star_{C_1} C_2$ and the pair $(\sigma_1, \psi \circ \beta_2)$ is coherent by the first equation above, we can construct $\sigma_2 : A_2 \to M$ such that

$$\sigma_2 \circ \alpha_{12} = \sigma_1 \qquad \text{and} \qquad \sigma_2 \circ \alpha_2 = \psi \circ \beta_2.$$

It follows that σ_2 satisfied the first of the desired equations. For the second we note that we have both equations

$$\pi \circ \sigma_2 \circ \alpha_2 = \pi \circ \psi \circ \beta_2 = \varphi \circ \delta_2 \circ \alpha_2;$$

$$\pi \circ \sigma_2 \circ \alpha_{12} = \pi \circ \sigma_1 = \varphi \circ \delta_2 \circ \alpha_{12}.$$

Since A_2 is generated by the images of α_2 and α_{12} it follows that $\pi \circ \sigma_2 = \varphi \circ \delta_2$, as desired. ∎

8.13. Corollary. *Given a conditionally (semi)projective diagram as below, to the left, and a morphism $C \to E$ we obtain a conditionally (semi)projective diagram as below, to the right, using the induced morphisms.*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & D \end{tikzcd} \quad \textit{gives} \quad \begin{tikzcd} E \arrow[r] \arrow[d] & E \bigstar_C B \arrow[d] \\ E \bigstar_C A \arrow[r] & E \bigstar_C D \end{tikzcd}$$

8.14. *Remark.* The result in 8.12 subsumes the versions given in [21, 5.1.2] (where $B_1 = B_2$ and $D_1 = D_2$ and also [21, 5.3.1] (where $B_i = C_i$ for $i = 1, 2$).

For later use (in 10.9) we state the following easy result about sequences of diagrams. If the sequences were finite we could actually replace projectivity with semiprojectivity throughout, but in the infinite case this will fail.

8.15. Proposition. *If we have a sequence of conditionally projective diagrams of C^*-algebras as below, to the left, and each morphism α_n is proper, we obtain the conditionally projective diagram below, to the right.*

$$\begin{tikzcd} C_n \arrow[r, "\beta_n"'] \arrow[d, "\alpha_n"] & B_n \arrow[d, "\gamma_n"] \\ A_n \arrow[r, "\delta_n"] & D_n \end{tikzcd} \quad \textit{gives} \quad \begin{tikzcd} \bigoplus C_n \arrow[r, "\oplus \beta_n"'] \arrow[d, "\oplus \alpha_n"] & \bigoplus B_n \arrow[d, "\oplus \gamma_n"] \\ \bigoplus A_n \arrow[r, "\oplus \delta_n"] & \bigoplus D_n \end{tikzcd}$$

Proof. Consider a quotient morphism $\pi : M \to Q$ between C^*-algebras and assume that we have a coherent pair of morphisms $\psi : \bigoplus B_n \to M$ and $\varphi : \bigoplus D_n \to Q$. Let M_n and Q_n denote the hereditary C^*-subalgebras of M and Q generated by $\psi(\beta_n(C_n))$ and $\varphi(\delta_n(A_n))$, respectively, and put $\pi_n = \pi \mid M_n$. Since Q_n is hereditarily generated by $\varphi(\delta_n(A_n))$ and α_n is proper it follows from Lemma 4.6 that π_n is a proper morphism. But any restriction of a surjective morphism to hereditary subalgebras, if proper, must actually be surjective, so $\pi(M_n) = Q_n$. As the nth diagram is conditionally projective there is a morphism $\sigma_n : A_n \to M_n$ such that $\sigma_n \circ \alpha_n = \psi \circ \beta_n$ and $\pi_n \circ \sigma_n = \varphi \circ \delta_n$. We can now define the morphism $\sigma = \bigoplus \sigma_n$ from $\bigoplus A_n$ into M, and evidently $\sigma \circ \bigoplus \alpha_n = \psi \circ \bigoplus \beta_n$ and $\pi \circ \sigma = \varphi \circ \bigoplus \delta_n$, as desired. ∎

9. PULLBACKS AND PUSHOUTS OF EXTENSIONS

We shall extend the two propositions 8.9 and 8.10 considerably (at the cost of some work and more loss of simplicity) by considering pullbacks and pushouts of triples of extensions. Note that although formulated for

C^*-algebras the results in 9.1 and 9.2 are valid in the category of abelian groups. By contrast, those in 9.3 and 9.4 use concepts (approximate units, proper morphisms) that are special to C^*-algebras (or at least to Banach algebras).

9.1. THEOREM. *Consider a commutative diagram whose rows are extensions of C^*-algebras*

$$\begin{tikzcd}
0 \arrow[r] & A_0 \arrow[r, hook] \arrow[d, "\alpha"] & A \arrow[r, "\pi"] \arrow[d, "\bar{\alpha}"] & A_1 \arrow[r] \arrow[d, "\tilde{\alpha}"] & 0 \\
0 \arrow[r] & C_0 \arrow[r, hook] & C \arrow[r, "\pi"] & C_1 \arrow[r] & 0 \\
0 \arrow[r] & B_0 \arrow[r, hook] \arrow[u, "\beta"] & B \arrow[r, "\pi"] \arrow[u, "\bar{\beta}"] & B_1 \arrow[r] \arrow[u, "\tilde{\beta}"] & 0
\end{tikzcd}$$

There is then a commutative diagram in box form obtained by forming the pullbacks $X_0 = A_0 \oplus_{C_0} B_0$, $X = A \oplus_C B$, and $X_1 = A_1 \oplus_{C_1} B_1$, and taking induced morphisms σ and ρ:

$$\begin{tikzcd}
X_0 \arrow[rr, "\sigma"] \arrow[dr, "\gamma"] \arrow[dd, "\delta"] & & X \arrow[rr, "\rho"] \arrow[dr, "\bar{\gamma}"] \arrow[dd, "\bar{\delta}"] & & X_1 \arrow[dr, "\tilde{\gamma}"] \arrow[dd, "\tilde{\delta}"] & \\
& B_0 \arrow[rr] \arrow[dd, "\beta"] & & B \arrow[rr, "\pi"] \arrow[dd, "\bar{\beta}"] & & B_1 \arrow[dd, "\tilde{\beta}"] \\
A_0 \arrow[rr] \arrow[dr, "\alpha"] & & A \arrow[rr, "\pi"] \arrow[dr, "\bar{\alpha}"] & & A_1 \arrow[dr, "\tilde{\alpha}"] & \\
& C_0 \arrow[rr] & & C \arrow[rr, "\pi"] & & C_1
\end{tikzcd}$$

We always have that σ is injective with $\sigma(X_0) = \ker \rho$. Moreover, $\rho(X) = X_1$ (so that we have an extension $X_0 \to X \to X_1$) if and only if

$$\bar{\alpha}(A) \cap \bar{\beta}(B) \cap C_0 = \alpha(A_0) \cap \bar{\beta}(B) + \bar{\alpha}(A) \cap \beta(B_0).$$

Proof. The two projections γ and δ of X_0, composed with the embeddings of A_0 and B_0 into A and B, respectively, form a coherent pair and therefore determine a morphism σ: $X_0 \to X$ with

$$\ker \sigma = \{(x, y) \in A_0 \oplus_{C_0} B_0 \mid x = 0, y = 0\} = \{0\}.$$

Similarly the projections $\bar{\gamma}$ and $\bar{\delta}$ of X composed with the quotient morphisms π of A and B onto A_1 and B_1, respectively, form a coherent pair and therefore determine a morphism $\rho\colon X \to X_1$. Here we find that

$$\begin{aligned}\ker \rho &= \{(a, b) \in A \oplus_C B \mid \pi(a) = \pi(b) = 0\} \\ &= \{(a, b) \in A_0 \times B_0 \mid \alpha(a) = \beta(b)\} = A_0 \oplus_{C_0} B_0.\end{aligned}$$

Assume now that the formally weaker condition

$$\bar{\alpha}(A) \cap \bar{\beta}(B) \cap C_0 \subset \alpha(A_0) + \beta(B_0) \tag{$*$}$$

is satisfied, and let us show that $\rho(X) = X_1$. If $(a_1, b_1) \in X_1$ then

$$a_1 \in \tilde{\alpha}^{-1}(\tilde{\beta}(B_1)) = \tilde{\alpha}^{-1}(\tilde{\beta}(\pi(B))) = \tilde{\alpha}^{-1}(\pi(\bar{\beta}(B))) = \pi(\bar{\alpha}^{-1}(\bar{\beta}(B))),$$

because the quotient morphisms are surjective. Similarly $b_1 \in \pi(\bar{\beta}^{-1}(\bar{\alpha}(A)))$. Thus we can find (a, b') and (a', b) in $A \times B$, such that

$$\pi(a) = a_1, \qquad \bar{\alpha}(a) = \bar{\beta}(b'), \qquad \pi(b) = b_1, \qquad \bar{\beta}(b) = \bar{\alpha}(a').$$

It follows that $(a - a', b' - b) \in X$. Moreover,

$$\pi(\bar{\alpha}(a - a')) = \tilde{\alpha}(\pi(a)) - \pi(\bar{\beta}(b)) = \tilde{\alpha}(a_1) - \tilde{\beta}(b_1) = 0,$$

so that

$$z = \bar{\alpha}(a - a') = \bar{\beta}(b' - b) \in \bar{\alpha}(A) \cap \bar{\beta}(B) \cap C_0.$$

By assumption there are elements x in A_0 and y in B_0 such that $z = \alpha(x) + \beta(y)$. But then

$$\bar{\alpha}(a - x) = z + \bar{\alpha}(a') - \alpha(x) = \alpha(x) + \beta(y) + \bar{\beta}(b) - \alpha(x) = \bar{\beta}(b + y),$$

so that $(a - x, b + y) \in X$, with $\rho(a - x, b + y) = (a_1, b_1)$.

Conversely, assume that $\rho(X) = X_1$. Note first that for any pair (a, b) in $A \times B$ such that $\bar{\alpha}(a) \in C_0$ and $\bar{\beta}(b) \in C_0$ we have $(\pi(a), \pi(b)) \in X_1$. By assumption we can therefore find (a', b') in X such that $\rho(a', b') = (\pi(a), \pi(b))$; i.e. $a - a' = x$ and $b - b' = y$ for some (x, y) $A_0 \times B_0$. It follows that $\bar{\alpha}(a) - \bar{\beta}(b) = \alpha(x) - \beta(y)$. Thus

$$\bar{\alpha}(A) \cap C_0 + \bar{\beta}(B) \cap C_0 \subset \alpha(A_0) + \beta(B_0).$$

In particular,

$$\begin{aligned}\bar{\alpha}(A)\cap\bar{\beta}(B)\cap C_0 &\subset (\alpha(A_0)+\beta(B_0))\cap\bar{\alpha}(A)\cap\bar{\alpha}(B)\\ &\subset \alpha(A_0)\cap\bar{\beta}(B)+\bar{\alpha}(A)\cap\beta(B_0)\\ &\subset \bar{\alpha}(A)\cap\bar{\beta}(B)\cap C_0. \quad \blacksquare\end{aligned}$$

9.2. PROPOSITION. *Consider a commutative diagram of C^*-algebras in the box form shown in Theorem 9.1 and assume that all four horizontal rows are extensions. Assume furthermore that the vertical square in the middle is a pullback and that*

$$\ker\tilde{\gamma}\cap\ker\tilde{\delta}=\{0\} \qquad \textit{and} \qquad \tilde{\alpha}(A)\cap\tilde{\beta}(B)\cap C_0\subset\alpha(A_0)+\beta(B_0).$$

Then both the left and right vertical squares are pullbacks.

Proof. With notations as in 9.1 we observe that the coherent pair (δ, γ) determines a morphism $\eta\colon X_0\to A_0\oplus_{C_0}B_0$. If $x_0\in\ker\eta$, then $\gamma(x_0)=\delta(x_0)=0$, whence also $\bar{\gamma}(x_0)=\bar{\delta}(x_0)=0$. But then $x_0=0$, since $X=A\oplus_C B$; proving that η is injective.

To show that η is surjective take (a_0, b_0) in $A_0\oplus_{C_0}B_0$. By assumption there is then an element x in X such that $\bar{\delta}(x)=a_0$ and $\bar{\gamma}(x)=b_0$. It follows that

$$\pi(x)\in\ker\tilde{\gamma}\cap\ker\tilde{\delta}=\{0\},$$

and thus $x\in X_0(=\ker\pi)$, as desired.

Concentrating now on the right vertical square we see that the coherent pair $(\tilde{\delta}, \tilde{\gamma})$ determines an injective morphism $\tilde{\eta}\colon X_1\to A_1\oplus_{C_1}B_1$. However, replacing X_1 with $A_1\oplus_{C_1}B_1$ in the diagram we see that the conditions in Theorem 9.1 are fulfilled, so that the induced morphism $\tilde{\eta}\circ\pi$ is the quotient morphism of X onto $A_1\oplus_{C_1}B_1=\tilde{\eta}(X_1)$. $\blacksquare$

9.3. THEOREM. *Consider a commutative diagram of extensions of C^*-algebras*

$$\begin{tikzcd}
0 \arrow[r] & A_0 \arrow[r, hook] & A \arrow[r, "\pi"] & A_1 \arrow[r] & 0\\
0 \arrow[r] & C_0 \arrow[r, hook] \arrow[u, "\alpha"] \arrow[d, "\beta"] & C \arrow[r, "\pi"] \arrow[u, "\bar{\alpha}"] \arrow[d, "\bar{\beta}"] & C_1 \arrow[r] \arrow[u, "\tilde{\alpha}"] \arrow[d, "\tilde{\beta}"] & 0\\
0 \arrow[r] & B_0 \arrow[r, hook] & B \arrow[r, "\pi"] & B_1 \arrow[r] & 0
\end{tikzcd}$$

There is then a commutative diagram in box form obtained by forming the pushouts $X_0 = A_0 \bigstar_{C_0} B_0$, $X = A \bigstar_C B$ *and* $X_1 = A_1 \bigstar_{C_1} B_1$ *and taking induced morphisms* σ *and* ρ:

$$
\begin{tikzcd}
C_0 \arrow[rr] \arrow[dr, "\beta"] \arrow[dd, "\alpha"'] & & C \arrow[rr, "\pi"] \arrow[dr, "\bar{\beta}"] \arrow[dd, "\bar{\alpha}"'] & & C_1 \arrow[dr, "\tilde{\beta}"] \arrow[dd, "\tilde{\alpha}"'] & \\
& B_0 \arrow[rr] \arrow[dd, "\gamma"] & & B \arrow[rr, "\pi"] \arrow[dd, "\bar{\gamma}"] & & B_1 \arrow[dd, "\tilde{\gamma}"] \\
A_0 \arrow[rr] \arrow[dr, "\delta"'] & & A \arrow[rr, "\pi"] \arrow[dr, "\bar{\delta}"'] & & A_1 \arrow[dr, "\tilde{\delta}"'] & \\
& X_0 \arrow[rr, "\sigma"] & & X \arrow[rr, "\rho"] & & X_1
\end{tikzcd}
$$

We always have that $\rho(X) = X_1$ *and* $\sigma(X_0) \subset \ker \rho$. *Moreover, if* α *and* β *are proper morphisms, then* $\sigma(X_0) = \ker \rho$ *and* σ *is injective, so that we have an extension* $X_0 \to X \to X_1$.

Proof. The two embeddings of A_0 and B_0 followed by the morphisms $\bar{\gamma}$ and $\bar{\delta}$, respectively, form a coherent pair and therefore determine a unique morphism $\sigma : X_0 \to X$ such that $\sigma \circ \delta = \bar{\delta}$ and $\sigma \circ \gamma = \bar{\gamma}$. Similarly, the two morphisms $\tilde{\delta} \circ \pi$ and $\tilde{\gamma} \circ \pi$ form a coherent pair and therefore determine a morphism $\rho : X \to X_1$ such that $\rho \circ \bar{\delta} = \tilde{\delta} \circ \pi$ and $\rho \circ \bar{\gamma} = \tilde{\gamma} \circ \pi$. Thus,

$$\rho(X) \supset \tilde{\delta}(\pi(A)) \cup \tilde{\gamma}(\pi(B)) = \tilde{\delta}(A_1) \cup \tilde{\gamma}(B_1),$$

and the latter set generates the pushout algebra X_1, proving that ρ is surjective. Moreover, $\rho \circ \sigma$ is zero both on $\delta(A_0)$ and on $\gamma(B_0)$, and since their union generates X_0, it follows that $\rho \circ \sigma = 0$, i.e., $\sigma(X_0) \subset \ker \rho$.

Assume now that α and β are proper morphisms. It follows from Lemma 4.6 that also γ and δ are proper, so that $X_0 = \delta(A_0)\, X_0 = \gamma(B_0)\, X_0$. Consequently,

$$\bar{\delta}(A)\, \sigma(X_0) = \bar{\delta}(A)\, \sigma(\delta(A_0)\, X_0) = \bar{\delta}(A)\, \bar{\delta}(A_0)\, \sigma(X_0) \subset \bar{\delta}(A_0)\, \sigma(X_0)$$

$$= \sigma(\delta(A_0)\, X_0) = \sigma(X_0).$$

Similarly, $\bar{\gamma}(B)\, \sigma(X_0) \subset \sigma(X_0)$, and since X is generated by $\bar{\delta}(A) \cup \bar{\gamma}(B)$ it follows that $\sigma(X_0)$ is an ideal in X.

By Theorem 2.5 the diagram to the left, below, is a pushout, and applying Theorem 8.3 to the box on the right in the large diagram above it follows that also the diagram to the right, below, is a pushout.

$$\begin{array}{ccc} C & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ A & \longrightarrow & A_1 \end{array} \qquad\qquad \begin{array}{ccc} B & \longrightarrow & B_1 \\ \downarrow{\scriptstyle \bar{\gamma}} & & \downarrow{\scriptstyle \tilde{\gamma}} \\ X & \xrightarrow{\rho} & X_1 \end{array}$$

But then, again by Theorem 2.5, $\ker \rho$ is generated as an ideal by $\bar{\gamma}(B_0)$. We have

$$\bar{\gamma}(B_0) \subset \sigma(X_0) \subset \ker \rho,$$

and since $\sigma(X_0)$ is an ideal in X it follows that $\sigma(X_0) = \ker \rho$.

Let now $\varphi : A_0 \to \mathbb{B}(\mathfrak{H})$ and $\psi : B_0 \to \mathbb{B}(\mathfrak{H})$ be an arbitrary coherent pair of representations (i.e. $\varphi \circ \alpha = \psi \circ \beta$), and assume without loss of generality that the ensuing representation $\varphi \star \psi : X_0 \to \mathbb{B}(\mathfrak{H})$ is nondegenerate. Since both δ and γ are proper morphisms this implies that also φ and ψ are nondegenerate. Since A_0 is and ideal in A there is a unique extension $\bar{\varphi} : A \to \mathbb{B}(\mathfrak{H})$, determined by

$$\bar{\varphi}(a)\, \varphi(a_0)\, \xi = \varphi(aa_0)\, \xi, \qquad a \in A, \quad a_0 \in A_0, \quad \xi \in \mathfrak{H}.$$

Similarly we have an extension $\bar{\psi} : B \to \mathbb{B}(\mathfrak{H})$. For any elements c in C, c_0 in C_0 and ξ in $\mathfrak{H}$ we compute

$$\begin{aligned} \bar{\varphi}(\bar{\alpha}(c))\, \varphi(\alpha(c_0))\, \xi &= \varphi(\bar{\alpha}(c)\, \alpha(c_0))\, \xi = \varphi(\alpha(cc_0))\, \xi = \psi(\beta(cc_0))\, \xi \\ &= \bar{\psi}(\bar{\beta}(c))\, \psi(\beta(c_0))\, \xi = \bar{\psi}(\bar{\beta}(c))\, \varphi(\alpha(c_0))\, \xi. \end{aligned}$$

Since α is a proper morphism and φ is nondegenerate we have $\varphi(\alpha(C_0))\, \mathfrak{H} = \mathfrak{H}$, whence $\bar{\varphi} \circ \bar{\alpha} = \bar{\psi} \circ \bar{\beta}$. The coherent pair $(\bar{\varphi}, \bar{\psi})$ therefore determines a representation $\bar{\varphi} \star \bar{\psi} : X \to \mathbb{B}(\mathfrak{H})$ such that $\bar{\varphi} = (\bar{\varphi} \star \bar{\psi}) \circ \bar{\delta}$ and $\bar{\psi} = (\bar{\varphi} \star \bar{\psi}) \circ \bar{\gamma}$. Restricting to A_0 and B_0 we see that $\varphi = (\bar{\varphi} \star \bar{\psi}) \circ \sigma \circ \delta$ and $\psi = (\bar{\varphi} \star \bar{\psi}) \circ \sigma \circ \gamma$, which implies that $\varphi \star \psi = (\bar{\varphi} \star \bar{\psi}) \circ \sigma$. Since φ and ψ were arbitrary, this proves that σ is injective, completing the argument. ∎

9.4. Proposition. *Consider a commutative diagram of C^*-algebras in the box form shown in Theorem 9.3, and assume that all four horizontal rows are extensions. Assume furthermore that the vertical square in the middle is a pushout and that α and β are both proper morphisms. Then both the right and the left vertical squares are pushouts.*

Proof. As in the proof of 9.3 we let $\varphi : A_0 \to \mathbb{B}(\mathfrak{H})$ and $\psi : B_0 \to \mathbb{B}(\mathfrak{H})$ be a coherent pair of representations (i.e., $\varphi \circ \alpha = \psi \circ \beta$), and without loss of generality we assume also that the ensuing representation $\varphi \star \psi$ of $A_0 \star_{C_0} B_0$ is nondegenerate. Since both α and β are proper morphisms, so are φ and ψ and $\varphi \circ \alpha (= \psi \circ \beta)$, and thus both φ and ψ are nondegenerate on $\mathfrak{H}$. There are therefor unique extensions $\bar{\varphi} : A \to \mathbb{B}(\mathfrak{H})$ and $\bar{\psi} : B \to \mathbb{B}(\mathfrak{H})$, and as in the proof of 9.3 we conclude that $\bar{\varphi} \circ \bar{\alpha} = \bar{\psi} \circ \bar{\beta}$. By assumption $X = A \star_C B$, so there is a morphism $\bar{\eta} : X \to \mathbb{B}(\mathfrak{H})$ such that $\bar{\varphi} = \bar{\eta} \circ \bar{\delta}$ and $\bar{\psi} = \bar{\eta} \circ \bar{\alpha}$. Taking $\eta = \bar{\eta} \mid X_0$ we see that $\varphi = \eta \circ \delta$ and $\psi = \eta \circ \alpha$, which proves that $X_0 = A_0 \star_{C_0} B_0$, as desired.

To show that also the vertical square on the right is a pushout we note that γ is proper (since α is), and thus by Theorem 2.5 both the right front and the right back square are pushouts. It follows by Theorem 8.3 that also the right vertical square is a pushout, as claimed. ∎

9.5. Examples. The conditions in Theorem 9.3 that both morphisms α and β be proper are probably stronger than necessary to ensure that $X_0 \to X \to X_1$ is an extension. After all, Proposition 8.10 gives an extension when only β is proper (but then with $B_0 = B$ and $C_0 = C$). However, *some* conditions are needed.

Consider the diagram below with its three extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathbb{C}^2 \arrow[r] & \mathbb{C} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[u, "\alpha"] \arrow[r, equal] \arrow[d, "\beta"] & \mathbb{C} \arrow[u, "\bar{\alpha}"] \arrow[r] \arrow[d, "\bar{\beta}"] & 0 \arrow[u, "\tilde{\alpha}"] \arrow[r] \arrow[d, "\tilde{\beta}"] & 0 \\
0 \arrow[r] & \mathbb{M}_2 \arrow[r, equal] & \mathbb{M}_2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Here $\beta(\mathbf{1}) = e_{11}$ (and $\bar{\beta} = \beta$), so this morphism is not proper (although $\mathrm{Id}(\beta(\mathbb{C})) = \mathbb{M}_2$). Otherwise all the conditions from Theorem 9.3 are fulfilled. In fact, this diagram is of the simple form already described in Proposition 8.10. However, the resulting sequence of amalgamated free products is given by

$$\mathbb{M}_2 \xrightarrow{\sigma} \mathbb{M}_2 \star_{\mathbb{C}} \mathbb{C}^2 \xrightarrow{\rho} \mathbb{C},$$

and as we saw in Example 5.4 this is not an extension, because $\sigma(\mathbb{M}_2)$ is not an ideal in the middle algebra (although it generates $\ker \rho$ as an ideal).

Another rich source of (counter-)examples arise from commutative diagrams of the form

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, equal] & X \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u] \arrow[d, "\beta"] & X \arrow[r, "\pi"] \arrow[u, equal] \arrow[d, "\bar{\beta}"] & B \arrow[r] \arrow[u] \arrow[d, "\tilde{\beta}"] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & X_1 \arrow[r, "\pi_1"] & B_1 \arrow[r] & 0
\end{tikzcd}$$

Here the second and third rows are general extensions, chosen so that β is a proper morphism and such that A and A_1 are essential ideals in X and X_1, respectively. If $\eta : B \to Q(A)$ and $\eta_1 : B_1 \to Q(A_1)$ are the Busby invariants for these extensions, and if $\beta_0 : Q(A) \to Q(A_1)$ denotes the canonical morphism between the corona algebras obtained from β, then $\beta_0 \circ \eta = \eta_1 \circ \tilde{\beta}$, cf. [20, Theorem 2.2].

We let X_0 be the extension of A_1 by B with Busby invariant $\beta_0 \circ \eta$, so that we get a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\beta"] & X \arrow[r, "\pi"'] \arrow[d] & B \arrow[d, equal] \\
A_1 \arrow[r] \arrow[d, equal] & X_0 \arrow[r] \arrow[d, "\sigma"] & B \arrow[d, "\tilde{\beta}"] \\
A_1 \arrow[r] & X_1 \arrow[r, "\pi_1"] & B_1
\end{tikzcd}$$

It follows from Theorem 2.4 that NW square in this diagram is a pushout (and so is the SE square by Theorem 2.5).

We can now read off the pushout sequence from the original diagram of extension

$$A_1 \star_A X = X_0, \qquad X_1 \star_X X = X_1, \qquad B_1 \star_B 0 = B_1/I,$$

where I is the closed ideal in B_1 generated by $\tilde{\beta}(B)$. The sequence

$$0 \to X_0 \xrightarrow{\sigma} X_1 \to B_1/I \to 0$$

will almost never be exact; and it is easy to construct examples where σ is not injective (because, say, $\tilde{\beta}$ need not be injective) and $\sigma(X_0)$ is not an ideal in X_1 (because $\tilde{\beta}(B)$ is not an ideal in B_1). Specifically one may take

$A = c_0$, $X = \ell^\infty$, $B = \ell^\infty / c_0$ and $A_1 = \mathbb{K}$, $X_1 = \mathbb{B}(\ell^2)$, $B_1 = Q(\ell^2)$; and let $\beta((\lambda_n)) = \sum \lambda_{2n} \otimes e_{nn}$. We find that $X_0 = (\mathbb{K} + \ell^\infty_{2\mathbb{N}}) \oplus \ell^\infty_{2\mathbb{N}-1}$, with $\ell^\infty_{2\mathbb{N}-1} = \ker \sigma$. Note that all of the conditions in Theorem 9.3 are satisfied, except that the morphism α: $A \to X$ is not proper.

9.6. *Remark.* The observant reader will have noticed that the results about the sequences $X_0 \to X \to X_1$ that hold in Theorems 9.1 and 9.3 without any extra conditions all serve to show that the diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ X \arrow[r] & X_1 \end{tikzcd}$$

is a pullback, respectively a pushout. This is no coincidence, but follows from proper applications of Propositions 8.5 and 8.7.

10. STABLE PUSHOUTS AND PULLBACKS

10.1. *Notations.* For a given Hilbert space $\mathfrak{H}$ let $\mathbb{K}(\mathfrak{H})$, or just $\mathbb{K}$, denote the C^*-algebra of compact operators on $\mathfrak{H}$, and let $\{e_{ij} \mid (i, j) \in d \times d\}$ be a system of matrix units for $\mathbb{K}$ corresponding to a fixed orthonormal basis for $\mathfrak{H}$, where $\operatorname{card}\{d\} = \dim \mathfrak{H}$.

Let $\mathbf{C}\mathbb{C} = C_0(]0, 1])$ and $\mathbf{C}\mathbb{K} = C_0(]0, 1], \mathbb{K})$ denote the cones over $\mathbb{C}$ and $\mathbb{K}$, respectively, and define a morphism $\iota \otimes e_{11}$ between them by setting $(\iota \otimes e_{11})(f) = f \otimes e_{11}$ for f in $C_0(]0, 1])$.

(Strictly speaking, the cone over a (unital) C^*-algebra A is the algebra $\mathbf{C}(A)^\sim$, but it has become customary to refer to the open cylinder $\mathbf{C}(A)$ over A as the cone. The same convention apply to the name (double) suspension for the algebra $\mathbf{S}(A) = C_0(]-1, 1[) \otimes A$, which in a more puritan world would be reserved for the extension $\mathbf{S}(A)^{\sim\sim}$ determined by $\mathbf{S}(A) \to \mathbf{S}(A)^{\sim\sim} \to \mathbb{C}^2$).

Given a σ-unital C^*-algebra A we can define a proper morphism $\alpha : \mathbf{C}\mathbb{C} \to A$ by choosing a strictly positive element h in A and with $\mathrm{id}(t) = t$ the identity function let $\alpha(\mathrm{id}) = h$, so that

$$\alpha(f) = f(h), \qquad f \in \mathbf{C}\mathbb{C}.$$

(Every proper morphism of $\mathbf{C}\mathbb{C}$ has this form.) We also define the morphisms

$$e_{11} \otimes \iota : A \to \mathbb{K} \otimes A$$

$$\alpha \otimes \iota : \mathbf{C}\mathbb{K} \to \mathbb{K} \otimes A$$

and we note that $(\alpha \otimes \imath)(f \otimes e_{ij}) = e_{ij} \otimes f(h)$. (The change of order among the tensor factors is dictated by the desire to represent both coning and stabilizing as functors $\mathbf{C}$ and $\mathbb{K}\otimes$ on $\mathscr{C}^*$.)

10.2. Proposition (Cf. [33, 6.2.2]). *With notations as above, the diagram*

$$\begin{array}{ccc} \mathbf{C}\mathbb{C} & \xrightarrow{e_{11}\otimes \imath} & \mathbf{C}\mathbb{K} \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha\otimes\imath} \\ A & \xrightarrow{e_{11}\otimes \imath} & \mathbb{K}\otimes A \end{array}$$

is a pushout, so that $\mathbb{K}\otimes A = \mathbf{C}\mathbb{K} \star_{\mathbf{C}\mathbb{C}} A$.

Proof. Evidently the diagram is commutative. To prove that it is a pushout, consider a pair $\varphi : A \to Y$ and $\psi : \mathbf{C}\mathbb{K} \to Y$ of coherent morphism $(\varphi \circ \alpha = \psi \circ (\imath \otimes e_{11}))$ into some C^*-algebra Y. Without loss of generality we may assume that Y is generated by $\varphi(A) \cup \psi(\mathbf{C}\mathbb{K})$, which by Lemma 4.6 implies that ψ is a proper morphism. Representing Y faithfully and non-degenerately on some Hilbert space $\mathfrak{K}$ we therefore have a unique extension of ψ to a unital morphism

$$\bar{\psi} : M(\mathbf{C}\mathbb{K}) \to \mathbb{B}(\mathfrak{K}),$$

cf. 7.1. Identifying $\mathbb{B}(\mathfrak{H})$ with a C^*-subalgebra of $M(\mathbf{C}\mathbb{K})$ via the map $x \to \mathbf{1} \otimes x$ we obtain the morphism $\rho : \mathbb{K} \to \mathbb{B}(\mathfrak{K})$, where $\rho(x) = \bar{\psi}(\mathbf{1} \otimes x)$.

With $h = \alpha(\mathrm{id})$ strictly positive in A we have

$$\varphi(h) = \varphi \circ \alpha(\mathrm{id}) = \psi(e_{11} \otimes \mathrm{id}) \leqslant \bar{\psi}(e_{11} \otimes \mathbf{1}) = \rho(e_{11}).$$

This implies that $\varphi(A) \subset \rho(e_{11})\, Y \rho(e_{11})$. We can therefore define a whole set of morphisms

$$\varphi_i : A \to Y, \qquad \varphi_i(a) = \rho(e_{i1})\, \varphi(a)\, \rho(e_{1i}).$$

As these have orthogonal ranges we now define

$$\bar{\varphi} : A \to \mathbb{B}(\mathfrak{K}), \qquad \bar{\varphi} = \bigoplus \varphi_i.$$

Since $\bar{\varphi}(a)\, \rho(e_{ij}) = \rho(e_{i1})\, \varphi(a)\, \rho(e_{1j})$ it follows that $\bar{\varphi}(A)$ commutes with $\rho(\mathbb{K})$, and we obtain the morphism $\sigma = \rho \otimes \bar{\varphi}$ of $\mathbb{K} \otimes A$ into $\mathbb{B}(\mathfrak{K})$, in fact into Y. If $a \in A$ then

$$\sigma(e_{11} \otimes a) = \rho(e_{11})\, \bar{\varphi}(a) = \rho(e_{11})\, \varphi_1(a) = \varphi(a);$$

and if $f \otimes e_{ij} \in \mathbf{C}\mathbb{K}$ then

$$\begin{aligned}\sigma(e_{ij} \otimes f(h)) &= \rho(e_{ij})\, \bar{\varphi}(f(h)) = \rho(e_{ij})\, \varphi_i(f(h)) \\ &= \bar{\psi}(\mathbf{1} \otimes e_{i1})\, \varphi(f(h))\, \bar{\psi}(\mathbf{1} \otimes e_{ij}) \\ &= \bar{\psi}(\mathbf{1} \otimes e_{i1})\, \psi(f \otimes e_{11})\, \bar{\psi}(\mathbf{1} \otimes e_{ij}) = \psi(f \otimes e_{ij}).\end{aligned}$$

We have shown that $\sigma \circ (e_{11} \otimes \iota) = \varphi$ and $\sigma \circ (\alpha \otimes \iota) = \psi$, and the proof is complete. ▮

10.3. *Remark.* If A is a unital C^*-algebra the same proof as above will show that

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\quad e_{11} \quad} & \mathbb{K} \\ \Big\downarrow{\scriptstyle \mathbf{1}} & & \Big\downarrow{\scriptstyle \mathbf{1} \otimes \iota} \\ A & \xrightarrow{e_{11} \otimes \iota} & \mathbb{K} \otimes A \end{array}$$

is a pushout diagram. In view of Remark 5.10—which expressly ruled out tensoring pushout diagrams such as

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \Big\downarrow & & \Big\downarrow \\ A & \longrightarrow & A \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{K} \\ \Big\downarrow & & \Big\downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{K} \end{array}$$

—this may come as a surprise. Evidently it is the embedding of $\mathbb{C}$ into multiples of a minimal projection (with central support $\mathbf{1}$) that makes the construction go through. That also indicates that there is not much hope for generalizing Proposition 10.2 from $\mathbb{M}_n$ and $\mathbb{K}$ to other C^*-algebras. However, focusing on the fact that $e_{11} \otimes \iota$ is an embedding of $\mathbb{C}$ as a full, hereditary C^*-subalgebra of $\mathbf{C}\mathbb{K}$, we see that 10.2 as well as Theorem 10.4 may be regarded as special cases of Theorem 5.11; cf. Remark 5.12.

The next result "explains" why stabilizing is a structure preserving operation. The diagram in Theorem 10.4 has appeared before; see [8, 3.15] and [6, 2.7].

10.4. Theorem. *If $\alpha : A \to B$ is a proper morphism between σ-unital C^*-algebras, then the diagram below is a pushout*:

$$\begin{array}{ccc} A & \xrightarrow{e_{11} \otimes \iota} & \mathbb{K} \otimes A \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \iota \otimes \alpha} \\ B & \xrightarrow{e_{11} \otimes \iota} & \mathbb{K} \otimes B \end{array}$$

Proof. If h is a strictly positive element in A, then $\alpha(h)$ is strictly positive in B, and we can define proper morphisms $\alpha_1 : \mathbf{C}\mathbb{C} \to A$ and

$\alpha_2 : \mathbf{C}\mathbb{C} \to B$ such that $\alpha_1(\mathrm{id}) = h$ and $\alpha_2 = \alpha \circ \alpha_1$. From this we obtain the commutative diagram

$$\begin{tikzcd}
 & A \arrow[rr, "e_{11}\otimes\iota"] \arrow[dd, "\alpha"] & & \mathbb{K}\otimes A \arrow[dd, "\iota\otimes\alpha"] \\
\mathbf{C}\mathbb{C} \arrow[ur, "\alpha_1"] \arrow[rr, "e_{11}\otimes\iota"] \arrow[dr, "\alpha_2"'] & & \mathbf{C}\mathbb{K} \arrow[ur, "\iota\otimes\alpha_1"] \arrow[dr, "\iota\otimes\alpha_2"] & \\
 & B \arrow[rr, "e_{11}\otimes\iota"] & & \mathbb{K}\otimes B
\end{tikzcd}$$

By Theorem 10.2 we know that the front faces of the prism are pushouts, and it follows from Corollary 8.4 that the hind face is a pushout as well. ∎

10.5. *Corona Extendibility.* Recall from [35, 1.1] (see also [20, Sect. 5] and [39, Sect. 8]) that a morphism $\alpha : A \to B$ between C^*-algebras is *corona extendible* if every morphism $\varphi : A \to Q(E)$ of A into the corona algebra of some σ-unital C^*-algebra E extends to a morphism $\bar{\varphi} : B \to Q(E)$, where $\bar{\varphi} \circ \alpha = \varphi$. By necessity such α must be injective. Rephrased in terms of extensions α is corona extendible if for every X in $\mathrm{ext}(E, A)$ there is a Y in $\mathrm{ext}(E, B)$ together with a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, hook] \arrow[d, equal] & X \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & E \arrow[r, hook] & Y \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

Corona extendible morphisms are rare. A useful weakening of the concept produces many more examples: If $\mathscr{B}$ is a class of unital C^*-algebras we say that $\alpha : A \to B$ is *weakly corona extendible* with respect to $\mathscr{B}$ if every morphism $\varphi : A \to Q(E)$ extends whenever

$$E = \bigoplus B_n, \qquad \{B_n\} \subset \mathscr{B}.$$

In applications $\mathscr{B}$ could be the class of all unital C^*-algebras of real rank zero (and/or of stable ranke one), or it could be the class of all matrix algebras (equivalently, the class of all finite-dimensional C^*-algebras). In the latter case we say that α is *matrically corona extendible*, cf. [20, Sect. 7].

10.6. Theorem (Cf. [21, 7.1.2]). *Let $\alpha: A \to B$ be a proper morphism between σ-unital C^*-algebras, and assume that α is corona extendible. Then also the morphism $\iota \otimes \alpha : X \otimes A \to X \otimes B$ is corona extendible for every separable, dual C^*-algebra X.*

Proof. Since X is separable and dual it has the form

$$X = \bigoplus \mathbb{K}(\mathfrak{H}_n)$$

for a sequence $(\mathfrak{H}_n)$ of separable Hilbert spaces.

Now let $\varphi : X \otimes A \to Q(E)$ be a morphism into some corona algebra $Q(E) = M(E)/E$ for a σ-unital C^*-algebra E. If h is a strictly positive element in A and k_n is strictly positive in $\mathbb{K}(\mathfrak{H}_n)$, then with $x_n = \varphi(k_n \otimes h)$ we have an orthogonal sequence in $Q(E)$.

For each n let $\varphi_n = \varphi \mid \mathbb{K}(\mathfrak{H}_n) \otimes A$. Since by Theorem 10.4 we have a pushout diagram and α is corona extendible, it follows that the morphism $\iota \otimes \alpha \mid \mathbb{K}(\mathfrak{H}_n) \otimes A$ is corona extendible, see [35, Lemma 3.3]. Thus we can find a morphism $\bar{\varphi}_n : \mathbb{K}(\mathfrak{H}_n) \otimes B \to Q(E)$ such that $\bar{\varphi}_n \circ (\iota \otimes \alpha) = \varphi_n$.

Since α is proper, $B = \alpha(A)\, B\alpha(A)$, and thus

$$\bar{\varphi}_n(\mathbb{K}(\mathfrak{H}_n) \otimes B) \subset (x_n Q(E)\, x_n)^{=}.$$

These algebras are pairwise orthogonal and we can therefore unambiguously define $\bar{\varphi} = \bigoplus \bar{\varphi}_n$ to obtain a morphism of $X \otimes B$ into $Q(E)$, and evidently

$$\bar{\varphi} \circ (\iota \otimes \alpha) = \bigoplus \bar{\varphi}_n \circ (\iota \otimes \alpha) = \bigoplus \varphi_n = \varphi. \quad \blacksquare$$

10.7. Corollary. *If $\alpha : A \to B$ is a proper morphism between σ-unital C^*-algebras and α is weakly corona extendible with respect to some class $\mathscr{B}$, then also the morphism $\iota \otimes \alpha : X \otimes A \to X \otimes B$ is weakly corona extendible for any separable, dual C^*-algebra X.*

Proof. The previous proof applies verbatim. $\blacksquare$

10.8. *Remark.* Both corona extendibility and conditional projectivity (cf. 8.11) are concerned with extensions of morphisms, but otherwise not related. They come together in [21, 5.3.2], where it is shown that in a commutative diagram of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \hookrightarrow & X & \longrightarrow & B & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & A & \hookrightarrow & X_1 & \longrightarrow & B_1 & \longrightarrow & 0
\end{array}$$

the morphism χ is corona extendible if β is conditionally projective.

10.9. THEOREM. *If we have a conditionally projective diagram of* C^**-algebras as below, to the left, with* α *a proper morphism, we obtain for every separable, dual* C^**-algebra* X *a conditionally projective diagram as below, to the right*

$$\begin{array}{ccc} C & \xrightarrow{\beta} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ A & \xrightarrow{\delta} & D \end{array} \qquad \textit{gives} \qquad \begin{array}{ccc} X\otimes C & \xrightarrow{\iota\otimes\beta} & X\otimes B \\ \downarrow{\scriptstyle\iota\otimes\alpha} & & \downarrow{\scriptstyle\iota\otimes\gamma} \\ X\otimes A & \xrightarrow{\iota\otimes\delta} & X\otimes D \end{array}$$

Proof. If $X = \mathbb{K}(\mathfrak{H})$ the result follows by combining Proposition 8.12 with Theorem 10.4.

In the general case we have $X = \bigoplus \mathbb{K}(\mathfrak{H}_n)$ for a sequence $(\mathfrak{H}_n)$ of separable Hilbert spaces. Since each morphism $\iota_n \otimes \alpha$ is proper, we can apply Proposition 8.15 to obtain the desired result.

10.10. *Full Corner Subalgebras.* Recall from [6, Sect. 1] that a hereditary C^*-subalgebra A of B is called a *corner* of B if $A = pBp$ for some projection p in $M(B)$. Like other hereditary subalgebras a corner is called *full* if it is not contained in any proper, closed ideal of B. Not all hereditary subalgebras are corners. Probably the simplest example of a full, hereditary C^*-subalgebras which is not a corner is obtained by taking $B = c \otimes \mathbb{M}_2$ (the algebra of convergent sequences of matrices) and letting A be the subalgebra of sequences (x_n) such that $\lim x_n \in \mathbb{C}e_{11}$.

Corners were devised by L. G. Brown as a tool for proving the stable isomorphism theorem in [6, 2.8]. The following concrete description of corners is an application of the techniques from [6] and [9]. The argument is essentially due to Brown (private communication).

From this point onward $\mathbb{K} = \mathbb{K}(\ell^2)$.

10.11. LEMMA. *Let* A *be a* σ*-unital* C^**-algebra and* q *a projection in* $M(\mathbb{K}\otimes A)$ *such that* $q(\mathbb{K}\otimes A)\,q$ *is a full corner in* $\mathbb{K}\otimes A$ *isomorphic to* A*. Then* $\bar{\alpha}(q)$ *is Murray–von Neumann equivalent to* $e_{11}\otimes 1_A$ *in* $M(\mathbb{K}\otimes A)$ *for some automorphism* α *of* $\mathbb{K}\otimes A$*.*

Proof. Let $B = \mathbb{K}\otimes A$ and identify A with $e_{11}\otimes A$ in B. Moreover, denote by $\varphi : A \to qBq$ the given isomorphism, and put $e = e_{11}\otimes 1_A$ in $M(B)$. Then the two left ideals $X = Be$ and $Y = Bq$ are both $B-A$ imprimitivity bimodules, cf. [42, 6.10]; X with the obvious products and Y with the right inner product and the right module action defined by

$$(bq \mid cq)_A = \varphi^{-1}(qb^*cq) \qquad \text{and} \qquad bq\cdot a = bq\varphi(a), \quad a\in A, \quad b, c\in B.$$

If $\hat{Y}$ denotes the adjoint $A-B$ bimodule—identified with the right ideal qB—we let $X \otimes_A \hat{Y}$ be the quotient space of $X \otimes \hat{Y}$ by the closed subspace spanned by elements

$$\{bea \otimes qc - be \otimes \varphi(a)\, qc \mid a \in A,\, b,\, c \in B\}.$$

Equipped with the obvious B-actions

$$b \cdot (b_1 e \otimes qb_2) = bb_1 e \otimes qb_2, \qquad (b_1 e \otimes qb_2) \cdot b = b_1 e \otimes qb_2 b,$$

and the inner products (on equivalence classes, really)

$$(b_1 e \otimes qc_1 \mid b_2 e \otimes qc_2)_\ell = b_1 e\varphi^{-1}(qc_1 c_2^* q)\, eb_2^*,$$
$$(b_1 e \otimes qc_1 \mid b_2 e \otimes qc_2)_r = c_1^* q\varphi(eb_1^* b_2 e)\, qc_2,$$

this, after completion, becomes a $B-B$ imprimitivity bimodule, cf. [42, 5.9]. It therefore induces a Rieffel–Morita equivalence of B with itself. Since B is stable, each Rieffel–Morita equivalence is equivalent to one associated with a specific automorphism of B, [9, 3.5]. Denoting by α^{-1} the one arising from $X \otimes_A \hat{Y}$ and replacing $\hat{Y}$ with $\hat{Y}_\alpha = pB$, where $p = \bar{\alpha}(q)$, and φ with $\psi = \alpha \circ \varphi$, so that we now have the left inner product $(pb \mid pc)_\ell = \psi^{-1}(pbc^*p)$ and the left A-action $a \cdot pb = \psi(a)\, pb$, we see that the $B-B$ imprimitivity bimodule $X \otimes_A \hat{Y}_\alpha$ induces the same Rieffel–Morita equivalence as B—which is the imprimitivity bimodule associated with the identity automorphism. Reasoning as in [9, 3.1] this means that $X \otimes_A \hat{Y}_\alpha$ and B are equivalent as imprimitivity bimodules. There is therefore a module isomorphism $f \colon X \otimes_A \hat{Y}_\alpha \to B$, i.e., a bounded linear map such that

$$bf(x) = f(b \cdot x), \qquad f(x)\, b = f(x \cdot b),$$
$$f(x)\, f(y)^* = (x \mid y)_\ell, \qquad f(x)^*\, f(y) = (x \mid y)_r,$$

for all x, y in $X \otimes_A \hat{Y}_\alpha$ and b in B.

Let (u_λ) be an approximate unit for A, and put $w_\lambda = ef(u_\lambda \otimes \psi(u_\lambda))\, p$. (Rather, w_λ is the image under f of the equivalence class in $X \otimes_A \hat{Y}_\alpha$ corresponding to this tensor element.) Then for a_1, a_2 in A we have

$$a_1 w_\lambda \psi(a_2) = f(a_1 u_\lambda \otimes \psi(u_\lambda a_2)) \to f(a_1 \otimes \psi(a_2)).$$

Moreover, using the properties of f:

$$\begin{aligned} f(a_1 \otimes \psi(a_2))\, f(a_1 \otimes \psi(a_2))^* &= (a_1 \otimes \psi(a_2) \mid a_1 \otimes \psi(a_2))_\ell \\ &= a_1 a_2 a_2^* a_1^*, \\ f(a_1 \otimes \psi(a_2))^*\, f(a_1 \otimes \psi(a_2)) &= (a_1 \otimes \psi(a_2) \mid a_1 \otimes \psi(a_2))_r \\ &= \psi(a_2^* a_1^* a_1 a_2). \end{aligned}$$

Since $w_\lambda w_\lambda^* = u_\lambda^4$ we can take a weak limit point w of the bounded net (w_λ)—having first represented B nondegenerately on some Hilbert space $\mathfrak{R}$. Then from the computations above we see that $a_1 w\psi(a_2) \in B$ for all a_1, a_2 in A. Since $w = ewp$, and since $Be = BeAe$ and $pB = p\psi(A)pB$, it follows that $w \in M(B)$. Finally,

$$a_1 w\psi(a_2 a_2^*)\, w^* a_1^* = a_1 a_2 a_2^* a_1^* \quad \text{and} \quad \psi(a_2^*)\, w^* a_1^* a_1 w\psi a_2 = \psi(a_2^* a_1^* a_1 a_2).$$

Replacing a_1 or a_2 with u_λ, and noting that (u_λ) converges strongly to e on $\mathfrak{R}$, whereas $(\psi(u_\lambda))$ converges strongly to p, we see that

$$w\psi(a)\, w^* = a, \qquad \text{and} \qquad w^* a w = \psi(a)$$

for every a in A. It follows that $ww^* = e$ and $w^* w = p\alpha(q)$, as desired. ∎

10.12. THEOREM. *Let A and B be σ-unital C^*-algebras with $A \subset B$. Then A is a full corner of B if and only if there is an injective morphism $\iota : B \to \mathbb{K} \otimes A$ that takes A onto $e_{11} \otimes A$ and B onto a hereditary C^*-subalgebra of $\mathbb{K} \otimes A$.*

Proof. If we have the embeddings $e_{11} \otimes A \subset B \subset \mathbb{K} \otimes A$, consider the full projection $p = e_{11} \otimes 1_A$ in $M(\mathbb{K} \otimes A)$, where 1_A denotes the unit for A in A^{**} (or any other unitization). Each element b in B has the form $b = \sum e_{ij} \otimes a_{ij}$, so if (u_n) is an approximate unit for A we have $(e_{11} \otimes u_n)\, b \in B$ with

$$(e_{11} \otimes u_n)\, b = \sum e_{1j} \otimes u_n a_{1j} \to \sum e_{1j} \otimes a_{1j} = pb.$$

Thus $p \in M(B)$ and $A = pBp$, so that A is a corner of B. If B is hereditary in $\mathbb{K} \otimes A$ and I is a closed ideal of B, then $I = B \cap J$ for a (unique) ideal J of $\mathbb{K} \otimes A$. If therefore $e_{11} \otimes A \subset I$, we have $e_{11} \otimes A \subset J$, whence $J = \mathbb{K} \otimes A$, since $e_{11} \otimes A$ is full in $\mathbb{K} \otimes A$. Thus $I = B$, so that $e_{11} \otimes A$ is full in B.

Conversely, if $A = pBp$ is a full corner of B there is by [6, 2.5] an isometry v in $M(\mathbb{K} \otimes B)$ such that $vv^* = \mathbf{1} \otimes p$, where $\mathbf{1}$ denotes the unit for $\mathbb{K}$ in $\mathbb{B}(\ell^2)$. The corner embedding $b \to e_{11} \otimes b$ of B into $\mathbb{K} \otimes B$, followed by the isomorphism $x \to vxv^*$ of $\mathbb{K} \otimes B$ onto $\mathbb{K} \otimes A$, gives an injective morphism $\kappa : B \to \mathbb{K} \otimes A$, such that $\kappa(B)$ is hereditary (in fact, $\kappa(B)$ is a corner of $\mathbb{K} \otimes A$), and $\kappa(A)$ is the full corner of $\mathbb{K} \otimes A$ determined by the projection $q = \bar{\kappa}(p)$ in $M(\mathbb{K} \otimes A)$. By Lemma 10.11 there is an automorphism α of $\mathbb{K} \otimes A$ such that $\bar{\alpha}(q)$ is Murray–von Neumann equivalent to $e_{11} \otimes 1_A$ in $M(\mathbb{K} \otimes A)$.

The two complementary projections $1 - e_{11} \otimes 1_A$ and $1 - \bar{\alpha}(q)$ are both equivalent to 1 in $M(\mathbb{K} \otimes A)$. For $1 - e_{11} \otimes 1_A$ this is evident. For $1 - \bar{\alpha}(q) = \bar{\alpha}(1 - q)$ it follows from the fact that $q = q_1 \sim q_n = v(e_{nn} \otimes p)\, v^*$ for all n. With $\bar{p} = v(\mathbf{1} \otimes p)\, v^*$ in $M(\mathbb{K} \otimes A)$ we get

$$1 - e_{11} \otimes 1_A \sim 1 = 1 - \bar{p} + \sum_{n=1}^{\infty} q_n \sim 1 - \bar{p} + \sum_{n=2}^{\infty} q_n = 1 - q.$$

Combining the two partial isometries obtained from above we obtain a unitary u in $M(\mathbb{K} \otimes A)$ such that $u\bar{\alpha}(q)\, u^* = e_{11} \otimes 1_A$. If we therefore define $\iota : B \to \mathbb{K} \otimes A$ by

$$\iota(b) = u\alpha(\kappa(b))\, u^* = u\alpha(v(e_{11} \otimes b)\, v^*)\, u^*, \qquad b \in B,$$

we obtain the desired hereditary embedding, since

$$\bar{\iota}(p) = u\bar{\alpha}(\bar{\kappa}(p))\, u^* = u\bar{\alpha}(q)\, u^* = e_{11} \otimes 1_A. \quad \blacksquare$$

10.13. COROLLARY. *If we have a pushout diagram of σ-unital C^*-algebras*

$$\begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\ A \arrow[r, "\delta"] & X \end{tikzcd}$$

in which α is proper and β is an embedding of C as a full corner of B, then, identifying C with $e_{11} \otimes C$ and regarding B as a hereditary subalgebra of $\mathbb{K} \otimes C$, we find that X is the hereditary C^*-subalgebra of $\mathbb{K} \otimes A$ generated by $(e_{11} \otimes A) \cup ((\iota \otimes \alpha)B)$. Moreover, $\gamma = \iota \otimes \alpha$ and $\delta = e_{11} \otimes \iota$; and if $C = pBp$ with p in $M(B)$, then $A = \bar{\gamma}(p)\, X \bar{\gamma}(p)$, where $\bar{\gamma}(p) \in M(X)$.

Proof. Combine Theorems 10.4 and 10.12 and Proposition 4.5. $\blacksquare$

10.14. *Remark.* Clearly there is a great variety of hereditary C^*-subalgebras of $\mathbb{K} \otimes A$ that will contain $e_{11} \otimes A$ as a full corner. If B is one such, then from simple matrix considerations we see that since the right ideal $R = (e_{11} \otimes 1_A)\, B$ is a left A-module it will have the form

$$R = (e_{11} \otimes 1_A)\, B = \sum e_{1n} \otimes L_n$$

for a sequence (L_n) of closed left ideals in A (with $L_1 = A$). Since $e_{11} \otimes 1_A$ is assumed to be a full projection in $M(B)$ we have $R^* \cdot R = B$ (in the sense of closed linear span of products); so the ideal sequence (L_n) determines B. In fact,

$$
\begin{aligned}
(e_{nn} \otimes 1_A)\, B(e_{mm} \otimes 1_A) &= (e_{nn} \otimes 1_A)\, R^* \cdot R(e_{mm} \otimes 1_A) \\
&= e_{nm} \otimes (L_n^* \cap L_m)
\end{aligned}
$$

for all n and m. Since B is hereditary in $\mathbb{K} \otimes A$, it follows that each subalgebra $e_{nn} \otimes L_n^* \cap L_n$ is contained in B, and therefore also each skew corner $e_{nm} \otimes L_n^* \cap L_m$ is contained in B. (Consider $(e_n + e_m)\, B(e_n + e_m)$.) We can therefore formally write

$$B = \bigoplus e_{nm} \otimes (L_n^* \cap L_m).$$

Note that the space of convergent matrices of the form

$$L_B = \bigoplus e_{nm} \otimes L_m$$

is the (unique) closed left ideal in $\mathbb{K} \otimes A$ such that $L_B^* \cap L_B = B$. Moreover, if q_n denotes the open projection in A^{**} supporting the hereditary C^*-algebra $L_n^* \cap L_n$, then $q = \sum e_{nn} \otimes q_n$ is the open projection in $(\mathbb{K} \otimes A)^{**}$ supporting B.

In the model construction for the embedding of B in $\mathbb{K} \otimes A$ given in Theorem 10.12, B becomes a (small) corner of $\mathbb{K} \otimes A$, so that the projection $q = \sum e_{nn} \otimes q_n$ defined above belongs to $M(\mathbb{K} \otimes A)$ and $1 - q$ is Murray–von Neumann equivalent to 1. This, however, need not be the most economical embedding. For example, if L denotes an arbitrary closed left ideal of A, and we define $B \subset \mathbb{M}_2(A)$ by

$$B = e_{11} \otimes A + e_{12} \otimes L + e_{21} \otimes L^* + e_{22} \otimes (L^* \cap L),$$

then A is a full corner of B; but B is not necessarily a corner of $\mathbb{M}_2(A)$.

10.15. *Stable C^*-Algebras.* In a recent paper, [25], Hjelmborg and Rørdam, provoked by a question from Houghton-Larsen, gave (among other) the following internal characterization of stability: A C^*-algebra A is stable if and only if for each positive element x in A and $\varepsilon > 0$, there is a unitary u in $\tilde{A}$, such that $\|xux\| < \varepsilon$. If desired, u can be chosen in the connected component of the identity $\mathbf{1}$ of $\mathscr{U}(\tilde{A})$. The question (still open) was whether any extension of stable (and separable or at least σ-unital) C^*-algebras is again stable, and this now seems highly probable.

We note in passing that Proposition 3.4 shows that *if* stability is a property which is closed under extensions, then it is also closed under pullbacks.

The contents of the next result arose in conversation with Houghton-Larsen.

10.16. PROPOSITION. *If we have a pushout diagram of C*-algebras*

$$\begin{tikzcd} C \arrow[r, "\beta"'] \arrow[d, "\alpha"] & B \arrow[d, "\gamma"] \\ A \arrow[r, "\delta"] & X \end{tikzcd}$$

in which A is stable and β is a proper morphism, then also X is stable.

Proof. Replacing A, B and C by the quotients $A/\ker\delta$, $B/\ker\gamma$ and $C/\ker(\delta\circ\alpha)$ we may assume that all morphisms are injective, cf. Theorem 4.4. (And we still have A stable and β proper.) Thus δ is a proper embedding of A into X by Lemma 4.6, and extends uniquely to a unital embedding $\bar{\delta}: M(A) \to M(X)$, cf. 7.1.

By assumption $A = \mathbb{K} \otimes A_0$ for some C^*-algebra A_0, so $e_{ij} \otimes \mathbf{1} \in M(A)$ for all e_{ij} in $\mathbb{K}$, giving us a family

$$\{v_{ij} = \bar{\delta}(e_{ij} \otimes \mathbf{1}) \mid (i, j) \in d \times d\}$$

of partial isometries in $M(X)$. Set $X_0 = v_{11} X v_{11}$, which is a hereditary C^*-subalgebra of X, and check directly that $X = \mathbb{K} \otimes X_0$, since $\sum v_{ii} = 1$ in $M(X)$, the sum being strictly convergent. ∎

10.17. *Remark.* The previous result is in keeping with the spirit of this paper, where the emphasis is on free products with "large" amalgamations. Nevertheless one may reasonably ask whether the result holds in the absence of properness. The opposite extreme to properness would seem to be free products, and here Ken Dykema gave the following rather devastating counterexample.

10.18. EXAMPLE (Dykema). The free product $\mathbb{K} \star \mathbb{K}$ has a unital representation and is consequently not stable.

Write the unit in $\mathbb{B}(\mathfrak{H})$ as $\sum p_n = \mathbf{1} = \sum q_n$, where all the projections involved are infinite dimensional (hence equivalent), and

$$p_1 = \sum_{n=2}^{\infty} q_n \qquad \text{and} \qquad q_1 = \sum_{n=2}^{\infty} p_n.$$

Choose partial isometries v_n and w_n such that

$$v_n v_n^* = p_1, \qquad v_n^* v_n = p_n, \qquad w_n w_n^* = q_1, \qquad w_n^* w_n = q_n.$$

Then both (v_n) and (w_n) will generate a copy of $\mathbb{K}$ acting (with infinite multiplicity) on $\mathfrak{H}$. Consequently $A = C^*((v_n) \cup (w_n))$ is a quotient of $\mathbb{K} \star \mathbb{K}$. However, $\mathbf{1} = p_1 + q_1 \in A$.

11. NCCW COMPLEXES

11.1. *Notations.* We put $\mathbb{I} = [-1, 1]$, but will frequently identify $\mathbb{I}^n$ with the closed unit ball in $\mathbb{R}^n$. Thus with $\mathbb{I}_0^n = (]-1, 1[)^n$ we will always identify $\mathbb{I}^n \backslash \mathbb{I}_0^n$ with the sphere $\mathbb{S}^{n-1}$.

For any C^*-algebra A we will use the abbreviations

$$\mathbb{I}^n A = C([-1, 1]^n, A), \qquad \mathbb{I}_0^n A = C_0(]-1, 1[^n, A), \qquad \mathbb{S}^n A = C(\mathbb{S}^n, A).$$

11.2. Definitions. The definition of noncommutative CW complexes is by induction on the topological dimension of the underlying space; cf. [21, 2.4].

In dimension zero an $NCCW$ complex A_0 is simply a C^*-algebra of finite (linear) dimension, corresponding to the decomposition of $A_0 = \bigoplus \mathbb{M}_{n(k)}$ as a finite collection of "noncommutative points," i.e., matrix algebras.

In dimension n an $NCCW$ complex is a sequence of C^*-algebras $\{A_0, A_1, ..., A_n\}$, where each A_k is obtained from the previous one by a pullback construction

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{I}_0^k F_k \arrow[r, hook] \arrow[d, equal] & A_k \arrow[r, "\pi"] \arrow[d, "\rho_k"] & A_{k-1} \arrow[r] \arrow[d, "\sigma_k"] & 0 \\
0 \arrow[r] & \mathbb{I}_0^k F_k \arrow[r, hook] & \mathbb{I}^k F_k \arrow[r, "\partial"] & \mathbb{S}^{k-1} F_k \arrow[r] & 0
\end{tikzcd}$$

Here both rows are extensions and F_k is some C^*-algebra of finite (linear) dimension. The boundary map ∂ is the obvious restriction morphism and σ_k (the *connecting morphism*) can be any morphism from A_{k-1} into the "model" $NCCW$ complex $\mathbb{S}^{k-1}F_k$. Finally, ρ_k and π are the projections on first and second coordinates, respectively, in the presentation of A_k as the restricted direct sum

$$A_k = \mathbb{I}^k F_k \bigoplus_{\mathbb{S}^{k-1} F_k} A_{k-1}.$$

By abuse of notation we shall often refer to the C^*-algebra A_n as an $NCCW$ complex, taking for granted the cellular structure that has been chosen for it.

It follows from the recursive definition that each $NCCW$ complex A_n of *topological dimension n* has a decreasing family of closed ideals

$$A_n = I_0 \supset I_1 \cdots \supset I_{n-1} \supset I_n \neq 0,$$

where $I_n = \mathbb{I}_0^n F_n$ and inductively $I_k / I_{k+1} = \mathbb{I}_0^k F_k$ (so that $I_k = I_{k+1}$ if $F_k = 0$). Together with these ideals come the canonical quotients

$$A_n / I_{k+1} = A_k, \qquad 0 \leqslant k \leqslant n-1,$$

where each A_k by assumption is an *NCCW* complex of topological dimension at most k.

Clearly an *NCCW* complex A_n is completely determined by the lower complex A_{n-1}, the n-cells $\mathbb{I}_0^n \mathbb{M}_j$ given by writing $F_n = \bigoplus \mathbb{M}_j$, and the connecting morphism

$$\sigma_n : A_{n-1} \to \mathbb{S}^{n-1} F_n.$$

Note that the kernel of σ_n is the isometric image under π of the kernel of the morphism $\rho_n : A_n \to \mathbb{I}^n F_n$, which is an ideal in A_n orthogonal to $\mathbb{I}_0^n F_n$. It follows that σ_n is injective if and only if $I_n = \mathbb{I}_0^n F_n$ is an essential ideal in A_n. We say in this case that A_n is *essentially of* (*topological*) *dimension n*. If each of the canonical lower complexes A_k, $0 \leqslant k \leqslant n$, are essentially of dimension k we say that A_n is a *proper* n-dimensional *NCCW* complex.

As an example of a proper n-dimensional *NCCW* complex take any finite-dimensional C^*-algebra F and consider the complex.

$$\{\mathbb{S}^0 F, \mathbb{S}^1 F, ..., \mathbb{S}^n F\}.$$

The connecting morphisms are obtained from the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{I}_0^k(F \oplus F) \arrow[r, hook] \arrow[d, equal] & \mathbb{S}^k F \arrow[r, "\pi"] \arrow[d, "\rho_k"] & \mathbb{S}^{k-1} F \arrow[r] \arrow[d, "\sigma_k"] & 0 \\
0 \arrow[r] & \mathbb{I}_0^k(F \oplus F) \arrow[r, hook] & \mathbb{I}^k(F \oplus F) \arrow[r, "\partial"] & \mathbb{S}^{k-1}(F \oplus F) \arrow[r] & 0
\end{tikzcd}$$

The embedding of $\mathbb{I}_0^k(F \oplus F)$ into $\mathbb{S}^k F$ is obtained by regarding $\mathbb{S}^{k-1}$ as the equator in $\mathbb{S}^k$, so that $\mathbb{S}^k \backslash \mathbb{S}^{k-1} = \mathbb{I}_0^k \cup \mathbb{I}_0^k$, and ρ_k embeds $\mathbb{S}^k F$ as functions on the two copies of $\mathbb{I}^k$ that agree on the boundary $\mathbb{S}^{k-1}$. Thus $\sigma_k(f)(s) = (f(s), f(s))$.

Note finally that an *NCCW* complex A_n is unital if and only if A_{n-1} is unital and the connecting morphism σ_n is also unital. It follows in this case that all the canonical quotients A_k are unital and that all the connecting morphisms $\sigma_k : A_{k-1} \to \mathbb{S}^{k-1} F_k$ are unital. We also see that for each non-unital *NCCW* complex $\{A_0, A_1, ..., A_n\}$ there is a unital *NCCW* complex $\{B_0, B_1, ..., B_n\}$, such that

(1) $B_k = A_k$ if both A_k and the linking morphism σ_k are unital;

(2) $B_k = \tilde{A}_k$ if A_k is nonunital (and the new linking morphism is the unitization of σ_k); and

(3) $B_k = A_k \oplus \mathbb{C}$ (forced unitization) if A_k is unital but σ_k is not (and the new linking morphism is σ_k adjusted on $\mathbb{C}$ to become unital).

11.3. Proposition. *Given NCCW complexes $\{A_0, ..., A_n\}$ and $\{B_0, ..., B_m\}$, with $m < n$, and a morphism $\rho : B_m \to A_m$, there is a joint NCCW complex $\{B_0, ..., B_n\}$, where $B_{k+1} = A_{k+1} \oplus_{A_k} B_k$ for each $k \geqslant m$.*

Proof. Consider the diagram

$$\begin{tikzcd} B_{m+1} \arrow[r] \arrow[d] & A_{m+1} \arrow[r] \arrow[d] & \mathbb{I}^{m+1} F_{m+1} \arrow[d] \\ B_m \arrow[r, "\rho"] & A_m \arrow[r, "\sigma"] & \mathbb{S}^m F_{m+1} \end{tikzcd}$$

where the left square is the pullback diagram that defines B_{m+1} and the right square is the pullback diagram that determines A_{m+1}. Concatenating the two pullbacks we again have a pullback by Proposition 2.7, and thus B_{m+1} is the *NCCW* complex obtained from B_m using the connecting morphism $\sigma \circ \rho$. The projection $B_{m+1} \to A_{m+1}$ can now be used instead of ρ, and we obtain the full complex $\{B_0, ..., B_n\}$ by recursion. ∎

11.4. Proposition. *Consider a commutative diagram of C^*-algebras*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, hook] \arrow[d, equal] & X \arrow[r, "\gamma"'] \arrow[d, "\delta"] & B \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & I \arrow[r, hook] & A \arrow[r, "\alpha"] & C \arrow[r] & 0 \end{tikzcd}$$

that describes a pullback algebra $X = A \oplus_C B$ arising from a diagram where α is surjective. Assume furthermore that I is an essential ideal in A. Writing $Z_B = Z(B) \cap \beta^{-1}(Z(C))$, the restriction of the morphisms α, β, γ, δ to the centers of the respective algebras then produce a pullback diagram

$$\begin{tikzcd} Z(X) \arrow[r, "\gamma"'] \arrow[d, "\delta"] & Z_B \arrow[d, "\beta"] \\ Z(A) \arrow[r, "\alpha"] & Z(C) \end{tikzcd}$$

Proof. Evidently $Z(A) \oplus_{Z(C)} Z_B \subset Z(X)$. To prove the converse inclusion take $x = (a, b)$ in $Z(X)$. Since I is an essential ideal in A we have inclusions $I \subset A \subset M(I)$. Identifying I with $\{(i, 0) \mid i \in I\}$ in X it follows that $a \in I'$; and since I is strictly dense in $M(I)$ this means that $a \in Z(M(I)) \cap A = Z(A)$.

On the other hand, $b \in Z(B)$ as γ is surjective. Since, after all, $(a, b) \in X$ we have $\alpha(a) = \beta(b)$, whence $\beta(b) \in Z(C)$; and thus $b \in Z(B) \cap \beta^{-1}(Z(C)) = Z_B$. ∎

11.5. *Remark.* The preceding result shows that the center of an *NCCW* complex A_n has the form $Z(A_n) = C_0(\Omega)$ for some finite *CW* complex Ω, provided that the C^*-algebra

$$Z_{n-1} = Z(A_{n-1}) \cap \sigma_n^{-1}(Z(\mathbb{S}^{n-1} F_n))$$

has the form $Z_{n-1} = C_0(\Omega_{n-1})$ for some finite *CW* complex Ω_{n-1}. This is usually quite easy to verify, and may be true in all cases.

11.6. *Simplicial Morphisms.* A morphism $\alpha\colon A_n \to B_m$ between *NCCW* complexes is called *simplicial* if the following two conditions are satisfied:

(i) If the sequences of canonical ideals for A_n and B_m are given as

$$I_n \subset I_{n-1} \cdots I_1 \subset I_0 = A_n,$$

$$J_m \subset J_{m-1} \cdots J_1 \subset J_0 = B_m,$$

then $\alpha(I_k) \subset J_k$ for all k. In particular, $\alpha(I_k) = 0$ if $k > m$.

(ii) For $0 \leqslant k \leqslant n$, let $I_k / I_{k+1} = \mathbb{I}_0^k F_k$ and $J_k / J_{k+1} = \mathbb{I}_0^k G_k$, and let $\tilde{\alpha}_k$ denote the induced morphism between these homogeneous algebras. There should then be a morphism $\varphi_k\colon F_k \to G_k$ and a homeomorphism ι_k of $\mathbb{I}^k$ such that $\tilde{\alpha}_k = \iota_k^* \otimes \varphi_k$.

Since we identify isomorphic C^*-algebras and homeomorphic spaces, we shall not (at this stage of the theory) need to restrict the class of homeomorphisms ι_k. In fact, we can quite often assume that ι_k is the identity map and write $\tilde{\alpha}_k = 1 \otimes \varphi_k$.

The classical *CW* complexes are designed so that higher cells are glued to (or their boundaries identified with) a lower dimensional structure in a combinatorial manner. For *NCCW* complexes we have to employ noncommutative combinatorics, exemplified by morphisms between finite-dimensional algebras (as pioneered by Bratteli in the diagrams for *AF* algebras). The simplicial morphisms between *NCCW* complexes are designed to be such noncommutative combinatorial assignments. Note that at the level of k-cells (of the form $\mathbb{I}^k \mathbb{M}_{m(k)}$) a simplicial morphism (of the form $\iota_k^* \otimes \varphi_k$) will either be an isomorphism (if $\varphi_k \mid \mathbb{M}_{m(k)}$ is injective) or zero (if $\mathbb{M}_{m(k)} \subset \ker \varphi_k$). It is therefore a noncommutative analogue of a simplicial map on a simplicial complex (except that we do not take ι_k to be piecewise linear).

Clearly it would be desirable to have a more general notion of cellular morphism, generalizing cellular maps on CW complexes, cf. Remarks 11.8. These would presumably be morphisms that only satisfy condition (i) above. Unfortunately, no theory is available for morphisms of this generality at the moment.

It is obvious from the definition that if $\alpha : A_n \to B_m$ and $\beta : B_m \to C_k$ are simplicial morphisms between $NCCW$ complexes, then $\beta \circ \alpha$ is also simplicial.

It also follows from the definition of a simplicial morphism α: $A_n \to B_m$ that a recursive definition must be possible, involving only the sequences (ι_k) and (φ_k) of homeomorphisms of $\mathbb{I}^k$ and morphisms between finite-dimensional algebras: If $\iota_0, ..., \iota_k$ and $\varphi_0, ..., \varphi_k$ are given and determine α as a morphism $\alpha : A_k \to B_k$, and if $\sigma : A_k \to \mathbb{S}^k F_{k+1}$ and $\tau : B_k \to \mathbb{S}^k G_{k+1}$ are the connecting morphisms determining A_{k+1} and B_{k+1}, respectively, then the morphism $\iota_{k+1}^* \otimes \varphi_{k+1}$ must appear in the commutative diagram

$$\begin{array}{ccc} A_k & \xrightarrow{\sigma} & \mathbb{S}^k F_{k+1} \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \iota_{k+1}^* \otimes \varphi_{k+1}} \\ B_k & \xrightarrow{\tau} & \mathbb{S}^k G_{k+1} \end{array}$$

cf. [20, Theorem 2.2]. Conversely, each such morphism can be used to "advance" α from level k to level $k+1$.

Finally we note that if $\{A_0, A_1, ..., A_n\}$ is an $NCCW$ complex and $\{A_0, A_1, ..., A_{n-1}\}$ is the complex one dimension lower, then the quotient map π: $A_n \to A_{n-1}$ is a simplicial morphism. Indeed, if

$$I_n \subset I_{n-1} \subset \cdots \subset I_1 \subset A_n,$$
$$J_{n-1} \subset \cdots \subset J_1 \subset A_{n-1}$$

are the canonical sequences of ideals for the two complexes (so that $I_n = \mathbb{I}_0^n F_n$ and $J_{n-1} = \mathbb{I}_0^{n-1} F_{n-1}$), then for each $k < n$ we have a commutative diagram of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_n & \hookrightarrow & I_k & \xrightarrow{\pi} & J_k & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\scriptstyle \rho_n} & & \Big\downarrow{\scriptstyle \sigma_n} & & \\ 0 & \longrightarrow & \mathbb{I}_0^n F_n & \hookrightarrow & \mathbb{I}^n F_n & \xrightarrow{\partial} & \mathbb{S}^{n-1} F_n & \longrightarrow & 0 \end{array}$$

Thus each I_k surjects onto J_k with $I_k / I_n = J_k$, and consequently

$$I_k / I_{k+1} = J_k / J_{k+1} = \mathbb{I}_0^k F_k$$

for every $k < n$, so that ι_k and φ_k are the identity maps.

11.7. Proposition. *If $\alpha\colon A_n \to B_m$ is a surjective simplicial morphism between NCCW complexes, then $n \geqslant m$ and $\alpha(I_k) = J_k$ for all k. Moreover, each of the induced morphisms*

$$\alpha_k : A_k \to B_k \qquad \text{and} \qquad \varphi_k : F_k \to G_k$$

is surjective. Here (I_k) and (J_k) are the series of canonical ideals for A_n and B_m, respectively, and we write

$$I_k/I_{k+1} = \mathbb{I}_0^k F_k, \qquad J_k/J_{k+1} = \mathbb{I}_0^k G_k, \qquad \text{and} \qquad \tilde{\alpha}_k = \iota_k^* \otimes \varphi_k.$$

Proof. With notations as in 11.6 we first note that by taking formal pullbacks over zero, i.e.,

$$\begin{tikzcd} A_{m+1} \arrow[r, equal] \arrow[d, "0"] & A_m \arrow[d, "0"] \\ \mathbb{I}^{m+1}\{0\} \arrow[r, "\partial"] & \mathbb{S}^m\{0\} \end{tikzcd}$$

(cf. Proposition 11.3) and similarly for B_{m+1}, we may assume that $m = n$. Any simplicial morphism originally given as $\alpha : A_n \to B_m$ will still be simplicial in the new setting. For if $n > m$, we already have $\alpha(I_k) = \{0\} (= J_k)$ for all $k > m$; and if $n < m$, we add the ideals $I_k = \{0\}$ for $k > n$, and evidently $\alpha(\{0\}) \subset J_k$.

Starting with $\alpha_n = \alpha$, we obtain the induced morphisms $\alpha_k\colon A_k \to B_k$ by an easy induction argument, using the commutative diagram

$$\begin{tikzcd} \mathbb{I}_0^k F_k \arrow[r, hook] \arrow[d] & A_k \arrow[r] \arrow[d] & A_{k-1} \arrow[d] \\ \mathbb{I}_0^k G_k \arrow[r, hook] & B_k \arrow[r] & B_{k-1} \end{tikzcd}$$

If α is surjective, then $\alpha_n(A_n) = \alpha(A_n) = B_n$; and from the diagram we see that this implies $\alpha_{n-1}(A_{n-1}) = B_{n-1}$, and thus by induction $\alpha_k(A_k) = B_k$ for all $k \leqslant n$.

If $g \in \mathbb{I}_0^k G_k$ there is therefore an element a_k in A_k such that $\alpha_k(a_k) = g$, and we may represent a_k as (f, a_{k-1}) in $\mathbb{I}^k F_k \oplus_{\mathbb{S}^{k-1} F_k} A_{k-1}$. Write $F_k = H \oplus K$, with $K = \ker \varphi_k$, and define $\tilde{f}$ in $\mathbb{I}^k H$ by $\tilde{f}(t) = ef(t)$, where e is the unit in H (identified with a central multiplier of $\mathbb{I}_0^k F_k$). Then

$$g = \alpha_k(a_k) = (\iota_k^* \otimes \varphi_k) f = (\iota_k^* \otimes \varphi_k) \tilde{f}.$$

Since φ_k is injective on H we conclude that actually $\tilde{f} \in \mathbb{I}_0^k H$. Consequently $\tilde{f} \in \mathbb{I}_0^k F_k$ with $(\iota_k^* \otimes \varphi_k) \tilde{f} = g$, so that $\iota_k^* \otimes \varphi_k$ is surjective. But then also φ_k

is surjective, as claimed. Moreover, $\alpha(I_k) = J_k$ for all k. In particular, $I_k \neq 0$ whenever $J_k \neq 0$, which in the original setting means that $m \leqslant n$. ∎

11.8. *Remarks.* Note that every simplicial extension

$$0 \to \ker \alpha \to A_n \to \alpha(A_n) \to 0$$

splits at the level of n-cells. Even for one-dimensional CW complexes this does not mean that all simplicial extensions split. Consider for example $A_1 = C(\mathbb{I} \cup \mathbb{I}) \oplus_{\mathbb{C}^4} \mathbb{C}^2$, where the connecting morphism $\sigma : \mathbb{C}^2 \to \partial(\mathbb{I} \cup \mathbb{I})$ is given by $\sigma(s, t) = (s, t, t, s)$. Thus, the two intervals $\mathbb{I} \cup \mathbb{I}$ are identified at their endpoints (in opposite ends), so as a C^*-algebra $A_1 = C(\mathbb{S})$. The parametrization is chosen such that the upper halfcircle is identified with $[-1, 1]$ by $s \to e^{i\pi(s+1)/2}$, whereas the lower halfcircle is identified with $[-1, 1]$ by $s \to e^{-i\pi(s+1)/2}$. Taking $B_1 = C(\mathbb{I}) \oplus_{\mathbb{C}^2} \mathbb{C}^2$, where $\tau : \mathbb{C}^2 \to \partial \mathbb{I}$ is given by $\tau(s, t) = (s, t)$, so that as a C^*-algebra $B_1 = C(\mathbb{I})$, we have a surjective simplicial morphism α: $A_1 \to B_1$. If $a = (f_1, f_2, s, t)$ in A_1 define $\alpha(a) = (f_1, s, t)$ in B_1. Thus $a \in \ker \alpha$ if and only if $f_1 = 0$ and $s = t = 0$; i.e. $\ker \alpha = C(\mathbb{I}) \oplus_{\mathbb{C}^2} \{0\} = C_0(\mathbb{I}_0)$. At the geometric level we have just decomposed the circle into an open and a closed halfcircle, but that decomposition is not a direct union.

Consider instead $C_1 = C(\mathbb{I}) \oplus_{\mathbb{C}^2} \mathbb{C}$ with the connecting morphism $s \to (s, s)$, so that as a C^*-algebra $C_1 = C(\mathbb{S})$. Here we have a simplicial embedding $\beta : C_1 \to A_1$. If $c = (f, s)$ in C_1 define $\beta(c) = (f, f, s, s)$ in A_1. Note that $\beta_0 : C_0 \to A_0$ is trivially given by $s \to (s, s)$, whereas $\tilde{\beta} : C_0(\mathbb{I}_0) \to C_0(\mathbb{I}_0 \cup \mathbb{I}_0)$ becomes $\tilde{\beta}(f) = (f, f) = (1 \otimes \varphi) f$, where $\varphi : \mathbb{C} \to \mathbb{C}^2$ simply is the unital morphism $s \to (s, s)$. The interesting point is that at the level of C^*-algebras the morphism $\beta : C(\mathbb{S}) \to C(\mathbb{S})$ is the transposed of the double covering map $z \to z^2$ on $\mathbb{S}$, i.e. $\beta(f)(z) = f(z^2)$. With our restrictive notion of simplicial morphism we can not realize this cellular map as a simplicial morphism $C_1 \to C_1$ or $A_1 \to A_1$.

11.9. Theorem. *If* α: $A_n \to B_m$ *is a simplicial morphism between NCCW complexes, then both* $\ker \alpha$ *and* $\alpha(A_n)$ *are NCCW complexes, and the embeddings* $\ker \alpha \subset A_n$ *and* $\alpha(A_n) \subset B_m$ *are simplicial. Moreover, if* C_k *is an NCCW subcomplex of* B_m, *then* $\alpha^{-1}(C_k)$ *is an NCCW subcomplex of* A_n.

Proof. With notations as in 11.6 we first note that by taking formal pullbacks over zero (as in the proof of 11.7), we may regard $B_m = B_n$ and $C_k = C_n$ as n-dimensional complexes.

Writing $\alpha_{n-1} : A_{n-1} \to B_{n-1}$ and $\iota_n^* \otimes \varphi_n : \mathbb{I}_0^n F_n \to \mathbb{I}_0^n G_n$ for the induced morphisms and setting $K_n = \ker \varphi_n$ and $H_n = \varphi_n(F_n)$ (so that $F_n = K_n \oplus H_n$) we obtain a commutative diagram

$$\begin{tikzcd}
\ker\alpha \arrow[rr] \arrow[dr] \arrow[dd] & & A_n \arrow[rr] \arrow[dr] \arrow[dd] & & \alpha(A_n) \arrow[rr] \arrow[dr] \arrow[dd] & & B_n \arrow[dr] \arrow[dd] & \\
& \ker\alpha_{n-1} \arrow[rr] \arrow[dd] & & A_{n-1} \arrow[rr] \arrow[dd] & & \alpha_{n-1}(A_{n-1}) \arrow[rr] \arrow[dd] & & B_{n-1} \arrow[dd] \\
\mathbb{I}^n K_n \arrow[rr] \arrow[dr] & & \mathbb{I}^n F_n \arrow[rr] \arrow[dr] & & \mathbb{I}^n H_n \arrow[rr] \arrow[dr] & & \mathbb{I}^n G_n \arrow[dr] & \\
& \mathbb{S}^{n-1} K_n \arrow[rr] & & \mathbb{S}^{n-1} F_n \arrow[rr] & & \mathbb{S}^{n-1} H_n \arrow[rr] & & \mathbb{S}^{n-1} G_n
\end{tikzcd}$$

Since $\alpha(A_n)$ is embedded in the pullback algebra B_n, it follows that the two morphisms into $\mathbb{I}^n H_n$ and $\alpha_{n-1}(A_{n-1})$ have no common nonzero kernel. Moreover, the morphism $\mathbb{I}^n K_n \to \mathbb{S}^{n-1} K_n$ is surjective. The conditions in Proposition 9.2 are therefore satisfied, and we conclude that both the first and the third vertical squares (from left) are pullbacks.

Assuming that both $\ker\alpha_{n-1}$ and $\alpha_{n-1}(A_{n-1})$ are *NCCW* complexes and that the embeddings $\ker\alpha_{n-1} \subset A_{n-1}$ and $\alpha_{n-1}(A_{n-1}) \subset B_{n-1}$ are simplicial, it follows that also $\ker\alpha$ and $\alpha(A_n)$ are *NCCW* subcomplexes of A_n and B_n, respectively. The argument can therefore be completed by induction.

To prove the other half of the theorem, let C_n be an *NCCW* subcomplex of B_n, and let $\mathbb{I}_0^n L_n$ denote its collection of n-cells. Consider the two diagrams below:

$$\begin{tikzcd}
D_n \arrow[rr] \arrow[dr] \arrow[dd] & & A_n \arrow[dr] \arrow[dd] & \\
& D_{n-1} \arrow[rr] \arrow[dd] & & A_{n-1} \arrow[dd] \\
\mathbb{I}^n M_n \arrow[rr] \arrow[dr] & & \mathbb{I}^n F_n \arrow[dr] & \\
& \mathbb{S}^{n-1} M_n \arrow[rr] & & \mathbb{S}^{n-1} F_n
\end{tikzcd}
\qquad
\begin{tikzcd}
C_n \arrow[rr] \arrow[dr] \arrow[dd] & & B_n \arrow[dr] \arrow[dd] & \\
& C_{n-1} \arrow[rr] \arrow[dd] & & B_{n-1} \arrow[dd] \\
\mathbb{I}^n L_n \arrow[rr] \arrow[dr] & & \mathbb{I}^n G_n \arrow[dr] & \\
& \mathbb{S}^{n-1} L_n \arrow[rr] & & \mathbb{S}^{n-1} G_n
\end{tikzcd}$$

The box diagram to the right simply expresses the simplicial embedding of C_n in B_n. To build up the diagram to the left we set $D_n = \alpha^{-1}(C_n)$ and $M_n = \varphi_n^{-1}(L_n) \subset F_n$. Then $\mathbb{I}_0^n M_n$ is an ideal in D_n via the identification

$$\mathbb{I}_0^n M_n = D_n \cap \mathbb{I}_0^n F_n = \alpha^{-1}(\mathbb{I}_0^n L_n) \cap \mathbb{I}_0^n F_n .$$

If $\alpha_{n-1} : A_{n-1} \to B_{n-1}$ therefore denotes the induced morphism, as above, then with $D_{n-1} = \alpha_{n-1}^{-1}(C_{n-1})$ we have an extension

$$0 \to \mathbb{I}_0^n M_n \to D_n \to D_{n-1} \to 0.$$

Since C_n is a subcomplex of B_n it follows by induction that we can write

$$C_n = \{b_n = (f, b_{n-1}) \in \mathbb{I}^n G_n \oplus_{\mathbb{S}^{n-1} G_n} B_{n-1} \mid f \in \mathbb{I}^n L_n, b_{n-1} \in C_{n-1}\}.$$

Consequently

$$D_n = \{a_n = (f, a_{n-1}) \in \mathbb{I}^n F_n \oplus_{\mathbb{S}^{n-1} F_n} A_{n-1} \mid f \in \mathbb{I}^n M_n, a_{n-1} \in D_{n-1}\},$$

which shows that $D_n = \mathbb{I}^n M_n \oplus_{\mathbb{S}^{n-1} M_n} D_{n-1}$.

Assuming that D_{n-1} is an *NCCW* subcomplex of A_{n-1} it follows by induction that D_n is an *NCCW* subcomplex of A_n. ∎

11.10. Theorem. *Consider a pullback diagram*

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & B_m \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \beta} \\ A_n & \xrightarrow{\alpha} & C_k \end{array}$$

in which A_n, B_m, and C_k are NCCW complexes of topological dimensions n, m and k, and α and β are simplicial morphisms. Then X is an NCCW complex of topological dimension $\max(n, m)$, *and both γ and δ are simplicial morphisms of X. Moreover, for each k we have $X_k = A_k \oplus_{C_k} B_k$ (where $A_k = A_n$ if $k > n$ and likewise for B_m and C_k).*

Proof. If $n = m = 0$, then $X \subset A_0 \oplus B_0$ and thus of finite linear dimension.

Assume therefore that the theorem has been established for all complexes of topological dimension less than n, and consider a pullback diagram with complexes A_n, B_m and C_k, where $k \leqslant m \leqslant n$. By taking pullbacks over zero we may formally regard B_m and C_k as n-dimensional complexes; i.e. we may assume that $k = m = n$. Furthermore, replacing C_n with $\alpha(A_n)$ and B_n with $\beta^{-1}(\alpha(A_n))$, we may assume that α is surjective, cf. Theorem 11.9 and Remark 3.2.

Consider the ideals $I_n = \mathbb{I}_0^n F_n$, $J_n = \mathbb{I}_0^n G_n$ and $K_n = \mathbb{I}_0^n H_n$ in A_n, B_n and C_n, respectively (where at least $I_n \neq 0$). We have morphisms $\varphi : F_n \to K_n$ and $\psi : G_n \to K_n$, such that $\alpha \mid I_n = \iota^* \otimes \varphi$ and $\beta \mid J_n = \kappa^* \otimes \psi$ for some homeomorphisms ι and κ of $\mathbb{I}^n$, and φ is surjective by Proposition 11.7. Taking quotients we obtain the natural simplicial morphisms

$$\alpha_{n-1} : A_{n-1} \to C_{n-1}, \qquad \beta_{n-1} : B_{n-1} \to C_{n-1},$$

from which we construct the *NCCW* complex

$$X_{n-1} = A_{n-1} \underset{C_{n-1}}{\oplus} B_{n-1}$$

by the induction hypothesis.

The morphisms φ and ψ determine the pullback diagram to the left, below, where $\dim R_n < \infty$, and by Theorem 3.9 this results in a new pullback to the right, which by repeated applications of Corollary 2.8 produces the pullback further below:

$$\begin{tikzcd} R_n \arrow[r, "\zeta"] \arrow[d, "\eta"] & G_n \arrow[d, "\psi"] \\ F_n \arrow[r, "\varphi"] & H_n \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} \mathbb{I}_0^n R_n \arrow[r, "1\otimes\zeta"] \arrow[d, "1\otimes\eta"] & \mathbb{I}_0^n G_n \arrow[d, "1\otimes\psi"] \\ \mathbb{I}_0^n F_n \arrow[r, "1\otimes\varphi"] & \mathbb{I}_0^n H_n \end{tikzcd}$$

$$\text{gives} \quad \begin{tikzcd} \mathbb{I}_0^n R_n \arrow[r, "\kappa^{-1*}\otimes\zeta"] \arrow[d, "\iota^{-1*}\otimes\eta"] & \mathbb{I}_0^n G_n \arrow[d, "\kappa^*\otimes\psi"] \\ \mathbb{I}_0^n F_n \arrow[r, "\iota^*\otimes\varphi"] & \mathbb{I}_0^n H_n \end{tikzcd}$$

We are now in a situation where Theorem 9.1 can be applied to the diagram

$$\begin{tikzcd} \mathbb{I}_0^n F_n \arrow[r] \arrow[d, "\iota^*\otimes\varphi"] & A_n \arrow[r] \arrow[d, "\alpha"] & A_{n-1} \arrow[d, "\alpha_{n-1}"] \\ \mathbb{I}_0^n H_n \arrow[r] & C_n \arrow[r] & C_{n-1} \\ \mathbb{I}_0^n G_n \arrow[r] \arrow[u, "\kappa^*\otimes\psi"] & B_n \arrow[r] \arrow[u, "\beta"] & B_{n-1}, \arrow[u, "\beta_{n-1}"] \end{tikzcd}$$

because $\iota^* \otimes \varphi$ is surjective. We obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{I}_0^n R_n \arrow[r, hook] \arrow[d, equal] & X \arrow[r, "\pi"'] \arrow[d, "\rho"] & X_{n-1} \arrow[r] \arrow[d, "\sigma"] & 0 \\ 0 \arrow[r] & \mathbb{I}_0^n R_n \arrow[r, hook] & \mathbb{I}^n R_n \arrow[r, "\partial"] & \mathbb{S}^{n-1} R_n \arrow[r] & 0 \end{tikzcd}$$

and by Theorem 9.1 also the upper row is an extension. Here the morphism ρ (hence also σ) is determined by the identifications

$$X = A_n \bigoplus_{C_n} B_n, \qquad \mathbb{I}^n R_n = \mathbb{I}^n \left(F_n \bigoplus_{H_n} G_n \right)$$

and the coherent pair of morphisms

$$X \to A_n \xrightarrow{\rho_A} \mathbb{I}^n F_n \qquad \text{and} \qquad X \to B_n \xrightarrow{\rho_B} \mathbb{I}^n G_n.$$

By the induction hypothesis we have

$$X_k = A_k \bigoplus_{C_k} B_k$$

for every $k < n$, and we have just shown that $X = A_n \oplus_{C_n} B_n$ is an *NCCW* complex extending X_{n-1} by one dimension.

The theorem now follows by induction. ∎

11.11. Corollary. *If A_n and B_n are NCCW complexes essentially of topological dimension n and $\alpha : A_n \to C_m$ and $\beta : B_n \to C_m$ are simplicial morphisms, then*

$$X_n = A_n \bigoplus_{C_m} B_n$$

is an NCCW complex essentially of topological dimension n. Moreover, if both A_n and B_n are proper, so is X_n.

Proof. We must show that the ideal in the extension

$$0 \to \mathbb{I}_0^n \left(F_n \bigoplus_{H_n} G_n \right) \to A_n \bigoplus_{C_n} B_n \to A_{n-1} \bigoplus_{C_{n-1}} B_{n-1} \to 0$$

is essential. But if $x = (a, b) \in A_n \oplus_{C_n} B_n$ and x annihilates the ideal, then

$$a \mathbb{I}_0^n F_n = 0 \qquad \text{and} \qquad b \mathbb{I}_0^n G_n = 0.$$

By assumption these ideals are essential in A_n and B_n, so $a = 0 = b$, whence $x = 0$.

If A_n and B_n are proper complexes, the argument above applies to every subcomplex $X_k = A_k \oplus_{C_k} B_k$ for $1 \leqslant k \leqslant n$.

11.12. Lemma. *If I and J are essential ideals in C^*-algebras A and B, respectively, then $I \otimes_{\min} J$ is an essential ideal in $A \otimes_{\min} B$.*

Proof. Choose faithful, nondegenerate representations $(\pi, \mathfrak{H})$ and $(\rho, \mathfrak{K})$ for I and J, respectively. Since I is an essential ideal in A there is a unique, faithful extension $\bar{\pi} : A \to \mathbb{B}(\mathfrak{H})$ determined by

$$\bar{\pi}(a)\, \pi(x)\, \xi = \pi(ax)\, \xi, \qquad a \in A, \quad x \in I, \quad \xi \in \mathfrak{H},$$

cf. [39, Theorem 5.9]. Similarly we have a faithful extension $\bar{\rho} : B \to \mathbb{B}(\mathfrak{K})$. It follows that $\bar{\pi} \otimes \bar{\rho}$ is a faithful representation of $A \otimes_{\min} B$ whose restriction, $\pi \otimes \rho$, is nondegenerate for $I \otimes_{\min} J$.

Discarding the morphisms we have inclusions

$$I \otimes_{\min} J \subset A \otimes_{\min} B \subset \mathbb{B}(\mathfrak{H} \otimes \mathfrak{K}),$$

where $I \otimes_{\min} J$ acts nondegenerately on $\mathfrak{H} \otimes \mathfrak{K}$. The annihilator of $I \otimes_{\min} J$ is therefore zero, and since evidently $I \otimes_{\min} J$ is an ideal in $A \otimes_{\min} B$, it is an essential ideal. ∎

11.13. Lemma. *If A_n is an NCCW complex of (essential) topological dimension n, then $A_n \otimes B_0$ is also an NCCW complex of (essential) topological dimension n for every C*-algebra B_0 of finite (linear) dimension.*

Proof. By Theorem 3.9 we have a pullback diagram

$$\begin{array}{ccc} A_n \otimes B_0 & \longrightarrow & A_{n-1} \otimes B_0 \\ \downarrow & & \downarrow \\ (\mathbb{I}^n F_n) \otimes B_0 & \longrightarrow & (\mathbb{S}^{n-1} F_n) \otimes B_0 \end{array}$$

and we note that if $\mathbb{I}_0^n F_n$ is an essential ideal in A_n then $(\mathbb{I}_0^n F_n) \otimes B_0 = \mathbb{I}_0^n (F_n \otimes B_0)$ is an essential ideal in $A_n \otimes B_0$. Since $\mathbb{I}^n F_n \otimes B_0 = \mathbb{I}^n (F_n \otimes B_0)$, and likewise $(\mathbb{S}^{n-1} F_n) \otimes B_0 = \mathbb{S}^{n-1}(F_n \otimes B_0)$, it follows that $A_n \otimes B_0$ is an *NCCW* complex of (essential) topological dimension n, provided that we know that $A_{n-1} \otimes B_0$ is an *NCCW* complex of dimension at most $n-1$. The result follows by induction. ∎

11.14. Theorem. *If A_n and B_m are NCCW complexes of (essential) topological dimensions n and m, then $A_n \otimes B_m$ is an NCCW complex of (essential) dimension $n+m$. Moreover, for each $k \leqslant n$ and $\ell \leqslant m$ the quotient map*

$$A_n \otimes B_m \to A_k \otimes B_\ell$$

onto the tensor products of subcomplexes is a simplicial morphism.

Proof. The argument is by induction on $n+m$, and we note from Lemma 11.13 that the theorem has been established for $n+m \leqslant 1$.

Assume now that the theorem is valid for all n, m with $n+m < n_0$ and take complexes A_n and B_m with $n+m = n_0$. By Theorem 3.8 we have a diagram of extensions, in which the right square is a pullback:

$$\begin{array}{ccccc} \mathbb{I}_0^n F_n \otimes \mathbb{I}_0^m G_m & \hookrightarrow & A_n \otimes B_m & \xrightarrow[\pi]{} & X_{n+m-1} \\ \Vert & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \sigma} \\ \mathbb{I}_0^n F_n \otimes \mathbb{I}_0^m G_m & \hookrightarrow & \mathbb{I}^n F_n \otimes \mathbb{I}^m G_m & \xrightarrow{\partial} & \mathbb{S}^{n+m-1}(F_n \otimes G_m) \end{array}$$

Note here that

$$\mathbb{I}_0^n F_n \otimes \mathbb{I}_0^m G_m = \mathbb{I}_0^{n+m}(F_n \otimes G_m) \qquad \text{and} \qquad \mathbb{I}^n F_n \otimes \mathbb{I}^m G_m = \mathbb{I}^{n+m}(F_n \otimes G_m).$$

Moreover, we have put

$$(A_{n-1} \otimes B_m) \bigoplus_{A_{n-1} \otimes B_{m-1}} (A_n \otimes B_{m-1}) = X_{n+m-1};$$

and we have identified

$$\begin{aligned}(\mathbb{S}^{n-1} F_n \otimes \mathbb{I}^m G_m) &\bigoplus_{\mathbb{S}^{n-1} F_n \otimes \mathbb{S}^{m-1} G_m} (\mathbb{I}^n F_n \otimes \mathbb{S}^{m-1} G_m) \\ &= \mathbb{S}^{n+m-1}(F_n \otimes G_m).\end{aligned}$$

The last equation—stripped of the irrelevant matrix factors—expresses the fact that any sphere $\mathbb{S}^{n+m-1}$ can be obtained from the two solid annuli $\mathbb{S}^{n-1} \times \mathbb{I}^m$ and $\mathbb{I}^n \times \mathbb{S}^{m-1}$, by identifying along the boundary $\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$. The connecting morphisms in the diagram are obvious: $\rho = \rho_A \otimes \rho_B$ and

$$\sigma = (\sigma_A \otimes \rho_B) \oplus (\rho_A \otimes \sigma_B).$$

We note in passing that if both $\mathbb{I}_0^n F_n$ and $\mathbb{I}_0^m G_m$ are essential ideals in A_n and B_m, respectively, then $\mathbb{I}_0^{n+m}(F_n \otimes G_m)$ is an essential ideal in $A_n \otimes B_m$ by Lemma 11.13, so the essential condition in the theorem is validated.

By the induction hypothesis both $A_{n-1} \otimes B_m$, and $A_{n-1} \otimes B_m$ are *NCCW* complexes and their quotient mappings down onto $A_{n-1} \otimes B_{m-1}$ are surjective simplicial morphisms. It follows from Theorem 11.10 that X_{n+m-1} is an *NCCW* complex (of topological dimension $n+m-1$), and from the diagram we see that the C^*-algebra $A_n \otimes B_n = X_{n+m}$ is the *NCCW* complex obtained from X_{n+m-1} via the standard construction.

$$\begin{tikzcd}
\mathbb{I}_0^{n+m} H \arrow[r, hook] \arrow[d, equal] & X_{n+m} \arrow[r, "\pi"'] \arrow[d, "\rho"] & X_{n+m-1} \arrow[d, "\sigma"] \\
\mathbb{I}_0^{n+m} H \arrow[r, hook] & \mathbb{I}^{n+m} H \arrow[r, "\partial"] & \mathbb{S}^{n+m-1} H
\end{tikzcd}$$

where $H = F_n \otimes G_m$. ∎

11.15. *Further Notation.* For any C^*-algebra A we recall the notations from [21, 2.1]:

$$\mathbf{S}A = C_0(]0, 1[, A), \qquad \mathbf{S}_1 A = \{f \in C([0, 1], A) \mid f(0) \in \mathbb{C}, f(1) = 0\},$$

$$\mathbf{C}A = C_0(]0, 1], A), \qquad \mathbf{C}_1 A = \{f \in C([0, 1], A) \mid f(0) \in \mathbb{C}\}.$$

Compared with 11.1 there is some redundancy in this terminology, since $\mathbf{S}A$ (the *suspension* of A) is isomorphic to $\mathbb{I}_0 A$; but several formulae become simpler when we can make the distinction. Note that the operations $\mathbf{C}_1$ and $\mathbf{S}_1$ only apply to unital C^*-algebras (identifying $\mathbb{C}$ with $\mathbb{C}\mathbf{1}$), and that $\mathbf{C}_1 A$ simply is the unitization of $\mathbf{C}A$. By contrast, $\mathbf{S}_1 A \neq (\mathbf{S}A)^\sim$, but the definition is designed so that $\mathbf{C}_1 A/\mathbf{S}_1 A = A$ (just as $\mathbf{C}A/\mathbf{S}A = A$).

For simplicity we now restrict attention to unital *NCCW* complexes, and we recall from 11.2 that this means that all the C^*-algebras in the complex $\{A_0, A_1, ..., A_n\}$ are unital. The general case could be handled by using the two-piece telescope $\mathbf{T}(\mathbb{C}, A)$ and its suspension ideal $\mathbf{S}(\mathbb{C}, A)$ in place of $\mathbf{C}_1 A$ and $\mathbf{S}_1 A$, cf. [21, Corollary 2.3.4]

We have shown earlier, [21, Corollary 2.3.5], that every *NCCW* complex of topological dimension one can be written as a pushout of standard "suspension-type" algebras—because each such is an extension by a finite-dimensional C^*-algebra. We now extend this result to higher dimensions.

11.16. THEOREM. *For every unital NCCW complex* $\{A_0, A_1, ..., A_n\}$ *there is a canonical sequence of commutative diagrams*

$$\begin{tikzcd}
\mathbf{S}_1 A_{k-1} \arrow[r, hook] \arrow[d, "\alpha"] & \mathbf{C}_1 A_{k-1} \arrow[r, "\partial_1"'] \arrow[d, "\gamma"] & A_{k-1} \arrow[d, equal] \\
\mathbb{I}_0^k F_k \arrow[r, hook] \arrow[d, equal] & A_k \arrow[r, "\pi"] \arrow[d, "\rho"] & A_{k-1} \arrow[d, "\sigma_k"] \\
\mathbb{I}_0^k F_k \arrow[r, hook] & \mathbb{I}^k F_k \arrow[r, "\partial"] & \mathbb{S}^{k-1} F_k
\end{tikzcd}$$

where each row is an extension. In particular, for $1 \leqslant k \leqslant n$ *we have*

$$A_k = \mathbf{C}_1 A_{k-1} \bigstar_{\mathbf{S}_1 A_{k-1}} \mathbb{I}_0^k F_k.$$

Proof. The notations are chosen so that the upper row in the diagram is an extension with ∂_1 evaluation of functions in the cone at 1. All we have to do is to define the morphisms α and γ so that the upper half of the diagram also commute, with α proper, since then the *NW* square will be a pushout by Theorem 2.4.

Identifying $\mathbb{I}^k$ with the closed unit ball, so that $\mathbb{I}^k \backslash \{0\} =]0, 1] \times \mathbb{S}^{k-1}$, we can use "polar" coordinates $s = t\theta$ for $\mathbb{I}^k \backslash \{0\}$ with t in $]0, 1]$ and θ in $\mathbb{S}^{k-1}$; and we define a morphism ψ: $\mathbf{C}A_{k-1} \to \mathbb{I}^k F_k$ by

$$\psi(f)(t\theta) = \sigma_k(f(t))(\theta), \qquad f \in \mathbf{C}A_{k-1}.$$

Note that $f(0)=0$, so that this is a meaningful definition. We then extend ψ to all of $\mathbf{C}_1 A_{k-1}$ by setting $\psi(\mathbf{1})=\mathbf{1}$.

Evidently $\partial \circ \psi = \sigma_k \circ \partial_1$ (recall that σ_k is a unital morphism), so the coherent pair (ψ, ∂_1) defines a unital morphism

$$\gamma : \mathbf{C}_1 A_{k-1} \to A_k$$

given by $\gamma(f) = (\psi(f), f(1))$. We take $\alpha = \gamma \mid \mathbf{S}_1 A_{k-1}$ (as we must), and since

$$\partial(\psi(f)) = \sigma_k f(1) = 0$$

for every f in $\mathbf{S}_1 A_{k-1}$, it follows that

$$\alpha(\mathbf{S}_1 A_{k-1}) \subset \ker \partial = \mathbb{I}_0^k F_k.$$

Finally, if we define e in $\mathbf{C}A_{k-1}$ by $e(t) = t\mathbf{1}$, then $\mathbf{1} - e$ is strictly positive in $\mathbf{S}_1 A_{k-1}$ and

$$\alpha(\mathbf{1} - e) = \mathbf{1} - \gamma(e) = (1 - |\mathrm{id}|)\,\mathbf{1}$$

computed in $\mathbb{I}_0^k F_k$, which is a strictly positive element, as desired. ∎

11.16. *Remark.* Another way of phrasing the preceding result (say, at level n) is that: given any unital morphism

$$\sigma : A_{n-1} \to \mathbb{S}^{n-1} F_n$$

between an $(n-1)$-dimensional *NCCW* complex and the model complex $\mathbb{S}^{n-1} F_n$ built over a finite-dimensional C^*-algebra F_n, we can find a commutative diagram of extensions

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{S}_1 A_{n-1} \arrow[r, hook] \arrow[d, "\alpha"] & \mathbf{C}_1 A_{n-1} \arrow[r, "\partial_1"'] \arrow[d, "\psi"] & A_{n-1} \arrow[r] \arrow[d, "\sigma"] & 0 \\
0 \arrow[r] & \mathbb{I}_0^n F_n \arrow[r, hook] & \mathbb{I}^n F_n \arrow[r, "\partial"] & \mathbb{S}^{n-1} F_n \arrow[r] & 0
\end{tikzcd}$$

with α proper, so that the pullback of the *SE* corner equals the pushout of the *NW* corner.

Evidently ψ is a simple (commutative) modification of σ. Taking $n=2$, $F = \mathbb{C}$ and $A_1 = C(\mathbb{S}^1)$ we may choose σ as the identity map. Then ψ is the transposed of the homeomorphism that identifies the unit disk with the cone over $\mathbb{S}^1$ (pulling the center of the disk upwards to the point of the cone).

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