HERMITIAN DUALS AND GENERIC REPRESENTATIONS FOR AFFINE HECKE ALGEBRAS

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Abstract. We further develop the abstract representation theory of affine Hecke algebras with arbitrary positive parameters. We establish analogues of several results that are known for reductive p-adic groups. These include: the relation between parabolic induction/restriction and Hermitian duals, Bernstein's second adjointness and generalizations of the Langlands classification. We check that, in the known cases of equivalences between module categories of affine Hecke algebras and Bernstein blocks for reductive p-adic groups, such equivalences preserve Hermitian duality.

We also initiate the study of generic representation of affine Hecke algebras. Based on an analysis of the Hecke algebras associated to generic Bernstein blocks for quasi-split reductive p-adic groups, we propose a fitting definition of genericity for modules over affine Hecke algebras. With that notion we prove special cases of the generalized injectivity conjecture, about generic subquotients of standard modules for affine Hecke algebras.

Contents

Introduction

Affine Hecke algebras typically arise in two different ways:

- from a presentation with generators and relations,
- from a Bernstein block of smooth representations of a reductive p-adic group.

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The former is more general because the q-parameters for roots of different lengths can be chosen independently, whereas for p-adic groups there is always some algebraic relation between the various q-parameters. Affine Hecke algebras are simpler than p-adic groups, and that has made it possible to derive many results about representations of reductive p-adic groups by studying Hecke algebras.

The motivation for this paper comes from two directions. Firstly, there are well-known results in the representation theory of p-adic groups for which no Hecke algebra version has been worked out. Here we are thinking mainly of more algebraic aspects, roughly speaking the parts of Renard's monograph [Ren] that also make sense for Hecke algebras. We want to prove analogues of those results using only Hecke algebras, that should be easier than for p-adic groups.

Secondly, we are interested in the generalized injectivity conjecture [CaSh], about generic subquotients of standard representations of quasi-split reductive p-adic groups. While this has been verified in many cases [Dij], it remains open in general. We hope that an approach via Hecke algebras can provide new insights in that conjecture.

Hermitian duals

In the representation theory of groups, contragredients of representations play a substantial role. Therefore it would be desirable to develop a notion of contragredient representations for Hecke algebras. While that can be done, there is a problem. Namely, given a smooth representation π in Bernstein block for a reductive p-adic group G, the contragredient $\pi^\vee$ need not lie in the same Bernstein block. So, if this Bernstein block would be equivalent to the module category of an affine Hecke algebra $\mathcal{H}$, a notion of contragredience for $\mathcal{H}$ would never agree with contragredience for smooth G-representations.

Instead, we prefer to use Hermitian duals of complex G-representations, that is, the contragredient of the complex conjugate of a representation. The main advantage is that Hermitian duality for reductive p-adic groups always sends representations in one Bernstein block to the same Bernstein block [Sol7, Lemma 2.2].

For an affine Hecke algebra $\mathcal{H}$, with underlying (extended) affine Weyl group $W \ltimes X$ and positive q-parameters, there is a natural conjugate-linear involution. In the Iwahori–Matsumoto presentation, it is given simply by $T_w^* = T_{w^{-1}}$ for all $w \in W \ltimes X$. The Hermitian dual of an $\mathcal{H}$-representation (π, V) is defined as the vector space $V^\dagger$ of conjugate-linear functions $V \to \mathbb{C}$, with the action

$$\pi^\dagger(h)\lambda(v) = \lambda(\pi(h^*)v) \qquad v \in V, \lambda \in V^\dagger. \tag{1}$$

Before we formulate our first result, let us point out that the affine Hecke algebras that arise from reductive p-adic groups are often of a slightly more general kind. Let Γ be a finite group acting on $\mathcal{H}$, preserving all the structure used to define $\mathcal{H}$. (See Section 8 for the precise setup.) Then we can form the crossed product $\mathcal{H} \rtimes \Gamma$, which is sometimes called an extended affine Hecke algebra. We may also involve a 2-cocycle $\natural : \Gamma^2 \to \mathbb{C}^\times$, which gives rise to a twisted affine Hecke algebra $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$. Of course Γ may be the trivial group, in which case $\mathcal{H} \rtimes \Gamma$ and $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ reduce to $\mathcal{H}$. We prove all our results first for $\mathcal{H}$, and we generalize them to $\mathcal{H} \rtimes \Gamma$ or $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ in Section 8.

Theorem A. (see Theorem 5.3 and Section 8)
Let G be a reductive group over a non-archimedean local field and let $\mathrm{Rep}(G)^{\mathfrak{s}}$ be a Bernstein block in the category of smooth complex G-representations. Suppose

that $\mathrm{Rep}(G)^{\mathfrak{s}}$ *is equivalent to the module category of a twisted affine Hecke algebra* $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$*, via a Morita equivalence as in* [Hei] *or* [Sol6, §10]*. Then the equivalence* $\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{Mod}(\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural])$ *preserves Hermitian duals.*

The Hermitian duals from (1) play a crucial role in our new results about representations of affine Hecke algebras, they are involved in the proofs of all the main results mentioned below.

Representation theory of affine Hecke algebras

For good notions of parabolic subalgebras, parabolic induction and parabolic restriction for $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ with Γ nontrivial, we need some conditions on subgroups of Γ. These are listed in Condition 8.1, which we assume the remainder of the introduction. In our setup, the root system R underlying $\mathcal{H}$ comes with a basis Δ, and parabolic subalgebras $\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]$ are parametrized bijectively by subsets $P \subset \Delta$.

Let w_Δ be the longest element of $W = W(R)$ and define $P^{op} = w_\Delta(-P)$. This is a subset of Δ, which plays the role that an opposite parabolic subgroup plays for reductive groups. There is a *-algebra isomorphism

$$\begin{array}{rccc} \psi_{\Delta P} : & \mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural] & \to & \mathcal{H}^{P^{op}} \rtimes \mathbb{C}[\Gamma_{P^{op}}, \natural] \\ & T_w & \mapsto & T_{w_\Delta w_P w w_P w_\Delta} \qquad\qquad w \in W_P \ltimes X, \end{array}$$

where w_P is the longest element of $W_P = W(R_P)$.

Theorem B. (see Propositions 2.5, 2.7 and Section 8)

(a) *Let* ρ *be a representation of* $\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]$*. Then* $\mathrm{ind}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]}(\rho^\dagger)$ *is canonically isomorphic to* $\mathrm{ind}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]}(\rho)^\dagger$.

(b) *Let* π *be an* $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$*-representation. There is a canonical isomorphism*

$$\mathrm{Res}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]}(\pi^\dagger) \cong \mathrm{Res}_{\mathcal{H}^{P^{op}} \rtimes \mathbb{C}[\Gamma_{P^{op}}, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]}(\pi)^\dagger \circ \psi_{\Delta P}.$$

For Hecke algebras it is easily seen that the parabolic restriction functor

$$\mathrm{Res}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]} : \mathrm{Mod}(\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]) \to \mathrm{Mod}(\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural])$$

is the right adjoint of the parabolic induction functor

$$\mathrm{ind}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]} : \mathrm{Mod}(\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]) \to \mathrm{Mod}(\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]).$$

Like for p-adic groups, it required more effort to find the second adjointness relation for parabolic induction. For graded Hecke algebras that had been achieved in [BaCi], the arguments for affine Hecke algebras are somewhat more complicated.

Theorem C. (see Theorem 3.1 and Section 8)

(a) *The left adjoint of* $\mathrm{ind}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]}$ *is*

$$\psi_{\Delta P}^* \circ \mathrm{Res}_{\mathcal{H}^{P^{op}} \rtimes \mathbb{C}[\Gamma_{P^{op}}, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]} : \pi \mapsto \mathrm{Res}_{\mathcal{H}^{P^{op}} \rtimes \mathbb{C}[\Gamma_{P^{op}}, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]}(\pi) \circ \psi_{\Delta P}.$$

(b) *The right adjoint of* $\mathrm{Res}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]}$ *is*

$$\mathrm{ind}_{\mathcal{H}^{P^{op}} \rtimes \mathbb{C}[\Gamma_{P^{op}}, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]} \circ \psi_{\Delta P *} : \rho \mapsto \mathrm{ind}_{\mathcal{H}^{P^{op}} \rtimes \mathbb{C}[\Gamma_{P^{op}}, \natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]}(\rho \circ \psi_{\Delta P}^{-1}).$$

This is useful in several ways, for instance to find a filtration of the functor parabolic induction followed by parabolic restriction (Proposition 8.3).

Recall that the Langlands classification for a reductive p-adic groups says:

(i) Every standard G-representation has a unique irreducible quotient.
(ii) This yields a bijection between the set of standard G-representations (up to isomorphism) and the set of irreducible smooth G-representations (also up to isomorphism).

By definition a standard G-representation is of the form $I_P^G(\tau \otimes \chi)$, where $P = MU$ is a parabolic subgroup of G, τ is an irreducible tempered M-representation and χ is an unramified character of M in positive position with respect to P. In [Ren] the positivity of χ was relaxed to a more algebraic regularity condition, such that (i) remains valid. Via contragredients or Hermitian duals, one can easily derive a version of the Langlands classification with subrepresentations instead of quotients.

For affine Hecke algebras the normal version of the Langlands classification is known from [Eve, Sol2], but variations like those mentioned above had not been worked out yet. We say that an $\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]$-representation π is $W\Gamma, P$-regular if: for all weights t of π and all $w \in W_P \Gamma_P D_+^{P,P}$, wt is not a weight of π, where

$$D_+^{P,P} = \{d \in W\Gamma : d(P) \subset R^+, d^{-1}(P) \subset R^+, d \notin \Gamma_P\}.$$

This notion relates to standard $\mathcal{H}$-modules in the following ways (Proposition 4.8).

- Suppose that an irreducible tempered $\mathcal{H}^P$-representation τ is twisted by a weight t in positive position for $\mathcal{H}^P$. Then $\tau \otimes t$ is W, P-regular.
- Suppose that an irreducible tempered $\mathcal{H}^P$-representation τ is twisted by a weight t in negative position for $\mathcal{H}^P$. Then $(\tau \otimes t) \circ \psi_{\Delta P}^{-1}$ is a W, P^{op}-regular $\mathcal{H}^{P^{op}}$-representation.

Theorem D. (see Theorem 4.7 and Section 8)
Let $P \subset \Delta$ and let π be an irreducible representation of $\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]$.

(a) *Suppose that π is $W\Gamma, P$-regular. Then $\mathrm{ind}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P,\natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma,\natural]}(\pi)$ has a unique irreducible quotient, namely $\mathrm{ind}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P,\natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma,\natural]}(\pi)$ modulo the kernel of the intertwining operator associated to $(w_\Delta w_P, P, \pi)$.*

(b) *Suppose that $\pi \circ \psi_{\Delta P}^{-1}$ is $W\Gamma, P^{op}$-regular. Then $\mathrm{ind}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P,\natural]}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma,\natural]}(\pi)$ has a unique irreducible subrepresentation, namely the image of the intertwining operator associated to $(w_P w_\Delta, P^{op}, \pi \circ \psi_{\Delta P}^{-1})$.*

Genericity of representations

For quasi-split reductive p-adic groups, the notion of genericity is well-known. For irreducible representations it is equivalent to the existence of a Whittaker model. It is especially useful for the normalization of intertwining operators, for γ-factors via the Langlands–Shahidi method and to select one member from an L-packet in the local Langlands correspondence.

For arbitrary connected reductive p-adic groups, a similar definition of genericity is available [BuHe]. In that generality it is convenient to consider representations which are simply generic, meaning that the multiplicity one property of Whittaker functionals holds by assumption.

For (extended) affine Hecke algebras no independent definition of genericity was known, so we provide one. The elements T_w with $w \in W\Gamma$ span a finite dimensional subalgebra $\mathcal{H}(W, q^\lambda) \rtimes \Gamma$ of $\mathcal{H} \rtimes \Gamma$. Let $\det_X$ be the determinant of the action of $W\Gamma$ on the lattice X. The Steinberg representation of $\mathcal{H}(W, q^\lambda) \rtimes \Gamma$ has dimension one and is defined by $\mathrm{St}(T_w) = \det_X(w)$. We say that a representation π of $\mathcal{H} \rtimes \Gamma$

is generic if its restriction to $\mathcal{H}(W, q^\lambda) \rtimes \Gamma$ contains St. This definition is justified by the following result.

Theorem E. (see Proposition 6.2 and Theorem A.1)
Let G be a connected reductive group over a non-archimedean local field. Let $\mathrm{Rep}(G)^{\mathfrak{s}}$ be a Bernstein block of smooth complex G-representations, such that the underlying supercuspidal representations are simply generic.

(*a*) $\mathrm{Rep}(G)^{\mathfrak{s}}$ *is equivalent to the module category of an extended affine Hecke algebra $\mathcal{H} \rtimes \Gamma$ with parameters in $\mathbb{R}_{\geq 1}$.*
(*b*) *With the normalizations from* [Sol7, §2]*, the equivalence* $\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{Mod}(\mathcal{H} \rtimes \Gamma)$ *preserves genericity.*

For affine Hecke algebras with q-parameters in $\mathbb{R}_{\geq 1}$, one can hope for a version of the generalized injectivity conjecture. Using our previous findings in the representation theory of Hecke algebras, we take some steps in that direction.

By definition the maximal commutative subalgebra $\mathcal{A} \cong \mathbb{C}[X]$ of $\mathcal{H}$ is the unique minimal parabolic subalgebra of $\mathcal{H} \rtimes \Gamma$. The basis Δ of R determines a positive cone in $\mathrm{Hom}(X, \mathbb{R}_{>0})$.

Theorem F. (see Propositions 7.6 and 8.6)
Let $\mathcal{H} \rtimes \Gamma$ be an extended affine Hecke algebra with q-parameters in $\mathbb{R}_{\geq 1}$.

(*a*) *For $t \in \mathrm{Hom}(X, \mathbb{C}^\times)$, the parabolically induced representation $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H} \rtimes \Gamma}(t)$ has a unique generic constituent, say π_t.*
(*b*) *When $|t|$ lies in the closure of the positive cone in $\mathrm{Hom}(X, \mathbb{R}_{>0})$, π_t is a subrepresentation of $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H} \rtimes \Gamma}(t)$.*
(*c*) *When $|t^{-1}|$ lies in the closure of the positive cone in $\mathrm{Hom}(X, \mathbb{R}_{>0})$, π_t is a quotient of $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H} \rtimes \Gamma}(t)$.*

In spite of this result, the generalized injectivity conjecture does not always hold for standard $\mathcal{H}$-representations that are induced from parabolic subalgebras other than $\mathcal{A}$, see Example 7.7. The problem seems to be that arbitrary q-parameters (in $\mathbb{R}_{\geq 1}$) offer too much freedom. We expect that the generalized injectivity conjecture does hold for affine Hecke algebras $\mathcal{H}$ whose q-parameters come from reductive p-adic groups. To all appearances such q-parameters are geometric in the sense of [Sol5, §5.3], so that algebro-geometric techniques to study representations of such Hecke algebras are available.

1. Preliminaries

We fix notations and recall a few basic notions about affine Hecke algebras. For more background we refer to [Lus, Opd2, Sol5].
Let R be a root system with basis Δ and positive roots R^+. Let $\mathcal{R} = (X, R, Y, R^\vee, \Delta)$ be a based root datum. It yields a Weyl group $W = W(R)$, with set of simple reflections $S = \{s_\alpha : \alpha \in \Delta\}$. For $\alpha \in R$ such that $\alpha^\vee \in R^\vee$ is maximal with respect to $\Delta^\vee$, we define the simple affine reflection

$$\begin{array}{rccl} s'_\alpha : & X & \to & X \\ & x & \mapsto & s_\alpha(x) + \alpha = x + (1 - \langle x, \alpha^\vee \rangle)\alpha. \end{array}$$

Then $S_{\mathrm{aff}} := S \cup \{s'_\alpha : \alpha^\vee \in R^\vee_{\max}\}$ is a set of Coxeter generators for the affine Weyl group

$$W_{\mathrm{aff}} = \langle S_{\mathrm{aff}} \rangle = W \ltimes \mathbb{Z}R.$$

It is a normal subgroup of the extended affine Weyl group $W(\mathcal{R}) = W \ltimes X$. The length function ℓ of W_{aff} extends naturally to $W \ltimes X$. Moreover the set of length zero elements $\Omega = \{w \in W \ltimes X : \ell(w) = 0\}$ is a group and

$$W \ltimes X = W_{\mathrm{aff}} \rtimes \Omega.$$

We fix $q \in \mathbb{R}_{>1}$ and we let $\lambda, \lambda^* : R \to \mathbb{R}$ be functions such that

- if $\alpha, \beta \in R$ are in the same W-orbit, then $\lambda(\alpha) = \lambda(\beta)$ and $\lambda^*(\alpha) = \lambda^*(\beta)$;
- if $\alpha^\vee \notin 2Y$, then $\lambda^*(\alpha) = \lambda(\alpha)$.

To every simple (affine) reflection we associate a q-parameter, by

$$q_{s_\alpha} = q^{\lambda(\alpha)} \quad \text{and} \quad q_{s'_\alpha} = q^{\lambda^*(\alpha)}.$$

The Iwahori–Hecke algebra $\mathcal{H}(W_{\mathrm{aff}}, \lambda, \lambda^*, q)$ can be presented as the vector space with basis $\{N_w : w \in W_{\mathrm{aff}}\}$ and multiplication rules (for $w \in W_{\mathrm{aff}}$ and $s \in S_{\mathrm{aff}}$)

$$N_w N_s = \begin{cases} N_{ws} & \text{if } \ell(ws) = \ell(w) + 1 \\ N_{ws} + (q_s^{1/2} - q_s^{-1/2}) N_w & \text{if } \ell(ws) = \ell(w) - 1 \end{cases}. \tag{1.1}$$

Notice that $q_s^{1/2}$ is unambiguous, because $q_s \in \mathbb{R}_{>0}$. The conjugation action of Ω on W_{aff} induces an action on $\mathcal{H}(W_{\mathrm{aff}}, \lambda, \lambda^*, q)$. That enables us to construct the affine Hecke algebra

$$\mathcal{H} := \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q) = \mathcal{H}(W_{\mathrm{aff}}, \lambda, \lambda^*, q) \rtimes \Omega,$$

which has a vector space basis $\{N_w : w \in W \ltimes X\}$. This is a version of the Iwahori–Matsumoto presentation of $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q)$. More common, and already used in [IwMa], is the same presentation expressed in terms of the basis $\{T_w : w \in W \ltimes X\}$, where $T_s = q_s^{1/2} N_s$ for $s \in S_{\mathrm{aff}}$.

There is another well-known presentation, due to Bernstein. To that end, we define elements θ_x ($x \in X$) by the following recipe. If $x = x_1 - x_2$ where $\langle x_1, \alpha^\vee \rangle \geq 0$ and $\langle x_2, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$, then $\theta_x = N_{x_1} N_{x_2}^{-1}$.

The set $\{\theta_x : x \in X\}$ spans a commutative subalgebra $\mathcal{A}$ of $\mathcal{H}$, canonically isomorphic with $\mathbb{C}[X]$. Let $\mathcal{H}(W, q^\lambda)$ be the Iwahori–Hecke algebra of W, with respect to the parameter function $q^\lambda : R \to \mathbb{R}_{>0}$. According to Bernstein, the multiplication maps

$$\mathcal{H}(W, q^\lambda) \otimes \mathcal{A} \; \to \; \mathcal{H} \; \leftarrow \; \mathcal{A} \otimes \mathcal{H}(W, q^\lambda) \tag{1.2}$$

are bijections. The cross relations for multiplication of elements of $\mathcal{H}(W, q^\lambda)$ and of $\mathcal{A}$ can be described explicitly. It follows from those relations that the centre of $\mathcal{H}$ is $\mathcal{A}^W$, where W acts on $\mathcal{A} \cong \mathbb{C}[X]$ via its canonical action on X.

For a set of simple roots $P \subset \Delta$ we have a parabolic subrootsystem $R_P \subset R$ and a parabolic subgroup $W_P = W(R_P)$. The parabolic subalgebra $\mathcal{H}^P \subset \mathcal{H}$ is generated by $\mathcal{A}$ and the N_w with $w \in W_P$. One can identify $\mathcal{H}^P$ with $\mathcal{H}(X, R_P, Y, R_P^\vee, P, \lambda, \lambda^*, q)$. In particular $\mathcal{H}^\emptyset = \mathcal{A}$ and $\mathcal{H}^\Delta = \mathcal{H}$.

We write $T = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$, this is a complex torus. It has subtori

$$T^P = \mathrm{Hom}_{\mathbb{Z}}(X / X \cap \mathbb{Q}P, \mathbb{C}^\times), \qquad T_P = \mathrm{Hom}_{\mathbb{Z}}(X / P^{\vee\perp}, \mathbb{C}^\times),$$

where $P^{\vee\perp} = \{x \in X : \langle x, \alpha^\vee \rangle = 0 \; \forall \alpha \in P\}$. Any $t \in T^P$ gives rise to an algebra automorphism

$$\begin{array}{llll} \psi_t : & \mathcal{H}^P & \to & \mathcal{H}^P \\ & N_w \theta_x & \mapsto & t(x) N_w \theta_x \qquad w \in W_P, x \in X. \end{array}$$

For an $\mathcal{H}^P$-representation π and $t \in T^P$ we write $\pi \otimes t = \pi \circ \psi_t$.

We define a conjugate-linear involution $*$ on $\mathcal{H}$ by

$$\Big(\sum\nolimits_{w \in W \ltimes X} z_w N_w\Big)^* = \sum\nolimits_{w \in W \ltimes X} \overline{z_w} N_{w^{-1}}.$$

Here we need $q_s \in \mathbb{R}_{>0}$ for all $s \in S_{\mathrm{aff}}$. We can regard $*$ as an $\mathbb{R}$-linear isomorphism from $\mathcal{H}$ to its opposite algebra. This involution interacts well with the trace

$$\tau : \mathcal{H} \to \mathbb{C}, \qquad \tau(N_w) = \begin{cases} 1 & \text{if } w = e \\ 0 & \text{if } w \in W(\mathcal{R}) \setminus \{e\} \end{cases}. \tag{1.3}$$

Namely, the formula

$$\langle h_1, h_2 \rangle = \tau(h_1 h_2^*) \qquad h_1, h_2 \in \mathcal{H} \tag{1.4}$$

defines an inner product on $\mathcal{H}$, linear in the first variable. The set $\{N_w : w \in W(\mathcal{R})\}$ is an orthonormal basis of $\mathcal{H}$ with this inner product. We note that (1.4) makes the left regular representation of $\mathcal{H}$ pre-unitary (i.e. a *-representation on an inner product space that need not be complete):

$$\langle h_1 h_2, h_3 \rangle = \tau(h_1 h_2 h_3^*) = \tau(h_2 h_3^* h_1) = \langle h_2, h_1^* h_3 \rangle \qquad h_1, h_2, h_3 \in \mathcal{H}. \tag{1.5}$$

The parabolic subalgebra $\mathcal{H}^P$ has its own involution $*_P$, which usually differs from $*|_{\mathcal{H}^P}$. In fact $\mathcal{H}^P$ is typically not a *-subalgebra of $\mathcal{H}$. Let w_P be the longest element of W_P. Recall that w_P has order two and that the set of positive roots made negative by w_P is precisely R_P^+. By [Opd1, Proposition 1.12]:

$$*_P\,(\theta_x) = N_{w_P} \theta_{-w_P(x)} N_{w_P}^{-1}. \tag{1.6}$$

For $P = \emptyset$ we get $w_\emptyset = 1$, so $*_\emptyset(\theta_x) = \theta_{-x}$.

For a subset $\tilde{W} \subset W$, let $\mathcal{H}(\tilde{W})$ be the linear subspace of $\mathcal{H}(W, q^\lambda)$ spanned by $\{N_w : w \in \tilde{W}\}$. Let

$$W^P = \{w \in W : w(P) \subset R^+\}$$

be the set of shortest length representatives for W/W_P. By (1.2) the multiplication map $\mathcal{H}(W^P) \otimes \mathcal{H}^P \to \mathcal{H}$ is a linear bijection. In particular every $h \in \mathcal{H}$ can be written as

$$h = \sum\nolimits_{w \in W^P} N_w h_w^P \quad \text{for unique } h_w^P \in \mathcal{H}^P. \tag{1.7}$$

The next result is analogous to [BaMo, Proposition 1.4] for graded Hecke algebras.

Lemma 1.1. *$(h^*)_e^P = (h_e^P)^{*_P}$ for all $h \in \mathcal{H}$.*

Proof. By conjugate-linearity it suffices to consider h of the form $N_w \theta_x$ with $w \in W$ and $x \in X$. From (1.6) we see that

$$h^* = N_{w_\Delta} \theta_{-w_\Delta(x)} N_{w_\Delta}^{-1} N_{w^{-1}} \quad \text{and} \quad (h_e^P)^{*_P} = \begin{cases} N_{w_P} \theta_{-w_P(x)} N_{w_P}^{-1} N_{w^{-1}} & w \in W_P, \\ 0 & w \notin W_P. \end{cases}$$

We recall from [Hum, §1.8] that

$$\ell(w^{-1}) + \ell(w w_\Delta) = \ell(w_\Delta) = \ell(w_\Delta w) + \ell(w^{-1}).$$

By definition of the multiplication in $\mathcal{H}(W, q^\lambda)$:

$$N_{w_\Delta w} N_{w^{-1}} = N_{w_\Delta} = N_{w^{-1}} N_{w w_\Delta}, \tag{1.8}$$

$$N_{w_\Delta} \theta_{-w_\Delta(x)} N_{w_\Delta}^{-1} N_{w^{-1}} = N_{w_\Delta} \theta_{-w_\Delta(x)} N_{w w_\Delta}^{-1}. \tag{1.9}$$

For a simple reflection $s \in W$, $N_s^{-1} = N_s + (q_s^{-1/2} - q_s^{1/2}) N_e$. That and the multiplication relations in the Bernstein presentation of $\mathcal{H}$ [Lus, §3] show that

$$\theta_y N_s^{-1} - N_s^{-1} \theta_{s(y)} \in \mathcal{A} \qquad \text{for all } y \in X. \tag{1.10}$$

We denote the Bruhat order on W by $\leq$. Applying (1.10) recursively, (1.9) can be expressed as $N_{w_\Delta} \sum_{v \in W, v \leq w w_\Delta} N_v^{-1} a_v$ for suitable $a_v \in \mathcal{A}$. By (1.8) that equals $\sum_{v \in W, v \leq w w_\Delta} N_{w_\Delta v^{-1}} a_v$. Here $v^{-1} \leq w_\Delta w^{-1}$, so $w_\Delta v^{-1} \geq w^{-1}$.

Suppose that $w \notin W_P$. Any reduced expression of w^{-1} contains simple reflections not in W_P, so the same goes for $w_\Delta v^{-1}$ with v as above. Hence $w_\Delta v^{-1} \notin W_P$, and it can be written as uw' with $u \in W^P \setminus \{e\}$ and $w' \in W_P$. Thus $N_{w_\Delta v^{-1}} a_v \in N_u \mathcal{H}^P$. That works for every $v \leq w w_\Delta$, showing that (1.9) lies in $\mathcal{H}(W^P \setminus \{e\}) \mathcal{H}^P$. In other words, $(h^*)^P_e = 0$.

Suppose that $w \in W_P$. We need to show that $\mathcal{H}(W^P \setminus \{e\}) \mathcal{H}^P$ contains

$$N_{w_\Delta} \theta_{-w_\Delta(x)} N_{w_\Delta}^{-1} N_{w^{-1}} - N_{w_P} \theta_{-w_P(x)} N_{w_P}^{-1} N_{w^{-1}}. \tag{1.11}$$

With (1.8) we rewrite this element as

$$\big(N_{w_\Delta} \theta_{-w_\Delta(x)} N_{w_P w_\Delta}^{-1} - N_{w_P} \theta_{-w_P(x)} \big) N_{w_P}^{-1} N_{w^{-1}}.$$

Reasoning as above we find

$$N_{w_\Delta} \theta_{-w_\Delta(x)} N_{w_P w_\Delta}^{-1} = \sum\nolimits_{v \in W, v \leq w_P w_\Delta} N_{w_\Delta v^{-1}} a_v .$$

Here $w_\Delta v^{-1} \geq w_P$, so this only belongs to W_P if $v = w_P w_\Delta$. From (1.10) one obtains

$$a_{w_P w_\Delta} = \theta_{w_P w_\Delta (-w_\Delta(x))} = \theta_{-w_P(x)} .$$

Then (1.11) reduces to

$$\sum\nolimits_{v \in W, v < w_P w_\Delta} N_{w_\Delta v^{-1}} a_v N_{w_P}^{-1} N_{w^{-1}} .$$

The same argument as in the case $w \notin W_P$ shows that this lies in $\mathcal{H}(W^P \setminus \{e\}) \mathcal{H}^P$. $\square$

2. Hermitian duals

For any complex vector space V, let $V^\dagger$ be space of conjugate-linear functions from V to $\mathbb{C}$. In case V has a topology, it is understood that $V^\dagger$ consists of the continuous conjugate-linear functionals on V. If (π, V_π) is an $\mathcal{H}$-representation, then $\mathcal{H}$ acts on $V_\pi^\dagger$ by

$$(h \cdot \lambda)(v) = \lambda(h^* v) \qquad h \in \mathcal{H}, v \in V_\pi, \lambda \in V_\pi^\dagger . \tag{2.1}$$

This defines the Hermitian dual $(\pi^\dagger, V_\pi^\dagger)$ of the $\mathcal{H}$-representation (π, V_π). For any $(\rho, V_\rho) \in \mathrm{Mod}(\mathcal{H})$ there is a conjugate-linear "transposition" isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{H}}(\pi, \rho^\dagger) & \cong & \mathrm{Hom}_{\mathcal{H}}(\rho, \pi^\dagger) \\ \phi & \mapsto & \phi^\dagger \end{array} \tag{2.2}$$

Here $\phi^\dagger$ sends $w \in V_\rho$ to $[v \mapsto \overline{\phi(v) w}]$ with $v \in V_\pi$.

Sometimes a representation is isomorphic to its Hermitian dual. For example, suppose that V_π is a Hilbert space and that the representation π is unitary:

$$\langle \pi(h) v, v' \rangle = \langle v, \pi(h^*) v' \rangle \qquad v, v' \in V_\pi, h \in \mathcal{H}.$$

Then $(\pi^\dagger, V_\pi^\dagger)$ can be identified with (π, V_π) via the inner product. Similarly we can consider the left regular representation of $\mathcal{H}$. Via τ, we can identify

$$\mathcal{H}^\dagger = \prod\nolimits_{w \in W(\mathcal{R})} \mathbb{C} N_w, \tag{2.3}$$

a completion of $\mathcal{H}$. This $\mathcal{H}^\dagger$ is naturally an $\mathcal{H}$-bimodule, and (1.5) remains valid for $h_1, h_3 \in \mathcal{H}$, $h_2 \in \mathcal{H}^\dagger$. Hence the Hermitian dual of $\mathcal{H}$ is $\mathcal{H}^\dagger$, with $\mathcal{H}$ acting by left multiplication on both. The projectivity of $\mathcal{H}$, in combination with (2.2), implies that $\mathrm{Hom}_{\mathcal{H}}(?, \mathcal{H}^\dagger)$ is an exact functor. In other words, $\mathcal{H}^\dagger$ is an injective $\mathcal{H}$-module.

The module $\mathcal{H}^\dagger$ enables us to describe Hermitian duals of modules induced from $\mathcal{H}(W, q^\lambda)$. We recall that, as $\mathcal{H}(W, q^\lambda)$ is finite dimensional and semisimple, all its irreducible modules appear in the (left) regular representation. In fact each irreducible module is the image of suitable a minimal idempotent.

Lemma 2.1. (*a*) *Let $V \in \mathrm{Mod}(\mathcal{H}(W, q^\lambda))$ be irreducible, and let $p_V \in \mathcal{H}(W, q^\lambda)$ be an idempotent so that $V \cong \mathcal{H}(W, q^\lambda) p_V$. The Hermitian dual of $\mathrm{ind}_{\mathcal{H}(W,q^\lambda)}^{\mathcal{H}} V$ is $\mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} V^\dagger$, with the pairing*

$$\langle h_1 \otimes p_V^*, h_2 \otimes p_V \rangle = \tau(h_1 p_V^* h_2^*) \qquad h_1 \in \mathcal{H}^\dagger, h_2 \in \mathcal{H}.$$

(*b*) *The functor $\mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} : \mathrm{Mod}(\mathcal{H}(W, q^\lambda)) \to \mathrm{Mod}(\mathcal{H})$ is right adjoint to the restriction functor $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}(W,q^\lambda)}$.*

Proof. (a) The $\mathcal{H}(W, q^\lambda)$-module $\mathcal{H}(W, q^\lambda) p_V^*$ is the Hermitian dual of $\mathcal{H}(W, q^\lambda) p_V$, with respect to the pairing

$$\langle h_1 p_V^*, h_2 p_V \rangle = \tau(h_1 p_V^* h_2^*) \qquad h_1, h_2 \in \mathcal{H}(W, q^\lambda).$$

Since $\mathcal{H}(W, q^\lambda) p_V$ is a direct summand of the left regular representation of $\mathcal{H}(W, q^\lambda)$, $\mathrm{ind}_{\mathcal{H}(W,q^\lambda)}^{\mathcal{H}} \mathcal{H}(W, q^\lambda) p_V$ is a direct summand of the $\mathcal{H}$-representation on

$$\mathrm{ind}_{\mathcal{H}(W,q^\lambda)}^{\mathcal{H}} \mathcal{H}(W, q^\lambda) = \mathcal{H}.$$

Similarly $\mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathcal{H}(W, q^\lambda) p_V^*$ is a direct summand of the $\mathcal{H}$-module $\mathcal{H}^\dagger$. For $h_1 \in \mathcal{H}^\dagger$ and $h_2 \in \mathcal{H}$ we have

$$\langle h_1 p_V^*, h_2 \rangle = \tau(h_1 p_V^* h_2^*) = \langle h_1 p_V^*, h_2 p_V \rangle.$$

Hence the Hermitian dual we want is

$$\mathcal{H}^\dagger \underset{\mathcal{H}(W,q^\lambda)}{\otimes} \mathcal{H}(W, q^\lambda) p_V^* \cong \mathcal{H}^\dagger \underset{\mathcal{H}(W,q^\lambda)}{\otimes} V^\dagger .$$

(b) Let $Y \in \mathrm{Mod}(\mathcal{H})$ and $V \in \mathrm{Mod}(\mathcal{H}(W, q^\lambda))$. We regard $\mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} V$ as a set of conjugate-linear maps from $\mathcal{H}$ to V. One checks readily that the maps

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{H}(W,q^\lambda)}(Y, V) & \longleftrightarrow & \mathrm{Hom}_{\mathcal{H}}(Y, \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} V) \\ f & \mapsto & \big[y \mapsto [h \mapsto f(h^* y)] \big] \\ [y \mapsto \phi(y)(1)] & \leftarrow\!\shortmid & \phi \end{array}$$

are natural bijections. □

The relation between Hermitian duals and tensoring with characters can be described easily:

Lemma 2.2. *Let (π, V_π) be an $\mathcal{H}^P$-representation and let $t \in T^P$. The Hermitian dual of $\pi \otimes t$ is $\pi^\dagger \otimes \bar{t}^{-1}$.*

Proof. Take $v \in V_\pi, \lambda \in V_\pi^\dagger, w \in W_P$ and $x \in X$. With (1.6) we compute

$$\begin{aligned}
\lambda\big((\pi \otimes t)(N_w\theta_x)v\big) &= \lambda\big(t(x)\pi(N_w\theta_x)v\big) = \overline{t(x)}\lambda\big(\pi(N_w\theta_x)v\big) \\
&= \overline{t(x)}\big(\pi^\dagger((N_w\theta_x)^{*_P})\lambda\big)(v) = \overline{t(x)}\big(\pi^\dagger((N_w\theta_x)^{*_P})\lambda\big)(v) \\
&= \overline{t(x)}\big(\pi^\dagger(N_{w_P}\theta_{-w_P(x)}N_{w_P}^{-1}N_{w^{-1}})\lambda\big)(v) \\
&= \big((\pi^\dagger \otimes \overline{w_P t}^{-1})(N_{w_P}\theta_{-w_P(x)}N_{w_P}^{-1}N_{w^{-1}})\lambda\big)(v) \\
&= \big((\pi^\dagger \otimes \overline{w_P t}^{-1})((N_w\theta_x)^{*_P})\lambda\big)(v).
\end{aligned}$$

This shows that $(\pi \otimes t)^\dagger = \pi^\dagger \otimes \overline{w_P t}^{-1}$. But $w_P t = t$ because $w_P \in W(R_P)$ and $t \in \mathrm{Hom}_{\mathbb{Z}}(X/X \cap \mathbb{Q}P, \mathbb{C}^\times)$. □

We want to find the relation between parabolic induction (from $\mathcal{H}^P$ to $\mathcal{H}$) and Hermitian duals. That will be achieved in a few steps, the first of which is making the relation between * and $*_P$ explicit.

Lemma 2.3. *For $w \in W_P$ and $x \in X$:*

$$* *_P (N_w\theta_x) = N_w N_{w_P w_\Delta}\theta_{w_\Delta w_P(x)} N_{w_P w_\Delta}^{-1}.$$

Proof. By definition

$$* *_P (N_w) = *(N_{w^{-1}}) = N_w.$$

From (1.6) and the anti-homomorphism property of * we obtain

$$* *_P(\theta_x) = *(N_{w_P}^{-1}) * (\theta_{-w_P(x)}) * (N_{w_P}) = N_{w_P}^{-1} N_{w_\Delta}\theta_{w_\Delta w_P(x)} N_{w_\Delta}^{-1} N_{w_P}. \tag{2.4}$$

We note that here the lengths of the involved elements of W add up:

$$\ell(w_P w_\Delta) + \ell(w_P) = |R^+ \setminus R_P^+| + |R_P^+| = |R^+| = \ell(w_\Delta)$$

Therefore $N_{w_P} N_{w_P w_\Delta} = N_{w_\Delta}$, and the right-hand side of (2.4) simplifies to $N_{w_P w_\Delta}\theta_{w_\Delta w_P(x)} N_{w_P w_\Delta}^{-1}$, proving the statement for θ_x.

To conclude, we use that $* *_P$ is an algebra homomorphism. □

For $h \in \mathcal{H}^\times$, let $\mathfrak{c}_h : \mathcal{H} \to \mathcal{H}$ denote conjugation with h. We define

$$\psi_{\Delta P} = \mathfrak{c}_{N_{w_P w_\Delta}^{-1}} \circ * *_P : \mathcal{H}^P \to \mathcal{H}.$$

Since $* *_P$ and $\mathfrak{c}_{N_{w_P w_\Delta}^{-1}}$ are injective algebra homomorphisms, so is $\psi_{\Delta P}$. We write

$$P^{op} = w_\Delta(-P).$$

This is a set of simple roots, it may or may not be equal to P. We note that $w_\Delta W_P w_\Delta = W_{P^{op}}$. In comparison with reductive groups, P^{op} replaces the notion of an opposite parabolic subgroup.

Lemma 2.4. (a) *For $w \in W_P$ and $x \in X$:*

$$\psi_{\Delta P}(N_w\theta_x) = N_{w_\Delta w_P w w_P w_\Delta}\theta_{w_\Delta w_P(x)}.$$

(b) *$\psi_{\Delta P}$ is an *-algebra isomorphism from $\mathcal{H}^P$ to $\mathcal{H}^{P^{op}}$, with inverse $\psi_{\Delta P^{op}}$.*

Proof. (a) Consider the algebra isomorphism

$$\begin{array}{llll} \psi_{w_P w_\Delta} : & \mathcal{H}^{P^{op}} & \to & \mathcal{H}^P \\ & N_{w'}\theta_x & \mapsto & N_{w_P w_\Delta w' w_\Delta w P}\theta_{w_P w_\Delta(x)} \qquad w' \in W_{P^{op}}, x \in X. \end{array} \tag{2.5}$$

Then $\psi_{\Delta P} \circ \psi_{w_P w_\Delta} : \mathcal{H}^{P^{op}} \to \mathcal{H}$ is an injective algebra homomorphism, and by Lemma 2.3:

$$\begin{aligned} \psi_{\Delta P} \circ \psi_{w_P w_\Delta}(N_{w'}\theta_x) &= \mathfrak{c}_{N^{-1}_{w_P w_\Delta}} \big(N_{w_P w_\Delta w' w_\Delta w_P} N_{w_P w_\Delta} \theta_x N^{-1}_{w_P w_\Delta} \big) \\ &= N^{-1}_{w_P w_\Delta} N_{w_P w_\Delta w' w_\Delta w_P} N_{w_P w_\Delta} \theta_x. \end{aligned} \tag{2.6}$$

Notice that $\psi_{\Delta P} \circ \psi_{w_P w_\Delta}$ is the identity on $\mathcal{A}$ and sends $\mathcal{H}(W_{P^{op}}, q^\lambda)$ bijectively to itself. For $\alpha \in P^{op}$, N_{s_α} commutes with the same elements of $\mathcal{A}$ as $\psi_{\Delta P} \circ \psi_{w_P w_\Delta}(N_{s_\alpha})$. That forces

$$\psi_{\Delta P} \circ \psi_{w_P w_\Delta}(N_{s_\alpha}) \in \mathbb{C} N_e + \mathbb{C} N_{s_\alpha}.$$

Furthermore $\psi_{\Delta P} \circ \psi_{w_P w_\Delta}(N_{s_\alpha})$ has the same eigenvalues $q_{s_\alpha}^{1/2}$ and $-q_{s_\alpha}^{-1/2}$ as N_{s_α}, so it can only be N_{s_α} or $-N_{s_\alpha}^{-1}$. The involved constructions work for any $q \in \mathbb{R}_{>0}$, and depend continuously on q. For $q = 1$ we see directly from (2.6) that $\psi_{\Delta P} \circ \psi_{w_P w_\Delta}(N_{s_\alpha}) = N_{s_\alpha}$. Hence $\psi_{\Delta P} \circ \psi_{w_P w_\Delta}(N_{s_\alpha})$ cannot be $-N_{s_\alpha}^{-1}$ for any $q \in \mathbb{R}_{>0}$.

We deduce that $\psi_{\Delta P} \circ \psi_{w_P w_\Delta}(N_{w'}) = N_{w'}$ for w' any simple reflection in $W_{P^{op}}$, and then the same follows for all $w' \in W_{P^{op}}$. Apply that to $w' = w_\Delta w_P w w_P w_\Delta$.

(b) By part (a) and (2.6), $\psi_{\Delta P} \circ \psi_{w_P w_\Delta}$ is the identity on $\mathcal{H}^{P^{op}}$. As $\psi_{w_P w_\Delta} : \mathcal{H}^{P^{op}} \to \mathcal{H}^P$ is an isomorphism, this shows that $\psi_{\Delta P}$ is its inverse. By construction $w_{P^{op}} = w_\Delta w_P w_\Delta$. From that, part (a) and (2.5) we see that $\psi_{w_P w_\Delta} = \psi_{\Delta P^{op}}$.

Further, from (2.5) and the definition of θ_x we obtain

$$\psi_{w_P w_\Delta}(N_{w'}) = N_{w_P w_\Delta w' w_\Delta w_P} \qquad \text{for all } w' \in W_{P^{op}} \rtimes X.$$

This shows that $\psi_{w_P w_\Delta}$ is in fact a *-isomorphism, and hence so it its inverse. □

Now we can relate Hermitian duals and parabolic restriction.

Proposition 2.5. *Let (π, V_π) be an $\mathcal{H}$-representation.*

(*a*) *The Hermitian dual of the $\mathcal{H}^P$-representation $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}(\pi)$ is isomorphic with $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\pi^\dagger) \circ \psi_{\Delta P}$.*

(*b*) $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}(\pi^\dagger) \cong \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\pi)^\dagger \circ \psi_{\Delta P}$.

Proof. By definition

$$\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}(\pi)^\dagger = \pi^\dagger \circ *\,*_P = \pi^\dagger \circ \mathfrak{c}_{N_{w_P w_\Delta}} \circ \psi_{\Delta P}.$$

Since $N_{w_P w_\Delta} \in \mathcal{H}^\times$, multiplication with $\pi^\dagger(N^{-1}_{w_P w_\Delta})$ provides an isomorphism from the right-hand side to $\pi^\dagger \circ \psi_{\Delta P}$. By Lemma 2.4.b, that can be regarded as $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\pi^\dagger) \circ \psi_{\Delta P}$.

(b) Start with part (a) for P^{op}. Composing the representations on both sides with $\psi^{-1}_{\Delta P^{op}} = \psi_{\Delta P}$ gives

$$\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}(\pi^\dagger) \cong \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\pi)^\dagger \circ \psi^{-1}_{\Delta P^{op}} = \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\pi)^\dagger \circ \psi_{\Delta P}. \qquad \square$$

We note that the pairing underlying Proposition 2.5.a is

$$\begin{array}{ccc} \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(V_\pi^\dagger) \times \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}(V_\pi) & \to & \mathbb{C} \\ (\lambda, v) & \mapsto & \pi^\dagger(N_{w_P w_\Delta})\lambda(v) = \lambda(\pi(N_{w_\Delta w_P})v). \end{array}$$

Similarly the pairing underlying Proposition 2.5.b is given by

$$\begin{array}{ccc} \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}(V_\pi^\dagger) \times \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(V_\pi) & \to & \mathbb{C} \\ (\lambda, v) & \mapsto & \pi^\dagger(N_{w_\Delta w_P})\lambda(v) = \lambda(\pi(N_{w_P w_\Delta})v). \end{array}$$

An important special case arise when $P = \emptyset$. Then $\psi_{\Delta\emptyset}(\theta_x) = \theta_{w_\Delta(x)}$ and Proposition 2.5 provides isomorphisms

$$\mathrm{Res}^{\mathcal H}_{\mathcal A}(\pi)^\dagger \cong \mathrm{Res}^{\mathcal H}_{\mathcal A}(\pi^\dagger) \circ \psi_{\Delta\emptyset} \quad \text{and} \quad \mathrm{Res}^{\mathcal H}_{\mathcal A}(\pi^\dagger) \cong \mathrm{Res}^{\mathcal H}_{\mathcal A}(\pi)^\dagger \circ \psi_{\Delta\emptyset}. \tag{2.7}$$

We move on to parabolic induction. Consider an $\mathcal H^P$-representation (ρ, V_ρ) and its induction $\mathrm{ind}_{\mathcal H^P}^{\mathcal H}(\rho)$. The underlying vector space is

$$\mathrm{ind}_{\mathcal H^P}^{\mathcal H}(V_\rho) = \mathcal H \otimes_{\mathcal H^P} V_\rho \cong \mathcal H(W^P) \otimes_{\mathbb C} V_\pi .$$

Here $W^P = \{w \in W : w(P) \subset R^+\}$ denotes the set of shortest length representatives for W/W_P and $\mathcal H(W^P)$ is the linear subspace of $\mathcal H(W, q^\lambda)$ spanned by the corresponding N_w. Following [Opd2, (4.24)] we define a sesquilinear pairing

$$\begin{array}{lll} \mathcal H(W^P) \otimes_{\mathbb C} V_\rho^\dagger \times \mathcal H(W^P) \otimes_{\mathbb C} V_\rho & \to & \mathbb C \\ \langle h' \otimes \lambda, h \otimes v \rangle & = & \tau(h' h^*) \lambda(v) \end{array} . \tag{2.8}$$

As preparation for a more general statement, we consider the left regular representation of $\mathcal H^P$. Clearly $\mathrm{ind}_{\mathcal H^P}^{\mathcal H}(\mathcal H^P) = \mathcal H$, and we already know $\mathcal H^\dagger$ from (2.3). Multiplication in $\mathcal H$ induces a linear bijection $m : \mathcal H(W^P) \otimes \mathcal H^P \to \mathcal H$. The transpose of m is the linear bijection

$$m^\dagger : \mathcal H^\dagger \to \mathcal H(W^P)^\dagger \otimes_{\mathbb C} \mathcal H^{P\dagger} \cong \mathcal H(W^P) \otimes_{\mathbb C} \mathcal H^{P\dagger} \cong \mathcal H \otimes_{\mathcal H^P} \mathcal H^{P\dagger}. \tag{2.9}$$

In the middle of (2.9) we identified $\mathcal H(W^P)^\dagger$ with $\mathcal H(W^P)$ via the inner product on $\mathcal H$. Notice that $\mathcal H^\dagger$ and $\mathcal H \otimes_{\mathcal H^P} \mathcal H^{P\dagger}$ independently carry $\mathcal H$-module structures, the latter induced from the $\mathcal H^P$-module structure of $\mathcal H^{P\dagger}$.

Lemma 2.6. *The map $m^\dagger : \mathcal H^\dagger \to \mathrm{ind}_{\mathcal H^P}^{\mathcal H}(\mathcal H^{P\dagger})$ is an isomorphism of $\mathcal H$-modules. In particular $\mathrm{ind}_{\mathcal H^P}^{\mathcal H}(\mathcal H^{P\dagger})$ with the pairing (2.8) is the Hermitian dual of $\mathrm{ind}_{\mathcal H^P}^{\mathcal H}(\mathcal H^P)$.*

Proof. Let $a, b \in W^P, h_1 \in \mathcal H^P$ and $\lambda \in \mathcal H^{P\dagger}$. For $h_2 \in \mathcal H$ there are elements

$$m^\dagger\big(h_2 \cdot (m^\dagger)^{-1}(N_a \otimes \lambda)\big) \quad \text{and} \quad h_2 \cdot (N_a \otimes \lambda) \quad \text{in} \quad \mathrm{ind}_{\mathcal H^P}^{\mathcal H}(\mathcal H^{P\dagger}).$$

We can compare them by pairing $N_b \otimes \mathcal H(W^P) \otimes \mathcal H^P$, as in (2.8). In the notation from (1.7), we compute

$$\begin{aligned} \langle h_2 \cdot (N_a \otimes \lambda), N_b \otimes h_1 \rangle = \langle h_2 N_a \cdot (1 \otimes \lambda), N_b \otimes h_1 \rangle = \\ \sum\nolimits_{v \in W^P} \langle N_v (h_2 N_a)^P_v \cdot (1 \otimes \lambda), N_b \otimes h_1 \rangle = \langle (h_2 N_a)^P_b \cdot \lambda, h_1 \rangle. \end{aligned} \tag{2.10}$$

We note that, for any $w \in W_P, v \in W^P$:

$$\langle N_b^* N_v, N_w \rangle = \langle N_u, N_b N_w \rangle = \langle N_u, N_{bw} \rangle = \begin{cases} 1 & v = b, w = e, \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

This implies

$$\big(N_b^* N_v (h_2 N_a)_v\big)^P_e = \begin{cases} (h_2 N_a)_v & b = v, \\ 0 & b \neq v. \end{cases} \tag{2.12}$$

With that the right hand side of (2.10) can be rewritten as

$$\langle (N_b^* h_2 N_a)^P_e \cdot \lambda, h_1 \rangle = \big\langle \lambda, \big((N_b^* h_2 N_a)^P_e \big)^{*_P} h_1 \big\rangle. \tag{2.13}$$

On the other hand

$$\begin{aligned}\big\langle m^\dagger\big(h_2\cdot(m^\dagger)^{-1}(N_a\otimes\lambda)\big), N_b\otimes h_1\big\rangle \;&=\; \big\langle h_2\cdot(m^\dagger)^{-1}(N_a\otimes\lambda), N_b h_1\big\rangle \\ &=\; \big\langle (m^\dagger)^{-1}(N_a\otimes\lambda), h_2^* N_b h_1\big\rangle \;=\; \langle N_a\otimes\lambda, m^{-1}(h_2^*N_b)h_1\rangle \\ &=\; \sum\nolimits_{v\in W^P}\langle N_a\otimes\lambda, N_v\otimes(h_2^*N_b)_v^P h_1\rangle \;=\; \langle\lambda,(h_2^*N_b)_a^P h_1\rangle\end{aligned} \tag{2.14}$$

Using (2.12) we identify the last expression in (2.14) with

$$\langle\lambda,(N_a^*h_2^*N_b)_e^P h_1\rangle = \big\langle\lambda,\big((N_b^*h_2N_a)^*\big)_e^P h_1\big\rangle.$$

Lemma 1.1 guarantees that this equals (2.13), which proves that the bijection $m^\dagger$ is an $\mathcal{H}$-module homomorphism.

By construction $m^{-1}:\mathcal{H}\to\mathcal{H}(W^P)\otimes\mathcal{H}^P$ and $m^\dagger$ transfer the pairing between $\mathcal{H}$ and $\mathcal{H}^\dagger$ to the pairing (2.8). Thus we realized $\mathcal{H}^\dagger=\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\mathcal{H}^P)^\dagger$ as $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\mathcal{H}^{P\dagger})$. □

In the special case $P=\emptyset$, Lemma 2.6 provides an isomorphism of $\mathcal{H}$-modules

$$\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}}(\mathcal{A}^\dagger)\cong\mathcal{H}^\dagger. \tag{2.15}$$

Here the embedding of $\mathcal{A}^\dagger$ in $\mathcal{H}^\dagger$ comes from (2.9):

$$\begin{array}{cccc}\imath: & \mathcal{A}^\dagger & \to & \mathcal{H}^\dagger \\ & a & \mapsto & (m^\dagger)^{-1}(N_e\otimes a)\end{array}. \tag{2.16}$$

The next result generalizes [Opd1, Theorem 2.20] and [Opd2, Proposition 4.19].

Proposition 2.7. *Let (ρ,V_ρ) be an $\mathcal{H}^P$-representation. The pairing (2.8) induces an isomorphism $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\rho^\dagger)\cong\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\rho)^\dagger$.*

Proof. We abbreviate $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}$ to ind for the duration of this proof.

Recall that $\{N_w : w\in W\}$ is an orthonormal basis of $\mathcal{H}(W,q^\lambda)$ for the inner product (1.4). Hence (2.8) identifies

$$\mathcal{H}(W^P)\otimes_{\mathbb{C}}V_\rho^\dagger\cong\mathcal{H}\otimes_{\mathcal{H}^P}(V_\rho^\dagger)\quad\text{with}\quad(\mathcal{H}(W^P)\otimes_{\mathbb{C}}V_\rho)^\dagger\cong(\mathcal{H}\otimes_{\mathcal{H}^P}V_\rho)^\dagger.$$

It remains to show that

$$\langle\mathrm{ind}(\pi^\dagger)(h^*)x,y\rangle\quad\text{equals}\quad\langle x,\mathrm{ind}(\pi)(h)y\rangle \tag{2.17}$$

for all $h\in\mathcal{H}, x\in\mathrm{ind}(V_\rho^\dagger), y\in\mathrm{ind}(V_\rho)$.

Choose a surjective $\mathcal{H}^P$-homomorphism $p:F\otimes\mathcal{H}^P\to V_\rho$, where $F\otimes\mathcal{H}^P$ is a free $\mathcal{H}^P$-module. Dually, that yields an injective $\mathcal{H}^P$-homomorphism $p^\dagger:V_\rho^\dagger\to(F\otimes\mathcal{H}^P)^\dagger$. For $v\in V_\rho$ with a preimage $\tilde v\in F\otimes\mathcal{H}^P$ and $\lambda\in V_\rho^\dagger$ with image $\tilde\lambda\in(F\otimes\mathcal{H}^P)^\dagger$, that means $\langle\lambda,v\rangle=\langle\tilde\lambda,\tilde v\rangle$.

With the functoriality of induction we obtain a surjective $\mathcal{H}$-homomorphism

$$\mathrm{ind}(p):F\otimes\mathcal{H}=\mathrm{ind}(F\otimes\mathcal{H}^P)\to\mathrm{ind}(V_\rho),$$

and an injective $\mathcal{H}$-homomorphism

$$\mathrm{ind}(p^\dagger):\mathrm{ind}(V_\rho^\dagger)\to\mathrm{ind}\big((F\otimes\mathcal{H}^P)^\dagger\big).$$

Now we encounter the minor complication that it is difficult to work with $(F\otimes\mathcal{H}^P)^\dagger$ when F has infinite dimension. We overcome that by playing it via finitely generated submodules. Choose a finite dimensional linear subspace $F_y\subset F$ such that

$$y\in\mathrm{ind}(p)(F_y\otimes\mathcal{H})=\mathrm{ind}\big(p(F_y\otimes\mathcal{H}^P)\big).$$

It follows from Lemma 2.6 that

$$\operatorname{ind}_{\mathcal{H}^P}^{\mathcal{H}}\big(F_y^\dagger \otimes \mathcal{H}^{P\dagger}\big) = F_y^\dagger \otimes \operatorname{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\mathcal{H}^{P\dagger}) \cong \operatorname{ind}_{\mathcal{H}^P}^{\mathcal{H}}(F_y \otimes \mathcal{H}^P)^\dagger, \tag{2.18}$$

with the pairing (2.8). The transpose of the inclusion $i_y : F_y \otimes \mathcal{H}^P \to F \otimes \mathcal{H}^P$ is the projection

$$i_y^\dagger : (F \otimes \mathcal{H}^P)^\dagger \to (F_y \otimes \mathcal{H}^P)^\dagger = F_y^\dagger \otimes (\mathcal{H}^P)^\dagger .$$

To these maps we can also apply ind. In that way (2.17) can be evaluated via the pairing of $\operatorname{ind}(F_y \otimes \mathcal{H}^P)$ with $\operatorname{ind}\big(F_y^\dagger \otimes (\mathcal{H}^P)^\dagger\big)$ given by (2.8). More explicitly:

$$\langle x, \operatorname{ind}(\pi)(h) y \rangle = \langle \operatorname{ind}(p^\dagger)(x), h \operatorname{ind}(p)(y) \rangle = \langle \operatorname{ind}(i_y^\dagger) \operatorname{ind}(p^\dagger)(x), h \operatorname{ind}(p)(y) \rangle .$$

By (2.18) the right-hand side equals

$$\begin{aligned} \langle h^* \operatorname{ind}(i_y^\dagger)\operatorname{ind}(p^\dagger)(x), \operatorname{ind}(p)(y) \rangle &= \langle \operatorname{ind}(i_y^\dagger)\operatorname{ind}(p^\dagger)(\operatorname{ind}(\pi^\dagger)(h^*)x), \operatorname{ind}(p)(y) \rangle \\ &= \langle \operatorname{ind}(\pi^\dagger)(h^*)x, y \rangle . \end{aligned}$$

This establishes (2.17). □

3. Second adjointness

For affine Hecke algebras the standard adjointness for parabolic induction reads

$$\operatorname{Hom}_{\mathcal{H}}\big(\operatorname{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\rho), \pi\big) \cong \operatorname{Hom}_{\mathcal{H}^P}\big(\rho, \operatorname{Res}_{\mathcal{H}^P}^{\mathcal{H}}(\pi)\big) \quad \rho \in \operatorname{Mod}(\mathcal{H}^P), \pi \in \operatorname{Mod}(\mathcal{H}). \tag{3.1}$$

This can be regarded as an instance of Frobenius reciprocity or of Hom-tensor duality (since $\operatorname{ind}_{\mathcal{H}^P}^{\mathcal{H}}(V_\rho) = \mathcal{H} \otimes_{\mathcal{H}^P} V_\rho$). In terms of reductive p-adic groups, normalized parabolic induction and normalized Jacquet restriction, (3.1) corresponds to Bernstein's second adjointness:

$$\operatorname{Hom}_G\big(I_P^G(\sigma), \tau\big) \cong \operatorname{Hom}_M\big(\sigma, J_{\overline{P}}^G(\tau)\big), \tag{3.2}$$

where $\sigma \in \operatorname{Rep}(M), \tau \in \operatorname{Rep}(G)$ and $P, \overline{P}$ are opposite parabolic subgroups of G with $P \cap \overline{P} = M$. The comparison between the two settings stems from [BuKu, Corollary 8.4], but one needs some modifications that lead to [Sol3, Condition 4.1]. By analogy, the first adjointness for p-adic groups (i.e. Frobenius reciprocity)

$$\operatorname{Hom}_M\big(J_P^G(\tau), \sigma\big) \cong \operatorname{Hom}_G\big(\tau, I_P^G(\sigma)\big) \tag{3.3}$$

should have a counterpart for affine Hecke algebras. In other words, we may expect that some form of parabolic restriction is left adjoint to some form of parabolic induction. By Frobenius reciprocity for co-induced modules:

$$\operatorname{Hom}_{\mathcal{H}^P}\big(\operatorname{Res}_{\mathcal{H}^P}^{\mathcal{H}}(\pi), \rho\big) \cong \operatorname{Hom}_{\mathcal{H}}\big(\pi, \operatorname{Hom}_{\mathcal{H}^P}(\mathcal{H}, \rho)\big), \tag{3.4}$$

where $\mathcal{H}$ acts on $\operatorname{Hom}_{\mathcal{H}^P}(\mathcal{H}, \pi)$ via right multiplication on $\mathcal{H}$. However, (3.4) is not yet satisfactory because it does not provide a left adjoint for parabolic induction.

For p-adic groups, one way to prove the second adjointness relation is via contragredients and Jacquet modules, see [Ren, §VI.9.6]. For graded Hecke algebras, a similar proof works with Hermitian duals instead of contragredients [BaCi, Lemma 3.8.1]. We follow the latter.

Theorem 3.1. *Let $P \subset \Delta$ and recall that $P^{op} = w_\Delta(-P)$.*

(a) The right adjoint of $\operatorname{Res}_{\mathcal{H}^P}^{\mathcal{H}}$ is $\operatorname{ind}_{\mathcal{H}^{P^{op}}}^{\mathcal{H}} \circ \psi_{\Delta P *}$.

(b) The left adjoint of $\operatorname{ind}_{\mathcal{H}^P}^{\mathcal{H}}$ is $\psi_{\Delta P}^* \circ \operatorname{Res}_{\mathcal{H}^{P^{op}}}^{\mathcal{H}}$.

Proof. (a) Let $(\pi, V_\pi) \in \mathrm{Mod}(\mathcal{H})$ and $(\rho, V_\rho) \in \mathrm{Mod}(\mathcal{H}^P)$. By the transposition isomorphism (2.2) and Proposition 2.5

$$\mathrm{Hom}_{\mathcal{H}^P}\big(\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}(\pi), \rho^\dagger\big) \cong \mathrm{Hom}_{\mathcal{H}^P}\big(\rho, \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}(\pi)^\dagger\big) \cong \mathrm{Hom}_{\mathcal{H}^P}\big(\rho, \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\pi^\dagger) \circ \psi_{\Delta P}\big). \tag{3.5}$$

We know from Lemma 2.4 that $\psi_{\Delta P} : \mathcal{H}^P \to \mathcal{H}^{P^{op}}$ is invertible, so that the right-hand side of (3.5) becomes isomorphic with

$$\mathrm{Hom}_{\mathcal{H}^{P^{op}}}\big(\rho \circ \psi_{\Delta P}^{-1}, \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\pi^\dagger)\big). \tag{3.6}$$

Now we apply Frobenius reciprocity in the form (3.1) and again the transposition isomorphism:

$$\cong \mathrm{Hom}_{\mathcal{H}}\big(\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\rho \circ \psi_{\Delta P}^{-1}), \pi^\dagger\big) \cong \mathrm{Hom}_{\mathcal{H}}\big(\pi, \mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\rho \circ \psi_{\Delta P}^{-1})^\dagger\big). \tag{3.7}$$

Using Proposition 2.7, we identify that with

$$\mathrm{Hom}_{\mathcal{H}}\big(\pi, \mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\rho^\dagger \circ \psi_{\Delta P}^{-1})\big) = \mathrm{Hom}_{\mathcal{H}}\big(\pi, \mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}} \circ \psi_{\Delta P}(\rho^\dagger)\big). \tag{3.8}$$

The first isomorphism in (3.5) and the second in (3.7) are conjugate-linear. The other above isomorphisms are complex linear, so the composition of (3.5)–(3.8) is again a complex linear bijection. That proves the desired adjointness relation for $(\pi, \rho' = \rho^\dagger)$, so whenever ρ' is the Hermitian dual of some $\mathcal{H}^P$-module. The same argument as in the analogous situation for reductive p-adic groups [Ren, p. 232] shows why that implies part (a) for all (π, ρ').
(b) Reverse the roles of P and P^{op} and apply part (a) with $\rho' = \psi^*_{\Delta P}(\rho) = \rho \circ \psi_{\Delta P}$. That gives isomorphisms

$$\mathrm{Hom}_{\mathcal{H}}\big(\pi, \mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^P}(\rho')\big) \cong \mathrm{Hom}_{\mathcal{H}^{P^{op}}}\big(\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\pi), \rho\big).$$

Left composition with $\psi_{\Delta P}^{-1}$ on both terms of the right-hand side makes this isomorphic with $\mathrm{Hom}_{\mathcal{H}^P}\big(\psi^*_{\Delta P}\,\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\pi), \rho'\big)$. □

Next we discuss a topic related to second adjointness, namely expressions for parabolic induction followed by parabolic restriction. In the setting of reductive p-adic groups this is known as Bernstein's geometric lemma [Ren, §VI.5.1]. A version for affine Hecke algebras should provide a filtration of the functor $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^Q}\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^P}$. Indeed that was achieved in [DeOp, §11], but restricted to tempered representations. Here we formulate that result in larger generality.

Let $P, Q \subset \Delta$ and let

$$W^{P,Q} = \{w \in W : w(Q) \subset R^+, w^{-1}(Q) \subset R^+\}$$

be the set of shortest length representatives of $W_P \backslash W / W_Q$. Each $d \in W^{P,Q}$ yields a bijection $d^{-1}(P) \cap Q \to P \cap d(Q)$ and an algebra isomorphism

$$\begin{array}{rlll} \psi_d : & \mathcal{H}^{d^{-1}(P)\cap Q} & \to & \mathcal{H}^{P\cap d(Q)} \\ & N_w \theta_x & \mapsto & N_{dwd^{-1}} \theta_{d(x)} \end{array}. \tag{3.9}$$

We choose a total ordering of $W^{P,Q}$ such that $\ell : W^{P,Q} \to \mathbb{Z}_{\geq 0}$ becomes a weakly increasing function. For $d \in W^{P,Q}$ and an $\mathcal{H}^Q$-representation (π, V_π), we consider the linear subspace

$$\big(\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^Q}\big)_{\leq d}(V_\pi) = \bigoplus\nolimits_{d' \in W^{P,Q}, d' \leq d} \mathcal{H}(W_P d' W_Q)\mathcal{A} \otimes_{\mathcal{H}^Q} V_\pi \tag{3.10}$$

of $\mathrm{ind}_{\mathcal{H}^Q}^{\mathcal{H}}(V_\pi)$. To analyse these subspaces, we need a result of Kilmoyer [Car, Theorem 2.7.4]:

$$W_P \cap dW_Q d^{-1} = W_{P \cap d(Q)} \qquad \text{for all } d \in W^{P,Q}. \tag{3.11}$$

Using that, the following is shown in [DeOp, (11.3)–(11.6)]:

Proposition 3.2. *For each $d \in W^{P,Q}$, $\big(\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P} \mathrm{ind}_{\mathcal{H}^Q}^{\mathcal{H}}\big)_{\leq d}(V_\pi)$ is an $\mathcal{H}^P$-submodule of $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P} \mathrm{ind}_{\mathcal{H}^Q}^{\mathcal{H}}(V_\pi)$. There is an isomorphism of $\mathcal{H}^P$-modules*

$$\big(\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P} \mathrm{ind}_{\mathcal{H}^Q}^{\mathcal{H}}\big)_{\leq d}(V_\pi) / \big(\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P} \mathrm{ind}_{\mathcal{H}^Q}^{\mathcal{H}}\big)_{< d}(V_\pi) \cong \mathrm{ind}_{\mathcal{H}^{P \cap d(Q)}}^{\mathcal{H}^P} \big(\psi_d \, \mathrm{Res}^{\mathcal{H}^Q}_{\mathcal{H}^{d^{-1}(P) \cap Q}}(V_\pi)\big),$$

where $< d$ means $\leq d'$ for the largest $d' \in W^{P,Q}$ which is smaller than d.

In other words, we have a filtration of the functor $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P} \mathrm{ind}_{\mathcal{H}^Q}^{\mathcal{H}}$, indexed by $W^{P,Q}$ and with successive subquotients $\mathrm{ind}_{\mathcal{H}^{P \cap d(Q)}}^{\mathcal{H}^P} \circ \psi_{d*} \circ \mathrm{Res}^{\mathcal{H}^Q}_{\mathcal{H}^{d^{-1}(P) \cap Q}}$.

Notice the analogy with Mackey's restriction-induction formula for representations of finite groups. From Proposition 3.2 and the two adjunctions, one can derive expressions for the Hom-space between two parabolically induced $\mathcal{H}$-representations.

4. Variations on the Langlands classification

The Langlands classification for a reductive group G over a local field [Lan, Ren] classifies irreducible admissible G-representations in terms of irreducible tempered representations of Levi subgroups of G. The analogous result for affine/graded Hecke algebras can be found in [Eve, Sol2]. Here we want to establish some useful variations, in particular with subrepresentations instead of quotients.

The complex torus T can be idenfied with the space $\mathrm{Irr}(\mathcal{A})$ of irreducible representations of $\mathcal{A} \cong \mathbb{C}[X] = \mathcal{O}(T)$. If (π, V_π) is an $\mathcal{H}$-representation, $t \in T$ and there exists $v \in V_\pi \setminus \{0\}$ such that

$$\pi(\theta_x) v = t(x) v \qquad \text{for all } x \in X,$$

then t is a called an $\mathcal{A}$-weight (or simply weight) of π. We denote the set of $\mathcal{A}$-weights of (π, V_π) by $\mathrm{Wt}(\pi)$ or $\mathrm{Wt}(V_\pi)$. If V_π has finite dimension, then there is a canonical decomposition in generalized $\mathcal{A}$-eigenspaces:

$$V_\pi = \bigoplus\nolimits_{t \in T} V_{\pi,t,\mathrm{gen}}. \tag{4.1}$$

The $t \in T$ for which $V_{\pi,t,\mathrm{gen}} \neq 0$ are precisely the $\mathcal{A}$-weights of π.

Lemma 4.1. *Let (π, V_π) be a finite dimensional $\mathcal{H}$-representation. Then* $\mathrm{Wt}(\pi^\dagger) = \big\{ \overline{w_\Delta t}^{-1} : t \in \mathrm{Wt}(\pi) \big\}$.

Proof. Let $s \in T$ be a weight of $\pi^\dagger$, with an eigenvector $\lambda \in V_\pi^\dagger \setminus \{0\}$. For any $v \in V, x \in X$ we compute, using (1.6),

$$s(x)\lambda(v) = (\pi^\dagger(\theta_x)\lambda)(v) = \lambda(\pi(\theta_x^*)v) = \lambda(\pi(N_{w_\Delta} \theta_{-w_\Delta(x)} N_{w_\Delta}^{-1})v). \tag{4.2}$$

Write $\pi' = \pi \circ \mathfrak{c}_{N_{w_\Delta}}$, so that

$$\begin{array}{ccc} (\pi, V_\pi) & \to & (\pi', V_\pi) \\ v & \mapsto & \pi(N_{w_\Delta}) v \end{array}$$

is an isomorphism of $\mathcal{H}$-representations. We can rewrite (4.2) as

$$\lambda(\overline{s(x)} v) = \lambda(\pi'(\theta_{-w_\Delta(x)}) v).$$

Equivalently, for each $x \in X, v \in V_\pi$ the kernel of λ contains

$$\pi'\big(\theta_{-w_\Delta(x)} - \overline{s(x)}\big)v = \pi'\big(\theta_{x'} - \overline{s}(-w_\Delta(x'))\big)v = \pi'\big(\theta_{x'} - \overline{w_\Delta s}^{-1}(x')\big)v,$$

where we abbreviated $x' = -w_\Delta(x)$. Thus $\pi'\big(\theta_{x'} - \overline{w_\Delta s}^{-1}(x')\big)$ is not surjective, for any $x' \in X$. Since V_π has finite dimension, we can use the decomposition, which shows that $\overline{w_\Delta s}^{-1}$ is a weight of π'. Via the isomorphism $\pi' \cong \pi$, it is also a weight of π.

Hence $s \mapsto \overline{w_\Delta s}^{-1}$ maps the weights of $\pi^\dagger$ to the weights of π. As $\dim V_\pi < \infty$, $\pi^{\dagger\dagger} = \pi$ and the same arguments apply with the roles of π and $\pi^\dagger$ exchanged. Therefore we have found a bijection between the set of weights of $\pi^\dagger$ and of $\dagger$, with inverse $t \mapsto \overline{w_\Delta t}^{-1}$. $\square$

For any $t \in T$ we have $|t| \in \mathrm{Hom}_\mathbb{Z}(X, \mathbb{R}_{>0})$ and $\log|t| \in \mathrm{Hom}_\mathbb{Z}(X, \mathbb{R}) = Y \otimes_\mathbb{Z} \mathbb{R}$. Given $P \subset \Delta$ we define the positive cones

$$(Y \otimes_\mathbb{Z} \mathbb{R})^{P+} = \{y \in Y \otimes_\mathbb{Z} \mathbb{R} : \langle \alpha, y\rangle = 0 \;\forall \alpha \in P, \langle \alpha, y \rangle > 0 \;\forall \alpha \in \Delta \setminus P\},$$
$$T^{P+} = \exp\big((Y \otimes_\mathbb{Z} \mathbb{R})^{P+}\big) \qquad \subset T^P.$$

The same can be done in $X \otimes_\mathbb{Z} \mathbb{R}$, and then taking anti-duals yields the obtuse negative cones

$$(Y \otimes_\mathbb{Z} \mathbb{R})^-_P = \big\{ \textstyle\sum_{\alpha \in P} c_\alpha \alpha^\vee : c_\alpha \in \mathbb{R}_{\leq 0}\big\},$$
$$T^-_P = \exp\big((Y \otimes_\mathbb{Z} \mathbb{R})^-_P\big) \qquad \subset T_P.$$

By definition, a finite dimensional $\mathcal{H}^P$-module V is tempered if $|t| \in T^-_P$ for all $t \in \mathrm{Wt}(V)$. Similarly we say that V is anti-tempered if $|t|^{-1} \in T^-_P$ for all $t \in \mathrm{Wt}(V)$. These two properties are preserved by taking Hermitian duals:

Lemma 4.2. *Let (π, V_π) be a finite dimensional $\mathcal{H}$-representation. If V_π is tempered (resp. anti-tempered), then $V_\pi^\dagger$ is tempered (resp. anti-tempered).*

Proof. Since $-w_\Delta$ stabilizes Δ, $w_\Delta s^{-1} \in T^{\Delta-}$ if and only if $s \in T^{\Delta-}$. Apply that to $s = |t|$ (resp. $s = |t|^{-1}$) for a weight t of V_π, and use Lemma 4.1. $\square$

The following result is an obvious generalization of the Langlands classification for affine Hecke algebras [Eve, Sol2].

Theorem 4.3. *Let $\pi \in \mathrm{Irr}(\mathcal{H}^P)$ and $t \in T^P$. Suppose that (i) or (ii) holds:*

(i) *π is tempered and $t \in T^{P+}$,*

(ii) *π is anti-tempered and $t^{-1} \in T^{P+}$.*

(*a*) *The $\mathcal{H}$-representation $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)$ has a unique irreducible quotient $L(P, \pi, t)$. It is the unique irreducible subquotient ρ of $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)$ which admits an injective $\mathcal{H}^P$-homomorphism $\pi \otimes t \to \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}(\rho)$.*

(*b*) *Every irreducible $\mathcal{H}$-representation is of the form $L(P, \pi, t)$, for unique (P, π, t) as in (i). This also holds with (ii) instead of (i).*

Proof. (i) The Langlands classification, as in [Eve, Theorem 2.1] and [Sol2, Theorem 2.2.4], states (a) and (b). Although the characterizing property of $L(P, \pi, t)$ is not made explicit in these sources, it plays an important role in [Eve, §2.7] and in [Sol2, proof of Theorem 2.2.4.a].

(ii) The same proof as for (i) applies, when we rewrite all the arguments in $Y \otimes_\mathbb{Z} \mathbb{R}$ with respect to $-\Delta$ instead of Δ. $\square$

An $\mathcal{H}$-representation $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)$ as in Theorem 4.3.i is called a standard module, and $L(P,\pi,t)$ is called its Langlands quotient. With the usage of Hermitian duals, we can deduce a version of Theorem 4.3 in terms of "Langlands" subrepresentations.

Proposition 4.4. *Let $\pi \in \mathrm{Irr}(\mathcal{H}^P)$ and $t \in T^P$. Suppose that (i) or (ii) holds:*

(i) *π is tempered and $t^{-1} \in T^{P+}$,*
(ii) *π is anti-tempered and $t \in T^{P+}$.*

(*a*) *The $\mathcal{H}$-representation $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)$ has a unique irreducible subrepresentation, which we call the Langlands subrepresentation $\tilde{L}(P,\pi,t)$. It is the unique irreducible subquotient σ of $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)$ that admits a surjective $\mathcal{H}^P$-homomorphism $\mathrm{Res}_{\mathcal{H}^{P^{op}}}^{\mathcal{H}}(\sigma) \circ \psi_{\Delta P} \to \pi \otimes t$.*

(*b*) *Every irreducible $\mathcal{H}$-representation is of the form $\tilde{L}(P,\pi,t)$ for unique (P,π,t) as in (i). This also holds with (ii) instead of (i).*

Proof. We assume (i). The proof when (ii) holds is completely analogous, only using the other assumption in Theorem 4.3.
(a) By Proposition 2.7 and Lemma 2.2

$$\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)^\dagger \cong \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}\big((\pi \otimes t)^\dagger\big) \cong \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi^\dagger \otimes \bar{t}^{-1}). \tag{4.3}$$

Here $\bar{t} = t$ because it is real-valued, and we know from Lemma 4.2 that $\pi^\dagger$ is tempered. Theorem 4.3.a says that (4.3) has a unique irreducible quotient $L(P,\pi^\dagger,t^{-1})$, which can be characterized by the existence of an injection

$$\pi^\dagger \otimes t^{-1} \to \mathrm{Res}_{\mathcal{H}^P}^{\mathcal{H}}\big(L(P,\pi^\dagger,t^{-1})\big).$$

Passing to Hermitian duals and using (2.2), we find that $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)$ has a unique irreducible subrepresentation $\sigma \cong L(P,\pi^\dagger,t^{-1})^\dagger$. Via Proposition 2.5.a, the characterizing property becomes a surjection $\mathrm{Res}_{\mathcal{H}^{P^{op}}}^{\mathcal{H}}(\sigma) \circ \psi_{\Delta P} \to \pi \otimes t$.
(b) Let $\tau \in \mathrm{Irr}(\mathcal{H})$. With Theorem 4.3.b we write $\tau^\dagger \cong L(P,\pi,t)$ for suitable $P \subset \Delta$, tempered $\pi \in \mathrm{Irr}(\mathcal{H}^P)$ and $t \in T^{P+}$. Then (2.2) gives an injection

$$\tau \cong L(P,\pi,t)^\dagger \to \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)^\dagger.$$

From the proof of part (a) we know that the right hand side is isomorphic with $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t^{-1})$, where (P,π,t^{-1}) is as in (i). Then part (a) says $\tau \cong \tilde{L}(P,\pi,t^{-1})$. The uniqueness in Theorem 4.3.b implies the uniqueness of (P,π,t^{-1}). $\square$

We would like to express the unique Langlands quotient or subrepresentation from Theorem 4.3 and Proposition 4.4 as the coimage or image of a suitable intertwining operator. To that end we establish the uniqueness (up to scalars) of those operators.

Lemma 4.5. *Let $\pi \in \mathrm{Irr}(\mathcal{H}^P)$ and assume that the W-stabilizer of t is contained in W_P for all $t \in \mathrm{Wt}(\pi)$. Then $\mathrm{Res}_{\mathcal{H}^P}^{\mathcal{H}}\mathrm{ind}_{\mathcal{H}^Q}^{\mathcal{H}}(\psi_{\gamma *}\pi)$ is a direct sum of $\mathcal{H}^P$-representations $\mathrm{ind}_{\mathcal{H}^{P \cap d(Q)}}^{\mathcal{H}^P}(\psi_{d*}\psi_{\gamma *}\pi)$ with $d \in W^{P,Q}$, whose sets of $Z(\mathcal{H}^P)$-weights are mutually disjoint.*

Proof. From Proposition 3.2.c we know that $\mathrm{Res}_{\mathcal{H}^P}^{\mathcal{H}}\mathrm{ind}_{\mathcal{H}^Q}^{\mathcal{H}}(\psi_{\gamma *}\pi)$ has a filtration with successive subquotients

$$\mathrm{ind}_{\mathcal{H}^{P \cap d(Q)}}^{\mathcal{H}^P}\big(\psi_d \,\mathrm{Res}_{\mathcal{H}^{d^{-1}(P) \cap Q}}^{\mathcal{H}^Q}(\psi_\gamma \,\pi)\big) = \mathrm{ind}_{\mathcal{H}^{P \cap d(Q)}}^{\mathcal{H}^P}(\psi_d \psi_\gamma \,\pi) \qquad d \in W^{P,Q}. \tag{4.4}$$

By construction $\mathrm{Wt}(\psi_{d*}\psi_{\gamma *}\pi) = d\gamma \mathrm{Wt}(\pi)$, and with [Opd2, Proposition 4.20] we obtain

$$\mathrm{Wt}\big(\mathrm{ind}_{\mathcal{H}^{P\cap d(Q)}}^{\mathcal{H}^P}(\psi_{d*}\psi_{\gamma *}\pi)\big) \subset W_P d\gamma \mathrm{Wt}(\pi). \tag{4.5}$$

Equivalently, the set of $Z(\mathcal{H}^P)$-weights of $\mathrm{ind}_{\mathcal{H}^{P\cap d(Q)}}^{\mathcal{H}^P}(\psi_{d*}\psi_{\gamma *}\pi)$ is contained in $W_P d\gamma \mathrm{Wt}(\pi)/W_P$. Suppose that $d, d' \in W^{P,Q}, t, t' \in \mathrm{Wt}(\pi)$ and

$$W_P d\gamma t \cap W_P d'\gamma t' \neq \emptyset.$$

Pick $w_1, w_2 \in W_P$ such that $w_1 d\gamma t = w_2 d'\gamma t'$. By the irreducibility of π, t and t' belong to the same W_P-orbit. Furthermore we assumed $W_t \subset W_P$, so

$$w_1 d\gamma W_P = w_2 d'\gamma W_P. \tag{4.6}$$

From that we obtain $\gamma^{-1}d^{-1}w_1^{-1}w_2 d'\gamma \in W_P$ and

$$d^{-1}w_1^{-1}w_2 d' \in \gamma W_P \gamma^{-1} = W_Q.$$

We note that now $w_1^{-1}w_2 d' \in W_P d' \cap dW_Q$. As $W^{P,Q}$ represents $W_P\backslash W/W_Q$, this shows that $d' = d$.

Thus, for different $d, d' \in W^{P,Q}$ the $\mathcal{H}^P$-representations (4.4) have disjoint sets of $\mathcal{A}$-weights and disjoint sets of $Z(\mathcal{H}^P)$-weights. In particular every extension of one of these modules by the other is a trivial extension. It follows that the aforementioned filtration of $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^Q}(\psi_{\gamma *}\pi)$ actually splits, and that $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^Q}(\psi_{\gamma *}\pi)$ is the direct sum of the modules (4.4). □

The conditions in Lemma 4.5 are often satisfied, but they do not cover all cases of Theorem 4.3. Inspired by [Ren, §VII.3.3], we say that an $\mathcal{H}^P$-representation π is W,P-regular if

$$wt \notin \mathrm{Wt}(\pi) \text{ for all } t \in \mathrm{Wt}(\pi) \text{ and } w \in W_P(W^{P,P} \setminus \{e\}). \tag{4.7}$$

Let $P, Q \subset \Delta$ and $\gamma \in W$, such that $\gamma(P) = Q$. Like in (3.9), there is an algebra isomorphism $\psi_\gamma : \mathcal{H}^P \to \mathcal{H}^Q$.

Lemma 4.6. *Let $\pi \in \mathrm{Irr}(\mathcal{H}^P)$ be W,P-regular.*

(*a*) *π has multiplicity one in $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^P}\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^Q}(\psi_{\gamma *}\pi)$, and is a direct summand of the latter.*

(*b*) $\dim \mathrm{Hom}_{\mathcal{H}}\big(\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^P}(\pi), \mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^Q}(\psi_{\gamma *}\pi)\big) = 1.$

Proof. (a) The element $\gamma^{-1} \in W$ belongs to $W^{P,Q}$ because $\gamma^{-1}(Q) \subset R^+$ and $\gamma(P) \subset R^+$. We follow the proof of Lemma 4.5, with $d' = \gamma^{-1}$. This time we cannot conclude (4.6), but our weaker assumption still provides a reasonable substitute. Namely, from $w_1 d\gamma t = w_2 t'$ we get $w_2^{-1}w_1 d\gamma t \in \mathrm{Wt}(\pi)$, which by the W,P-regularity of π implies

$$w_2^{-1}w_1 d\gamma \notin W_P(W^{P,P} \setminus \{e\}). \tag{4.8}$$

Notice that $d\gamma(P) = d(Q) \subset R^+$, which says that $d\gamma \in W^P$. By [Car, Proposition 2.7.5] we can write $d\gamma = a\tilde{d}$ with $\tilde{d} \in W^{P,P}$ and $a \in W_P$. It follows that $W_P d\gamma = W_P\tilde{d}$. Then (4.8) forces

$$\tilde{d} = e \quad \text{and} \quad d\gamma \in W_P \cap W^P = \{e\}.$$

Hence the representations (4.4) with $d \neq \gamma^{-1}$ do not have the central character of $\pi \in \mathrm{Irr}(\mathcal{H}^P)$ as $Z(\mathcal{H}^P)$-weight. Like in the proof of Lemma 4.5, this entails that π

appears with multiplicity one in $\mathrm{Res}^{\mathcal H}_{\mathcal H^P} \mathrm{ind}^{\mathcal H}_{\mathcal H^Q}(\psi_{\gamma *}\pi)$, as a direct summand.
(b) By Frobenius reciprocity

$$\mathrm{Hom}_{\mathcal H}\big(\mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi), \mathrm{ind}^{\mathcal H}_{\mathcal H^Q}(\psi_{\gamma *}\,\pi)\big) \cong \mathrm{Hom}_{\mathcal H^P}\big(\pi, \mathrm{Res}^{\mathcal H}_{\mathcal H^P} \mathrm{ind}^{\mathcal H}_{\mathcal H^Q}(\psi_{\gamma *}\pi)\big).$$

Now apply part (a). □

Lemma 4.6 tells us that, whenever $\pi \in \mathrm{Irr}(\mathcal H^P)$ is W,P-regular, there exists a nonzero intertwining operator

$$I(\gamma, P, \pi) : \mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi) \to \mathrm{ind}^{\mathcal H}_{\mathcal H^Q}(\psi_{\gamma *}\,\pi), \tag{4.9}$$

unique up to scalars. The ΔP-genericity is only a very mild restriction. Namely, for every finite dimensional $\mathcal H^P$-representation τ there exists a Zariski-open nonempty subset $T^P_\tau \subset T^P$ such that $\tau \otimes t$ is ΔP-generic for all $t \in T^P_\tau$.

The next result and its proof are similar to [Ren, Théorème VII.4.2].

Theorem 4.7. *Let $P \subset \Delta$ and $\pi \in \mathrm{Irr}(\mathcal H^P)$.*

(a) Suppose that π is W,P-regular. Then $\mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi)$ has a unique irreducible quotient, namely

$$\mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi)/\ker I(w_\Delta w_P, P, \pi) \cong \mathrm{im}\, I(w_\Delta w_P, P, \pi).$$

(b) Suppose that $\psi_{w_\Delta w_P *}\pi$ is W,P^{op}-regular. Then $\mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi)$ has a unique irreducible subrepresentation, namely the image of

$$I(w_P w_\Delta, P^{op}, \psi_{w_\Delta w_P *}\pi) : \mathrm{ind}^{\mathcal H}_{\mathcal H^{P^{op}}}(\psi_{w_\Delta w_P *}\pi) \to \mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi).$$

Proof. (a) Let ρ be any quotient $\mathcal H$-representation of $\mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi)$. The quotient map gives a nonzero element of

$$\mathrm{Hom}_{\mathcal H}(\mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi), \rho) \cong \mathrm{Hom}_{\mathcal H^P}(\pi, \mathrm{Res}^{\mathcal H}_{\mathcal H^P}\rho),$$

so π is a subrepresentation of $\mathrm{Res}^{\mathcal H}_{\mathcal H^P}\rho$. By Lemma 4.6.a with $\gamma = e$, π is a direct summand of $\mathrm{Res}^{\mathcal H}_{\mathcal H^P}\rho$, and appears with multiplicity one. The projection $\mathrm{Res}^{\mathcal H}_{\mathcal H^P}\rho \to \pi$ and the adjunction from Theorem 3.1.a yield a nonzero $\mathcal H$-homomorphism from ρ to $\mathrm{ind}^{\mathcal H}_{\mathcal H^{P^{op}}}(\psi_{\Delta P *}\pi)$. Thus we have $\mathcal H$-homomorphisms

$$\mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi) \to \rho \to \mathrm{ind}^{\mathcal H}_{\mathcal H^{P^{op}}}(\psi_{\Delta P *}\pi). \tag{4.10}$$

Suppose now that ρ is irreducible. Then the second map in (4.10) is injective, and first map is surjective by definition, so the composition of the two maps in (4.10) is nonzero. Lemma 4.6.a guarantees that (4.10) is a multiple of $I(w_\Delta w_P, P, \pi)$. In particular

$$\ker I(w_\Delta w_P, P, \pi) = \ker (4.10) = \ker \big(\mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi) \to \rho\big).$$

We conclude that ρ equals $\mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi \otimes t)/\ker I(w_\Delta w_P, P, \pi)$.
(b) Let σ be any subrepresentation of $\mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi)$. The inclusion map and Theorem 3.1.b give a nonzero element of

$$\mathrm{Hom}_{\mathcal H}(\sigma, \mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi)) \cong \mathrm{Hom}_{\mathcal H^P}\big(\psi_{\Delta P^{op} *}(\mathrm{Res}^{\mathcal H}_{\mathcal H^{P^{op}}}\sigma), \pi\big) \cong \mathrm{Hom}_{\mathcal H^{P^{op}}}(\sigma, \psi_{\Delta P *}\pi).$$

In the setting of Lemma 4.6 we take $P^{op}, P, \psi_{\Delta P^{op}}, \psi_{\Delta P}\pi$ in the roles of, respectively, P, Q, γ, π. Then Lemma 4.6.a says that $\psi_{\Delta P *}\pi$ appears with multiplicity one in

$$\mathrm{Res}^{\mathcal H}_{\mathcal H^{P^{op}}} \mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\psi_{\Delta P^{op}} \psi_{\Delta P}\pi) = \mathrm{Res}^{\mathcal H}_{\mathcal H^{P^{op}}} \mathrm{ind}^{\mathcal H}_{\mathcal H^P}(\pi), \tag{4.11}$$

as a direct summand. Since $\psi_{\Delta P}(\pi)$ appears in the subrepresentation $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}\sigma$ of (4.11), it is also a direct summand thereof. In particular there exists a nonzero element of

$$\mathrm{Hom}_{\mathcal{H}^{P^{op}}}(\psi_{\Delta P*}\pi, \mathrm{Res}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}\sigma) \cong \mathrm{Hom}_{\mathcal{H}^P}(\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\psi_{\Delta P*}\pi), \sigma).$$

Thus we have nonzero $\mathcal{H}$-homomorphisms

$$\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\psi_{\Delta P*}\pi) \to \sigma \to \mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^P}(\pi). \tag{4.12}$$

Now we assume that σ is irreducible. Then the first map in (4.12) is surjective and the second map is injective, so their composition is nonzero. The same argument as for (4.10) shows that σ is isomorphic to

$$\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^{P^{op}}}(\psi_{\Delta P*}\pi)/\ker I(w_P w_\Delta, P^{op}, \psi_{w_\Delta w_P *}\pi) \cong \mathrm{im}\, I(w_P w_\Delta, P^{op}, \psi_{w_\Delta w_P *}\pi). \quad \Box$$

With Theorem 4.7 and Langlands' geometric lemmas [Lan, §4], we can provide alternative proofs of Theorem 4.3 and Proposition 4.4.

Proposition 4.8. *Let $P \subset \Delta, \pi \in \mathrm{Irr}(\mathcal{H}^P)$ and $t \in T^P$.*

(a) *Suppose that*
 (i) *π is tempered and $t \in T^{P+}$ or*
 (ii) *π is anti-tempered and $t^{-1} \in T^{P+}$.*
 Then $\pi \otimes t$ is W,P-regular and the Langlands quotient $L(P, \pi \otimes t)$ from Theorem 4.3 equals $\mathrm{ind}^{\mathcal{H}}_{\mathcal{H}^P}(\pi \otimes t)/\ker I(w_\Delta w_P, P, \pi \otimes t)$.

(b) *Suppose that*
 (iii) *π is tempered and $t^{-1} \in T^{P+}$ or*
 (iv) *π is anti-tempered and $t \in T^{P+}$.*
 Then $\psi_{w_\Delta w_P *}(\pi \otimes t)$ is W,P^{op}-regular and the Langlands subrepresentation $\tilde{L}(P, \pi, t)$ from Proposition 4.4 is the image of $I\big(w_P w_\Delta, P^{op}, \psi_{w_\Delta w_P *}(\pi \otimes t)\big)$.

Proof. First we establish the regularity in all four cases.
(i) Recall from [Lan, Lemma 4.4] that every $\lambda \in Y \otimes_{\mathbb{Z}} \mathbb{R}$ can be expressed uniquely as

$$\lambda = \lambda_- + \lambda_+, \quad \text{where} \quad \lambda \in (Y \otimes_{\mathbb{Z}} \mathbb{R})_{Q-},\ \lambda_+ \in (Y \otimes_{\mathbb{Z}} \mathbb{R})^{Q+},\ Q = Q(\lambda) \subset \Delta.$$

For any $s \in \mathrm{Wt}(\pi)$ we have

$$\log|st|_+ = \log|t|,\ \log|st|_- = \log|s| \text{ and } Q(\log|st|) = P. \tag{4.13}$$

Assume that $w_1 \in W_P, w_2 \in W^{P,P} \setminus \{e\}$ and $w_1 w_2 st = s't \in \mathrm{Wt}(\pi \otimes t)$. Then

$$w_2 st = w_1^{-1} s' t \quad \text{and} \quad \log|w_2 st|_+ = \log|w_1^{-1} s' t|_+. \tag{4.14}$$

By [KrRa, p. 38] $\log|w_1^{-1}s't|_+ \geq \log|s't|_+$ by [KrRa, (2.13)] $\log|w_2 st|_+ < \log|st|_+$. Together with (4.13) we obtain

$$\log|w_2 st|_+ < \log|st|_+ = \log|s't| \leq \log|w_1^{-1}s't|_+.$$

That contradicts (4.13) and hence our assumption is untenable. In other words, $\pi \otimes t$ is W,P-regular.
(ii) Notice that the proof of (i) only involves root systems and Weyl groups, no Hecke algebras. It can also be applied to the current $\pi \otimes t$, when we replace the basis Δ of R by $-\Delta$.
(iii) As $\psi_{w_\Delta w_P} : \mathcal{H}^P \to \mathcal{H}^{P^{op}}$ is an isomorphism that respects all the structure of

these affine Hecke algebras, $\psi_{w_\Delta w_P}(\pi)$ is tempered and $w_\Delta w_P(t) \in T^{P^{op}}$. For $\alpha \in \Delta$ there are equalities

(4.15)
$$\langle \alpha, \log |w_\Delta w_P t| \rangle = \langle w_\Delta(\alpha), \log |w_P t| \rangle = \langle w_\Delta(\alpha), \log |t| \rangle = \langle -w_\Delta(\alpha), \log |t^{-1}| \rangle,$$

where we used $t \in T^P$ in the second step. If $\alpha \in \Delta \setminus P^{op}$, then $-w_\Delta \alpha \in \Delta \setminus P$ and (4.15) is strictly positive because $t^{-1} \in T^{P+}$. Therefore $w_\Delta w_P(t) \in T^{P^{op}+}$. Now part (i) says that

$$\psi_{w_\Delta w_P *}(\pi \otimes t) = \psi_{w_\Delta w_P *}(\pi) \otimes w_\Delta w_P(t)$$

is W,P^{op}-regular.
(iv) With the same method as for (iii) this can be reduced to part (ii).
In the cases (i) and (ii), Theorem 4.7.a says that $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)$ has a unique irreducible quotient, which moreover has the given shape. In the cases (iii) and (iv) Theorem 4.7.b says that $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)$ has a unique irreducible subrepresentation, namely the image of the indicated intertwining operator. □

5. Comparison with Hermitian duals for reductive p-adic groups

Consider a non-archimedean local field F and a reductive group G over F, connected in the Zariski topology. We briefly call G a reductive p-adic group. As is well-known, affine Hecke algebras often arise from Bernstein blocks in the category $\mathrm{Rep}(G)$ of smooth complex G-representations. In such a situation there are two notions of a Hermitian dual: in $\mathrm{Rep}(G)$ and in the module category of the appropriate affine Hecke algebra. We will show that in many such cases the two Hermitian duals agree.

Let M be a Levi factor of a parabolic subgroup P of G, and let $\sigma \in \mathrm{Irr}(G)$ be supercuspidal. The inertial equivalence class $\mathfrak{s} = [M, \sigma]_G$ determines a Bernstein block of $\mathrm{Rep}(G)^{\mathfrak{s}}$ in $\mathrm{Rep}(G)$. By tensoring with a suitable unramified character we may assume that σ is unitary. Then its smooth Hermitian dual $\sigma^\dagger$ can be identified with σ itself. It is not hard to show that the smooth Hermitian dual functor stabilizes every Bernstein block in $\mathrm{Rep}(G)$, see [Sol7, Lemma 2.2].

One way to relate $\mathrm{Rep}(G)^{\mathfrak{s}}$ to Hecke algebras stems from [Hei]. Let $M^1 \subset M$ be the subgroup generated by all compact subgroups of M, and let σ_1 be an irreducible subrepresentation of $\mathrm{Res}^M_{M^1}(\sigma)$. Let ind denote smooth induction with compact supports, in contrast to Ind, which will denote smooth induction without any support condition.

Then $\Pi_{\mathfrak{s}_M} = \mathrm{ind}_{M^1}^M(\sigma_1)$ is a progenerator of $\mathrm{Rep}(M)^{\mathfrak{s}_M}$, where $\mathfrak{s}_M = [M,\sigma]_M$. Moreover $\Pi_{\mathfrak{s}} = I_P^G \Pi_{\mathfrak{s}_M}$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$, see [Ren, §VI.10.1]. As worked out in [Roc, Theorem 1.8.2.1], there is an equivalence of categories

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \to & \mathrm{Mod}(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}) \\ \pi & \mapsto & \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi) \end{array} . \tag{5.1}$$

It is known from [Sol6] that $\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}$ is always very similar to an affine Hecke algebra. To stay within the setting of the paper we assume in the remainder of this section:

Condition 5.1. $\mathrm{Res}^M_{M^1}(\sigma)$ is multiplicity-free and $\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}$ is isomorphic to an affine Hecke algebra $\mathcal{H}$ with q-parameters in $\mathbb{R}_{\geq 1}$, via an isomorphism as in [Hei] or [Sol6, §10.2].

This condition and [Sol6, §10] imply that $\mathrm{End}_M(\Pi_{\mathfrak{s}_M})^{op}$ is isomorphic to the minimal parabolic subalgebra $\mathcal{H}^{\emptyset} \cong \mathbb{C}[X]$ of $\mathcal{H}$. By [Ren, (IV.2.1.2)] there are isomorphisms

(5.2)
$$\Pi_{\mathfrak{s}}^\dagger = I_P^G\big(\mathrm{ind}_{M^1}^M(\sigma_1)\big)^\dagger \cong I_P^G\big(\mathrm{ind}_{M^1}^M(\sigma_1)^\dagger\big) \cong I_P^G\big(\mathrm{Ind}_{M^1}^M(\sigma_1^\dagger)\big) \cong I_P^G\big(\mathrm{Ind}_{M^1}^M(\sigma_1)\big).$$

In particular $\Pi_{\mathfrak{s}}^\dagger$ contains $\Pi_{\mathfrak{s}}$ as a dense submodule. The action of $\mathcal{H}^{op} = \mathrm{End}_G(\Pi_{\mathfrak{s}})$ on $\Pi_{\mathfrak{s}}$ extends to an action on $\Pi_{\mathfrak{s}}^\dagger$ in the following way. Write $v \in \Pi_{\mathfrak{s}}^\dagger$ as a limit of elements $v_n \in \Pi_{\mathfrak{s}}$. For $h \in \mathcal{H}^{op}$ we define $h \cdot v = \lim_{n\to\infty} h \cdot v_n$.

Then $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \Pi_{\mathfrak{s}}^\dagger)$ becomes an $\mathcal{H} \times \mathcal{H}^{op}$-module with action

$$h \cdot f \cdot h' = h' \circ f \circ h \qquad h, h' \in \mathcal{H} \cong \mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}, f : \Pi_{\mathfrak{s}} \to \Pi_{\mathfrak{s}}^\dagger. \tag{5.3}$$

Proposition 5.2. *Assuming Condition 5.1,* $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \Pi_{\mathfrak{s}}^\dagger)$ *is isomorphic to* $\mathcal{H}^\dagger$ *as* $\mathcal{H}$*-bimodules.*

Proof. First we consider the supercuspidal case, so with $\Pi_{\mathfrak{s}_M} = \mathrm{ind}_{M^1}^M(\sigma_1)$. With Frobenius reciprocity we compute

(5.4)
$$\mathrm{Hom}_M\big(\Pi_{\mathfrak{s}_M}, \Pi_{\mathfrak{s}_M}^\dagger\big) = \mathrm{Hom}_M\big(\mathrm{ind}_{M^1}^M(\sigma), \mathrm{ind}_{M^1}^M(\sigma)^\dagger\big) \cong \mathrm{Hom}_{M^1}\big(\sigma, \mathrm{ind}_{M^1}^M(\sigma_1)^\dagger\big).$$

Hermitian duals turn ind into Ind, so the right-hand side of (5.4) is also

$$\mathrm{Hom}_{M^1}\big(\sigma_1, \mathrm{Ind}_{M^1}^M(\sigma_1^\dagger)\big) \cong \mathrm{Hom}_{M^1}\big(\sigma_1, \mathrm{Ind}_{M^1}^M(\sigma_1)\big). \tag{5.5}$$

An analogous computation (with ind instead of Ind) applies to $\mathrm{End}_M(\Pi_{\mathfrak{s}_M})$. We note that the set

$$M^\sigma = \{m \in M : m \cdot \sigma_1 \cong \sigma_1\}$$

is a finite index subgroup of M which does not depend on the choice of σ_1, see [Roc, §1.6]. With the Mackey decomposition we obtain

$$\mathcal{A} \cong \mathrm{End}_M(\Pi_{\mathfrak{s}_M})^{op} \cong \bigoplus\nolimits_{m \in M^\sigma / M^1} \mathrm{Hom}_{M^1}(\sigma_1, m \cdot \sigma_1),$$

so in particular $X = M^\sigma / M^1$. Similarly (5.4) and (5.5) become

$$\mathrm{Hom}_{M^1}\big(\sigma_1, \mathrm{Ind}_{M^1}^M(\sigma_1)\big) \cong \prod\nolimits_{m \in M^\sigma / M^1} \mathrm{Hom}_{M^1}(\sigma_1, m \cdot \sigma_1). \tag{5.6}$$

This is isomorphic to $\mathcal{A}^\dagger \cong \prod_{x \in X} \mathbb{C}\{x\}$ as $\mathbb{C}[X]$-bimodule, so

$$\mathrm{Hom}_M(\Pi_{\mathfrak{s}_M}, \Pi_{\mathfrak{s}_M}^\dagger) \cong \mathcal{A}^\dagger. \tag{5.7}$$

In the non-supercuspidal case (5.2) gives an $\mathcal{H}$-isomorphism

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \Pi_{\mathfrak{s}}^\dagger) \cong \mathrm{Hom}_G\big(\Pi_{\mathfrak{s}}, I_P^G(\mathrm{ind}_{M^1}^M(\sigma_1)^\dagger)\big). \tag{5.8}$$

By [Roc, Proposition 1.8.5.1] the right-hand side is naturally isomorphic with

$$\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}} \mathrm{Hom}_M\big(\Pi_{\mathfrak{s}_M}, \mathrm{ind}_{M^1}^M(\sigma_1)^\dagger\big).$$

From the supercuspidal case we know that this $\mathcal{H}$-module is isomorphic with $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}}(\mathcal{A}^\dagger)$, which by Lemma 2.6 is isomorphic with $\mathcal{H}^\dagger$. It remains to see that the resulting $\mathcal{H}$-module isomorphism

$$\phi : \mathcal{H}^\dagger \to \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \Pi_{\mathfrak{s}}^\dagger) \tag{5.9}$$

is an isomorphism of $\mathcal{H}$-bimodules. We already knew from (2.3) that $\mathcal{H} \subset \mathcal{H}^\dagger$. On the $\mathcal{H}$-submodule $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \Pi_{\mathfrak{s}})$ of $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \Pi_{\mathfrak{s}}^\dagger)$, the isomorphisms (5.8), (5.4), (5.5) and (5.6) become just the identity. Hence (5.9) extends the given algebra isomorphism $\mathcal{H} \cong \mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}$.

By Proposition 2.7, any element h^+ of $\mathcal{H}^\dagger$ admits a unique expression as

$$h^+ = \sum\nolimits_{w \in W} \sum\nolimits_{x \in X} c_{w,x} N_w \imath(\theta_x) \qquad c_{w,x} \in \mathbb{C}, \tag{5.10}$$

with $\imath$ as in (2.16). The element $h^+_w := \sum_{x \in X} c_{w,x} \theta_x$ belongs to $\mathcal{A}^\dagger$, which via (5.7) and I_P^G embeds naturally in $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \Pi_{\mathfrak{s}}^\dagger)$. From (5.6) we see that

$$\phi\imath(h^+_w) = \phi\imath\big(\sum\nolimits_{x \in X} c_{w,x}\theta_x\big) = \sum\nolimits_{x \in X} c_{w,x} \phi\imath(\theta_x).$$

Using the $\mathcal{H}$-linearity of ϕ we find

$$\begin{aligned} \phi(h^+) = \phi\big(\sum_{w \in W} N_w \imath(h^+_w)\big) &= \sum_{w \in W} N_w \cdot \phi\imath(h^+_w) = \sum_{w \in W} N_w \cdot \sum_{x \in X} c_{w,x}\phi\imath(\theta_x) \\ = \sum_{w \in W} \phi(N_w) \sum_{x \in X} c_{w,x}\phi\imath(\theta_x) &= \sum_{w \in W}\sum_{x \in X} c_{w,x} \phi(N_w \imath(\theta_x)). \end{aligned} \tag{5.11}$$

This shows that ϕ commutes with infinite sums of elements of $\mathcal{H}$ in the form (5.10). Hence ϕ commutes with limits of sequences in $\mathcal{H}$ that converge in $\mathcal{H}^\dagger$. Let $(h_n^+)_{n=1}^\infty$ be a sequence in $\mathcal{H}$ with limit $h^+ \in \mathcal{H}^\dagger$. For any $h \in \mathcal{H}$, (5.11) yields

$$\phi(h^+ h) = \phi\big(\lim_{n \to \infty} h_n^+ h\big) = \lim_{n \to \infty} \phi(h_n^+ h).$$

As $\phi|_{\mathcal{H}}$ is an algebra homomorphism, the right-hand side equals

$$\lim_{n \to \infty} \phi(h_n^+)\phi(h) = \lim_{n \to \infty} \phi(h_n^+) \cdot h = \phi(h^+) \cdot h,$$

where the dot indicates the right action of $\mathcal{H}$ on $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \Pi_{\mathfrak{s}}^\dagger)$ from (5.3). $\square$

Proposition 5.2 serves as the starting point for the next result. It says that the Hermitian dual functors in $\mathrm{Rep}(G)^{\mathfrak{s}}$ and in $\mathrm{Mod}(\mathcal{H})$ match via (5.1).

Theorem 5.3. *Let $\pi \in \mathrm{Rep}(G)^{\mathfrak{s}}$ and assume Condition 5.1. Then the $\mathcal{H}$-modules $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger)$ and $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)^\dagger$ are isomorphic.*

Proof. First we consider the special case where $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)$ is a finitely generated $\mathcal{H}$-module. Since $\mathcal{H}$ is Noetherian, there exists a projective resolution

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi) \xleftarrow{d_0} \mathcal{H} \otimes F_0 \xleftarrow{d_1} \mathcal{H} \otimes F_1 \leftarrow \cdots \tag{5.12}$$

where each $\mathcal{H} \otimes F_i$ is a free $\mathcal{H}$-module with a finite dimensional multiplicity space F_i. We note that here $d_i (i > 0)$ is determined entirely by the map

$$d_i|_{F_i} : F_i \cong \mathbb{C}1 \otimes F_i \to \mathcal{H} \otimes F_{i-1}.$$

The conjugate-transpose of (5.12) is an injective resolution

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)^\dagger \xrightarrow{d_0^\dagger} \mathcal{H}^\dagger \otimes F_0^\dagger \xrightarrow{d_1^\dagger} \mathcal{H}^\dagger \otimes F_1^\dagger \to \cdots \tag{5.13}$$

Via the equivalence of categories (5.1), (5.12) becomes a projective resolution of G-representations

$$\pi \xleftarrow{d_0} \Pi_{\mathfrak{s}} \otimes F_0 \xleftarrow{d_1} \Pi_{\mathfrak{s}} \otimes F_1 \leftarrow \cdots \tag{5.14}$$

Here $d_i (i > 0)$ is determined by the map $d_i|_{F_i} : F_i \to \mathrm{End}_G(\Pi_{\mathfrak{s}}) \otimes F_{i-1}$ given by $d_i|_{F_i}$ above composed with $\mathcal{H} \cong \mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}$. The conjugate-transpose of (5.14) is the injective resolution

$$\pi^\dagger \xrightarrow{d_0^\dagger} \Pi_{\mathfrak{s}}^\dagger \otimes F_0^\dagger \xrightarrow{d_1^\dagger} \Pi_{\mathfrak{s}}^\dagger \otimes F_1^\dagger \to \cdots \tag{5.15}$$

in $\mathrm{Rep}(G)^{\mathfrak{s}}$. Again applying (5.1), we obtain an injective resolution

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger) \xrightarrow{\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, d_0^\dagger)} \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \Pi_{\mathfrak{s}}^\dagger) \otimes F_0^\dagger \xrightarrow{\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, d_1^\dagger)} \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \Pi_{\mathfrak{s}}^\dagger) \otimes F_1^\dagger \to \cdots \tag{5.16}$$

The maps $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, d_i^\dagger)$ are still determined by $d_i|_{F_i}$. To (5.16) we can apply Proposition 5.2, that yields

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger) \to \mathcal{H}^\dagger \otimes F_0^\dagger \to \mathcal{H}^\dagger \otimes F_1^\dagger \to \cdots \tag{5.17}$$

The maps in this sequence (except the leftmost) are induced by $d_i|_{F_i}$, so they equal the maps in (5.13). We deduce isomorphisms of $\mathcal{H}$-modules

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)^\dagger \cong \ker\big(\mathcal{H}^\dagger \otimes F_0^\dagger \to \mathcal{H}^\dagger \otimes F_1^\dagger\big) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger).$$

Now we consider the general case. Write $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)$ as the direct limit of its finitely generated submodules $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi_i)$, where i runs through some index set. Then $\pi \cong \varinjlim \pi_i$ and $\pi^\dagger \cong \varprojlim \pi_i^\dagger$, which gives

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger) \cong \varprojlim \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi_i^\dagger).$$

By (5.17), the right-hand side is isomorphic to

$$\varprojlim \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi_i)^\dagger \cong \big(\varinjlim \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi_i)\big)^\dagger \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)^\dagger. \qquad \square$$

Theorem 5.3 implies among others that the equivalence of categories (5.1) sends Hermitian representations (i.e. $\pi^\dagger \cong \pi$) to Hermitian representations.

6. Equivalent characterizations of genericity

Recall that a G-representation π is called generic if there exist

- a nondegenerate character ξ of the unipotent radical U of a minimal parabolic subgroup B of G,
- a nonzero U-homomorphism from π to ξ.

More precisely, π is (U, ξ)-generic if

$$\mathrm{Hom}_G\big(\pi, \mathrm{Ind}_U^G(\xi)\big) \cong \mathrm{Hom}_U(\pi, \xi) \quad \text{is nonzero.}$$

Following [BuHe], we say that π is simply generic if $\dim \mathrm{Hom}_U(\pi, xi) = 1$. For irreducible representations of quasi-split reductive p-adic groups, simple genericity is equivalent to genericity [Shal, Rod].

From [BuHe, (2.1.1)] we know that $\mathrm{ind}_U^G(\xi)^\dagger \cong \mathrm{Ind}_U^G(\xi)$, and hence there is a natural conjugate-linear isomorphism

$$\mathrm{Hom}_G\big(\pi, \mathrm{Ind}_U^G(\xi)\big) \cong \mathrm{Hom}_G(\mathrm{ind}_U^G(\xi), \pi^\dagger). \tag{6.1}$$

By (5.1) the right-hand side of (6.1) is isomorphic to

$$\mathrm{Hom}_{\mathrm{End}_G(\Pi_{\mathfrak{s}})}\big(\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)), \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi^\dagger)\big) \tag{6.2}$$

Recall that $\mathfrak{s} = [M,\sigma]_G$ and notice that ξ restricts to a nondegenerate character of $U \cap M$. Suppose now that $\pi \in \mathrm{Rep}(G)^{\mathfrak{s}}$ and assume Condition 5.1. By Theorem 5.3, (6.2) is isomorphic to

$$\mathrm{Hom}_{\mathcal{H}}\big(\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)), \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \pi)^\dagger\big).$$

The explicit structure of affine Hecke algebras (in comparison with reductive p-adic groups) will make it possible to characterize genericity of π much more simply than above. Recall that the Steinberg representation of $\mathrm{St} : \mathcal{H}(W,q^\lambda) \to \mathbb{C}$ is given by $\mathrm{St}(N_s) = -q_s^{-1/2}$ for every simple reflection $s \in W$.

Part (a) of the next result stems from [BuHe]. We include it because it compares well with part (b), which generalizes [ChSa, MiPa].

Theorem 6.1. *(a) Suppose that σ is not $(U \cap M, \xi)$-generic. Then $\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) = 0$ and no object of $\mathrm{Rep}(G)^{\mathfrak{s}}$ is (U,ξ)-generic.*
(b) Assume that σ is simply $(U \cap M, \xi)$-generic and assume Condition 5.1. Then

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathrm{ind}_{\mathcal{H}(W,q^\lambda)}^{\mathcal{H}}(\mathrm{St}) \quad \text{as} \quad \mathcal{H}\text{-modules.}$$

Proof. Recall that $\Pi_{\mathfrak{s}} = I_P^G(\mathrm{ind}_{M^1}^M(\sigma_1))$ for an irreducible subrepresentation σ_1 of $\mathrm{Res}^M_{M^1}(\sigma_1)$. By [BuHe, Theorem 2.2] there is a natural isomorphism $J_{\overline{P}}^G \mathrm{ind}_U^G(\xi) \cong \mathrm{ind}_{U\cap M}^M(\xi)$. With Bernstein's second adjointness we find

$$\begin{aligned} \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) &= \mathrm{Hom}_G\big(I_P^G(\mathrm{ind}_{M^1}^M(\sigma_1)), \mathrm{ind}_U^G(\xi)\big) \\ &\cong \mathrm{Hom}_M\big(\mathrm{ind}_{M^1}^M(\sigma_1), \mathrm{ind}_{U\cap M}^M(\xi)\big). \end{aligned} \tag{6.3}$$

(a) The non-genericity of σ implies, by [BuHe, Corollary 4.2], that the component of $\mathrm{ind}_{U\cap M}^M(\xi)$ in $\mathrm{Rep}(M)^{\mathfrak{s}_M}$ is zero. Hence (6.3) reduces to 0. That and (6.1)–(6.2) imply the second claim of part (a).
(b) Since $U \cap M \subset M^1$, there is a unique irreducible constituent σ_1 of $\mathrm{Res}^M_{M^1}(\sigma)$ such that $\mathrm{Hom}_{U\cap M}(\sigma_1, \xi)$ is nonzero. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{U\cap M}(\sigma_1,\xi) = 1 \text{ and } \sigma_1 \text{ appears with multiplicity 1 in } \mathrm{Res}^M_{M^1}(\sigma). \tag{6.4}$$

Now [BuHe, Proposition 9.2] says that

$$\mathrm{ind}_{U\cap M}^M(\xi) \cong \mathrm{ind}_{M^1}^M(\sigma_1).$$

By Condition 5.1 $\mathrm{End}_M(\mathfrak{s}_M) \cong \mathcal{A} \cong \mathbb{C}[X]$. Hence (6.3) is isomorphic to

$$\mathrm{End}_M(\mathrm{ind}_{M^1}^M(\sigma_1)) \cong \mathrm{End}_M(\Pi_{\mathfrak{s}_M}) \cong \mathbb{C}[X] \tag{6.5}$$

as modules for $\mathrm{End}_M(\mathfrak{s}_M) \cong \mathbb{C}[X]$. That makes our setup is almost the same as in [Sol7, §2], which means that the arguments from there remain valid in our setting. Then [Sol7, Lemma 3.1] proves the theorem. □

The Hermitian duals of the representations in Theorem 6.1 can be described in various ways, which has interesting consequences.

Proposition 6.2. *Assume Condition 5.1 and that σ is simply $(U \cap M, \xi)$-generic.*
(a) There are isomorphisms of $\mathcal{H}$-modules

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{Ind}_U^G(\xi)) \cong \mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi))^\dagger \cong \mathrm{ind}_{\mathcal{H}(W,q^\lambda)}^{\mathcal{H}}(\mathrm{St})^\dagger \cong \mathcal{H}^\dagger \underset{\mathcal{H}(W,q^\lambda)}{\otimes} \mathrm{St}.$$

(*b*) *For* $\pi \in \mathrm{Rep}(G)^{\mathfrak{s}}$ *there are isomorphisms of complex vector spaces*

$$\begin{aligned}\mathrm{Hom}_U(\pi,\xi) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G(\xi)) &\cong \mathrm{Hom}_{\mathcal{H}}\big(\mathrm{Hom}_G(\Pi_{\mathfrak{s}},\pi), \mathcal{H}^\dagger \underset{\mathcal{H}(W,q^\lambda)}{\otimes} \mathrm{St}\big) \\ &\cong \mathrm{Hom}_{\mathcal{H}(W,q^\lambda)}(\mathrm{Hom}_G(\Pi_{\mathfrak{s}},\pi), \mathrm{St}).\end{aligned}$$

(*c*) *A representation* $\pi \in \mathrm{Rep}(G)^{\mathfrak{s}}$ *is* (U,ξ)*-generic if and only if the* $\mathcal{H}(W,q^\lambda)$*-module* $\mathrm{Hom}_G(\Pi_{\mathfrak{s}},\pi)$ *contains* St.

Proof. (a) The first isomorphism is an instance of Theorem 5.3, the second is Theorem 6.1 and the third is Lemma 2.1.a for $V = \mathrm{St} = \mathrm{St}^\dagger$.
(b) The first isomorphism is a version of Frobenius reciprocity and the third is Lemma 2.1.b. The second isomorphism follows from the equivalence of categories (5.1) and part (a).
(c) This follows from part (b) and the semisimplicity of $\mathcal{H}(W,q^\lambda)$. □

We note that Proposition 6.2.c is almost the same as [Sol7, Theorem 3.4]. The latter was only proven for representations of finite length, and did not include Proposition 6.2.a,b.

7. Generic representations of affine Hecke algebras

Let us return to a more general setting, where $\mathcal{H}$ is an affine Hecke algebra with q-parameters in $\mathbb{R}_{\geq 1}$, but $\mathcal{H}$ does not have to come from a reductive p-adic group. Motivated by Proposition 6.2, we put

Definition 7.1. An $\mathcal{H}$-module V is generic if and only if $\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}(W,q^\lambda)} V$ contains St.

From this definition the multiplicity one property of generic constituents of standard modules, as in [Shal, Rod] for quasi-split reductive p-adic groups, follows quickly.

Lemma 7.2. *Let* $P \subset \Delta$ *and let* $V \in \mathrm{Mod}(\mathcal{H}^P)$.
(*a*) V *is generic if and only if* $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}} V$ *is generic.*
(*b*) *Suppose* V *is irreducible and generic. Then* $\dim \mathrm{Hom}_{\mathcal{H}(W,q^\lambda)}(\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}} V, \mathrm{St}) = 1$ *and* $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}} V$ *has a unique generic irreducible subquotient. This constituent appears with multiplicity one in* $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}} V$.

Proof. (a) The Bernstein presentation of $\mathcal{H}$ shows that

$$\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}(W,q^\lambda)}\big(\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}} V\big) = \mathrm{ind}_{\mathcal{H}(W_P,q^\lambda)}^{\mathcal{H}(W,q^\lambda)}\big(\mathrm{Res}^{\mathcal{H}^P}_{\mathcal{H}(W_P,q^\lambda)} V\big).$$

Then by Frobenius reciprocity

$$\mathrm{Hom}_{\mathcal{H}(W,q^\lambda)}(\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}} V, \mathrm{St}) \cong \mathrm{Hom}_{\mathcal{H}(W_P,q^\lambda)}(V, \mathrm{St}). \tag{7.1}$$

(b) By [Sol7, Lemma 3.5]

$$\dim \mathrm{Hom}_{\mathcal{H}(W,q^\lambda)}(\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}} V, \mathrm{St}) \leq 1,$$

and by part (a) it is not 0. In view of the semisimplicity of $\mathcal{H}(W,q^\lambda)$, this shows that $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}} V$ contains a unique copy of St, say $\mathbb{C}v$. It follows that $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}} V$ has a generic irreducible subquotient, which appears with multiplicity one. It can be described as $\mathcal{H}v$ modulo the maximal submodule that does not contain v. □

We will investigate when the generic constituent of $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}} V$ is a quotient or a subrepresentation. That is related to the generalized injectivity conjecture [CaSh] about representations of reductive p-adic groups. It asserts that the generic irreducible subquotient of a generic standard representation is always a subrepresentation.

The last isomorphism in Proposition 6.2.b provides a useful alternative (but equivalent) condition for genericity of an $\mathcal{H}$-representation π, namely that

$$\mathrm{Hom}_{\mathcal{H}}(\pi, \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}) \text{ is nonzero.}$$

By Proposition 2.7 there are isomorphisms of $\mathcal{A}$-modules

$$\begin{aligned} \mathrm{Res}^{\mathcal{H}}_{\mathcal{A}}(\mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}) &\cong \mathrm{Res}^{\mathcal{H}}_{\mathcal{A}}(\mathrm{ind}^{\mathcal{H}}_{\mathcal{A}}(\mathcal{A}^\dagger) \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}) \\ &\cong \mathcal{A}^\dagger \otimes_{\mathbb{C}} \mathcal{H}(W,q^\lambda) \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St} \cong \mathcal{A}^\dagger. \end{aligned} \tag{7.2}$$

The composed isomorphism is given explicitly by

$$\begin{array}{ccc} \mathcal{A}^\dagger & \to & \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St} \\ h & \mapsto & \imath(h) \otimes 1 \end{array} ,$$

where $\imath$ is as in (2.16). For $t \in T$ we write

$$f_t = \imath\big(\sum\nolimits_{x \in X} x(t)^{-1} \theta_x \big) \in \mathcal{H}^\dagger .$$

In these terms

$$\mathrm{Hom}_{\mathcal{H}}\big(\mathrm{ind}^{\mathcal{H}}_{\mathcal{A}}(t), \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}\big) \cong \mathrm{Hom}_{\mathcal{A}}\big(t, \mathcal{A}^\dagger\big) \cong \mathbb{C} f_t . \tag{7.3}$$

Lemma 7.3. *Let $P \subset \Delta$ and let π be an irreducible generic $\mathcal{H}^P$-representation.*

(*a*) *For any $t \in T^P$: $\dim \mathrm{Hom}_{\mathcal{H}}\big(\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t), \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}\big) = 1$.*

(*b*) *Let $s \in T$ be a weight of π. Then the image of $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)$ in $\mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}$, via a nonzero $\mathcal{H}$-homomorphism as in part (a), is generated as $\mathcal{H}$-module by $f_{st} \otimes 1$.*

Proof. (a) This follows from Lemmas 2.1.b and 7.2.b.
(b) For any weight s of π, Frobenius reciprocity yields a nonzero (and hence surjective) $\mathcal{H}^P$-homomorphism $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}^P}(s) \to \pi$. Hence the unique (up to scalars) $\mathcal{H}^P$-homomorphism $\pi \to \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}$ can be inflated to a nonzero $\mathcal{H}$-homomorphism $\phi_s : \mathrm{ind}_{\mathcal{A}}^{\mathcal{H}^P}(s) \to \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}$. Similarly the unique (up to scalars) homomorphism $\pi \otimes t \to \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}$ inflates to

$$\phi_{st} : \mathrm{ind}_{\mathcal{A}}^{\mathcal{H}^P}(st) \to \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}.$$

By (7.3) we may assume that

$$\phi_{st}(h \otimes 1) = h(f_{st} \otimes 1).$$

Let v_s be the image of $N_e \otimes 1 \in \mathrm{ind}_{\mathcal{A}}^{\mathcal{H}^P}(s)$ in π, by irreducibility it generates π. Then ϕ_{st} factors through a homomorphism $\pi \otimes t \to \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}$ that sends v_s to $f_{st} \otimes 1$. Frobenius reciprocity produces a homomorphism

$$\mathfrak{Wh}(P, \pi \otimes t, v_s) : \begin{array}{ccc} \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t) & \to & \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St} \\ \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)(h) v_s & \mapsto & h(f_{st} \otimes 1) \end{array} . \tag{7.4}$$

Clearly the image of $\mathfrak{Wh}(P, \pi \otimes t, v_s)$ is generated by $f_{st} \otimes 1$. □

The $\mathfrak{Wh}(P, \pi \otimes t, v_s)$ with $t \in T^P$ form an algebraic family of $\mathcal{H}$-homomorphisms, in the sense that for any fixed $h \in \mathcal{H}$ the image $\mathfrak{Wh}(P, \pi \otimes t, v_s)(h \otimes v_s)$ is a regular function of t.

To analyse the unique irreducible generic subquotient of $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)$, which exists by Lemma 7.2.b, we use a version of Shahidi's local constant [Shah]. Let us set up the intertwining operators $I(\gamma, \pi \otimes t)$ more systematically. For $\alpha \in \Delta$ we define $\imath^\circ_{s_\alpha} \in \mathbb{C}(T)^W \otimes_{\mathcal{O}(T)^W} \mathcal{H}$ by

$$1 + q^{\lambda(\alpha)/2} N_{s_\alpha} = (1 + \imath^\circ_{s_\alpha}) \frac{\theta_\alpha q^{(\lambda(\alpha)+\lambda^*(\alpha))/2} - 1}{\theta_\alpha - 1} \frac{\theta_\alpha q^{(\lambda(\alpha)-\lambda^*(\alpha))/2} + 1}{\theta_\alpha + 1}. \tag{7.5}$$

By [Lus, §5], $s_\alpha \mapsto \imath^\circ_{s_\alpha}$ extends to a group homomorphism

$$W \to \big(\mathbb{C}(T)^W \otimes_{\mathcal{O}(T)^W} \mathcal{H}\big)^\times : w \mapsto \imath^\circ_w. \tag{7.6}$$

These elements provide an algebra isomorphism

$$\begin{array}{ccc} \mathbb{C}(T) \rtimes W & \to & \mathbb{C}(T)^W \otimes_{\mathcal{O}(T)^W} \mathcal{H} \\ fw & \mapsto & f\imath^\circ_w \end{array}.$$

Assume that $P, w(P) \subset \Delta$. It follows from (7.5) that

$$\imath^\circ_w N_{s_\alpha} \imath^\circ_{w^{-1}} = N_{s_{w(\alpha)}} \text{ for all } \alpha \in P.$$

Hence the isomorphism $\psi_w : \mathcal{H}^P \to \mathcal{H}^{w(P)}$ equals conjugation by $\imath^\circ_w$ in $\mathbb{C}(T)^W \otimes_{\mathcal{O}(T)^W} \mathcal{H}$. Let $t \in T^P$ and $(\pi, V_\pi) \in \mathrm{Mod}(\mathcal{H}^P)$. Consider the bijection

$$\begin{array}{ccc} \mathbb{C}(T)^W \otimes_{\mathcal{O}(T)^W} \mathcal{H} \otimes_{\mathcal{H}^P} V_\pi & \to & \mathbb{C}(T)^W \otimes_{\mathcal{O}(T)^W} \mathcal{H} \otimes_{\mathcal{H}^{w(P)}} V_\pi \\ h \otimes v & \mapsto & h\imath^\circ_{w^{-1}} \otimes v \end{array},$$

where V_π is endowed with the representation $\pi \otimes t$. For t in a Zariski-open dense subset of T^P, this defines an intertwining operator

$$I(w, P, \pi, t) : \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t) \to \mathrm{ind}_{\mathcal{H}^{w(P)}}^{\mathcal{H}}(\psi_w(\pi \otimes t)),$$

which is rational as function of $t \in T^P$. By (7.6), whenever $\gamma w(P) \subset \Delta$:

$$I(\gamma, w(P), \psi_w(\pi), w(t)) \circ I(w, P, \pi, t) = I(\gamma w, P, \pi, t). \tag{7.7}$$

The Whittaker functionals from (7.4) satisfy

$$\mathfrak{Wh}(w(P), \psi_w(\pi \otimes t), v_s) \circ I(w, P, \pi, t) \in \mathrm{Hom}_{\mathcal{H}}\big(\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t), \mathcal{H}^\dagger \otimes_{\mathcal{H}(W,q^\lambda)} \mathrm{St}\big),$$

at least away from the poles of $I(w, P, \pi, t)$. By Lemma 7.3 there exists a unique $C(w, P, \pi, t) \in \mathbb{C} \cup \{\infty\}$ such that

$$\mathfrak{Wh}(P, \pi \otimes t, v_s) = C(w, P, \pi, t) \mathfrak{Wh}(w(P), \psi_w(\pi \otimes t), v_s) \circ I(w, P, \pi, t). \tag{7.8}$$

For $\gamma = w^{-1}$, (7.7) implies

$$C(w^{-1}, w(P), \psi_w(t), w(t)) = C(w, P, \pi, t)^{-1}. \tag{7.9}$$

Notice that $C(w, P, \pi, ?)$ is a rational function on T^P, because all the other terms in (7.8) are so. This $C(w, P, \pi, t)$ is the local constant for affine Hecke algebras, analogous to [Shah]. We note that this is based on the normalized intertwining operators that involve $\imath^\circ_w$. An even stronger analogy with [Shah] can be obtained by using intertwining operators based on the elements $\imath_w$ from [Opd2, (4.1)]. The normalization of the intertwining operators does not affect the poles of the local constants.

Lemma 7.4. *Let $\pi \in \mathrm{Irr}(\mathcal{H}^P)$ be generic. Then $C(w,P,\pi,?)$ is regular at $t \in T^P$ if and only if* $\ker I(w,P,\pi,t) \subset \ker \mathfrak{Wh}(P,\pi\otimes t, v_s)$.

Proof. Suppose that $C(w,P,\pi,t) = \infty$. Then

$$\mathfrak{Wh}(w(P),\psi_w(\pi\otimes t),v_s)\circ I(w,P,\pi,t) = 0,$$

so $\mathrm{im}\, I(w,P,\pi,t)$ is not generic. In the short exact sequence

$$0 \to \ker I(w,P,\pi,t) \to \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi\otimes t) \to \mathrm{im}\, I(w,P,\pi,t) \to 0,$$

the middle term is generic by Lemma 7.3. Hence $\mathfrak{Wh}(w,P,\pi\otimes t,v_s)$ does not vanish on $\ker I(w,P,\pi,t)$, and the latter is generic.

When $C(w,P,\pi,t) \in \mathbb{C}^\times$, the equivalence is clear from (7.8).

Suppose that $C(w,P,\pi,t) = 0$. Then $I(w,P,\pi,t)$ has a pole at $t \in T^P$, caused by a pole of $\imath^\circ_{w^{-1}}$. Let f be a holomorphic function on a neighborhood U of t in T^P, such that $f(t')I(w,P,\pi,t')$ is regular and nonzero on U. We can replace (7.8) by

$$\mathfrak{Wh}(P,\pi\otimes t,v_s) = C\,\mathfrak{Wh}(w(P),\psi_w(\pi\otimes t),v_s)\circ f(t)I(w,P,\pi,t), \tag{7.10}$$

for some $C \in \mathbb{C}\cup\{\infty\}$. As $\mathfrak{Wh}(P,\pi\otimes t,v_s)$ is nonzero, so is C. Then (7.10) shows that $\mathfrak{Wh}(w(P),\psi_w(\pi\otimes t),v_s)$ is nonzero on $\mathrm{im}\, f(t)I(w,P,\pi,t)$. Therefore $C \neq \infty$, and we conclude that $\mathfrak{Wh}(P,\pi\otimes t,v_s)$ factors through $f(t)I(w,P,\pi,t)$. In particular

$$\ker I(w,P,\pi,t) \;\subset\; \ker f(t)I(w,P,\pi,t) \;\subset\; \ker \mathfrak{Wh}(P,\pi\otimes t,v_s). \qquad \square$$

Next we prove some cases of the generalized injectivity conjecture for affine Hecke algebras. For that we need $q_s \geq 1$ for all $s \in S_{\mathrm{aff}}$, otherwise the statement would be false (the Steinberg representation of $\mathcal{H}$ would violate it).

Theorem 7.5. *Assume that $\lambda(\alpha) \geq \lambda^*(\alpha) \geq 0$ for all $\alpha \in R$. Let $t \in T^P$ and let $\pi \in \mathrm{Irr}(\mathcal{H}^P)$ be generic, tempered and anti-tempered.*

(*a*) *When $t^{-1} \in T^{P+}$, the unique generic irreducible subquotient of $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi\otimes t)$ is*

$$L(P,\pi\otimes t) = \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi\otimes t)\big/\ker I(w_\Delta w_P,P,\pi\otimes t).$$

(*b*) *When $t \in T^{P+}$, the unique generic irreducible subquotient of $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi\otimes t)$ is its unique irreducible subrepresentation $\tilde{L}(P,\pi,t)$.*

Proof. (a) In view of Proposition 4.8.ii and Lemma 7.4, it suffices to check that $C(w_\Delta w_P,P,\pi,?)$ does not have a pole at t. By (7.9), it is equivalent to show that

$$C(w_P w_\Delta, P^{op}, \psi_{w_\Delta w_P}(\pi), w_\Delta w_P(t)) \text{ is nonzero.} \tag{7.11}$$

Suppose that (7.11) is zero. By (7.8), $I(w_P w_\Delta, P^{op}, \psi_{w_\Delta w_P}(\pi), w_\Delta w_P(t))$ has a pole at t. It is known from [Opd2, Theorem 4.33.i] that every such t is a zero of

$$\big(\alpha(w_\Delta w_P(rt)) - q^{(\lambda(\alpha)+\lambda^*(\alpha))/2}\big)\big(\alpha(w_\Delta w_P(rt)) + q^{(\lambda(\alpha)-\lambda^*(\alpha))/2}\big),$$

for some weight r of π and $-\alpha, w_P w_\Delta(\alpha) \in R^+$. Equivalently, such a t satisfies

$$\big(\beta(rt) - q^{(\lambda(\alpha)+\lambda^*(\alpha))/2}\big)\big(\beta(rt) + q^{(\lambda(\alpha)-\lambda^*(\alpha))/2}\big) = 0 \tag{7.12}$$

for some weight r of π and $\beta, -w_\Delta w_P(\beta) \in R^+$. The eligible β are precisely the roots in $R^+ \setminus R_P^+$. From $t^{-1} \in T^{P+}$ we get $\beta(t) \in (0,1)$ for all $\beta \in R^+ \setminus R_P^+$. By the temperedness and anti-temperedness of π, $|r| = 1$. Hence $|\beta(rt)| < 1$, which in combination with $\lambda(\alpha) \geq \lambda^*(\alpha) \geq 0$ implies that (7.12) never holds under our assumptions. Hence (7.11) is valid.

(b) In the proof of Proposition 4.8.iii we checked that $\psi_{w_\Delta w_P}(\pi)$ is again tempered and anti-tempered, and that $w_\Delta w_P(t)^{-1} \in T^{P^{op}+}$. By part (a)

$$\mathrm{ind}_{\mathcal{H}^{P^{op}}}^{\mathcal{H}} \psi_{w_\Delta w_P}(\pi \otimes t) / \ker I(w_P w_\Delta, \psi_{w_\Delta w_P}(\pi \otimes t)) \cong \mathrm{im}\, I(w_P w_\Delta, \psi_{w_\Delta w_P}(\pi \otimes t))$$

is generic. This is an irreducible subrepresentation of $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\pi \otimes t)$, and from Proposition 4.8.iv we know that there exists only one such subquotient. □

One interesting application of Theorem 7.5 concerns the induction of suitable characters of $\mathcal{A}$.

Proposition 7.6. *Suppose that $\lambda(\alpha) \geq \lambda^*(\alpha) \geq 0$ for all $\alpha \in R$, and let $t \in T$.*

(*a*) *Suppose $|t^{-1}|$ lies in the closure of $T^{\emptyset+}$. Then the unique generic irreducible constituent of $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}}(t)$ is a quotient.*

(*b*) *Suppose $|t|$ lies in the closure of $T^{\emptyset+}$. Then the unique generic irreducible constituent of $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}}(t)$ is a subrepresentation.*

Proof. Write $P = \{\alpha \in \Delta : |\alpha(t)| = 1\}$. Then $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}^P}(t|t|^{-1})$ is an $\mathcal{H}^P$-representation all whose $\mathcal{A}$-weights belong to $\mathrm{Hom}_{\mathbb{Z}}(X, S^1)$, so it is both tempered and anti-tempered. By [Sol2, Proposition 3.1.4.a] $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}^P}(t|t|^{-1})$ is completely reducible, say a direct sum of irreducible subrepresentations ρ_i. As $|t| \in T^P$:

$$\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}^P}(t) = \mathrm{ind}_{\mathcal{A}}^{\mathcal{H}^P}(t|t|^{-1}) \otimes |t| = \bigoplus\nolimits_i \rho_i \otimes |t|, \tag{7.13}$$

where all the weights of $\rho_i \otimes |t|$ have absolute value $|t|$. Lemma 7.3.a guarantees that exactly one of the $\rho_i \otimes |t|$ is generic, say it is $\rho_1 \otimes |t|$. Now the arguments for the two parts diverge:

(a) By Theorem 7.5.a $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\rho_1 \otimes |t|)$ has a generic irreducible quotient π. In view of (7.13) and the transitivity of parabolic induction, π is also a quotient of $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}}(t)$.

(b) By Theorem 7.5.b $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\rho_1 \otimes |t|)$ has a generic irreducible subrepresentation π. By (7.13) π is also a subrepresentation of $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}}(t)$. □

Proposition 7.6 is a Hecke algebra version of the generalized injectivity conjecture for inductions of supercuspidal representations of reductive p-adic groups [CaSh, Theorem 1]. For the current status of the generalized injectivity conjecture we refer to [Dij]. Possibly our Hecke algebra interpretation can be useful to establish more cases. However, the generalized injectivity conjecture does not hold for all affine Hecke algebras with parameters $\lambda(\alpha) \geq \lambda^*(\alpha) \geq 0$, as witnessed by the next example.

Example 7.7. Consider the based root datum

$$\mathcal{R} = \big(\mathbb{Z}^2, B_2, \mathbb{Z}^2, C_2, \{\alpha = e_1 - e_2, \beta = e_2\}\big).$$

We take $\lambda(\beta) = \lambda^*(\beta) = 1$ and $\lambda(\alpha) = \lambda^*(\alpha) = 6$. (It would also work with any number > 2 instead of 6.) The algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q)$ has a one-dimensional discrete series representation δ given by:

- $\mathcal{A}$ acts on δ via the weight $t_\delta = (q^{-5}, q)$,
- $\delta(N_{s_\alpha}) = -q^{-3}$ and $\delta(N_{s_\beta}) = q^{1/2}$.

As $W_{t_\delta} = \{e\}$, δ is the unique irreducible $\mathcal{H}$-representation with $\mathcal{A}$-weight t_δ. Notice that δ is not generic. We will show that δ occurs as the unique irreducible subrepresentation of a standard module.

Let St_α be the Steinberg representation of $\mathcal{H}^{\{\alpha\}}$, a generic discrete series representation with $\mathcal{A}$-weight (q^{-3}, q^3). For $(q^2, q^2) \in T^{\{\alpha\}+}$, $\mathrm{St}_\alpha \otimes (q^2, q^2)$ has the unique $\mathcal{A}$-weight $t = (q^{-1}, q^5)$. By [Sol5, Lemma 3.3] the standard module

$$\pi = \mathrm{ind}_{\mathcal{H}^{\{\alpha\}}}^{\mathcal{H}} \big(\mathrm{St}_\alpha \otimes (q^2, q^2) \big)$$

has set of $\mathcal{A}$-weights

$$W^{\{\alpha\}} t = \{ t, s_\beta t, s_\alpha s_\beta t, s_\beta s_\alpha s_\beta t = t_\delta \}.$$

By considering the invertibility of intertwining operators, one sees that $t, s_\beta t$ and $s_\alpha s_\beta t$ sit together in one irreducible subquotient of π. That representation involves the maximal weight t of π, so by Theorem 4.3.a it is the Langlands quotient $L\big(\{\alpha\}, \mathrm{St}_\alpha, (q^2, q^2)\big)$. Further t_δ is not a weight of $L\big(\{\alpha\}, \mathrm{St}_\alpha, (q^2, q^2)\big)$, because the only irreducible $\mathcal{H}$-representation with that property is δ. Thus π is reducible and has a subquotient δ, which is in fact a subrepresentation because it equals the kernel of $\pi \to L\big(\{\alpha\}, \mathrm{St}_\alpha, (q^2, q^2)\big)$. Lemma 7.2.b says that π has a unique generic irreducible constituent and it is not δ, so it must be the Langlands quotient $L\big(\{\alpha\}, \mathrm{St}_\alpha, (q^2, q^2)\big)$.

A weaker version of the generalized injectivity conjecture is known as the standard module conjecture [CaSh]. It asserts that the Langlands quotient of a generic standard representation is generic if and only if that standard module is irreducible. This has been proven for all quasi-split reductive p-adic groups [HeMu, HeOp]. Using Section 6, one can deduce the standard module conjecture for all affine Hecke algebras whose parameters come from a generic Bernstein component for a quasi-split reductive p-adic group.

Nevertheless, our above counterexample to the generalized injectivity conjecture is also a counterexample to the standard module conjecture for affine Hecke algebras with arbitrary parameters ≥ 1.

8. Affine Hecke algebras extended with finite groups

For comparison with reductive p-adic groups it is useful to consider a slightly larger class of algebras. Let Γ be a finite group acting on the based root datum $\mathcal{R} = (X, R, Y, R^\vee, \Delta)$. Then Γ acts on W by

$$\gamma(s_\alpha) = s_{\gamma\alpha} = \gamma s_\alpha \gamma^{-1} \qquad \alpha \in R,$$

where the conjugation takes place in $\mathrm{Aut}_{\mathbb{Z}}(X)$. This yields a semidirect product $(X \rtimes W) \rtimes \Gamma$. We also suppose that Γ acts on $\mathcal{A} \cong \mathbb{C}[X] \cong \mathcal{O}(T)$, such that the induced action on $\mathcal{A}^\times / \mathbb{C}^\times \cong X$ recovers the given action on X. Thus Γ acts on $T = \mathrm{Irr}(\mathcal{A})$, but it need not fix a point of T.

Further we assume that the label functions $\lambda, \lambda^* : R \to \mathbb{R}$ are Γ-invariant. Then Γ acts on $\mathcal{H}$ by the algebra automorphisms

$$\gamma(N_w \theta_x) = N_{\gamma(w)} \gamma(\theta_x) \qquad \gamma \in \Gamma, w \in W, x \in X.$$

The algebra $\mathcal{H} \rtimes \Gamma = \Gamma \ltimes \mathcal{H}$ has an Iwahori–Matsumoto basis $\{N_w : w \in (X \rtimes W) \rtimes \Gamma\}$ and a Bernstein basis $\{\theta_x N_w : x \in X, w \in W \rtimes \Gamma\}$. The length function of $X \rtimes W$ extends naturally to $X \rtimes (W \rtimes \Gamma)$, and then it becomes zero on Γ. The involution $*$ of $\mathcal{H}$ extends to $\mathcal{H} \rtimes \Gamma$ by $N_\gamma^* = N_{\gamma^{-1}}$ for $\gamma \in \Gamma$. We extend the trace τ of $\mathcal{H}$ to $\mathcal{H} \rtimes \Gamma$ by defining $\tau|_{\mathcal{H} N_\gamma} = 0$ for all $\gamma \in \Gamma \setminus \{e\}$.

More generally we can involve a 2-cocycle $\natural : \Gamma^2 \to \mathbb{C}^\times$. It gives rise to a twisted group algebra $\mathbb{C}[\Gamma, \natural]$, with multiplication rules

$$N_\gamma \cdot N_{\gamma'} = \natural(\gamma, \gamma') N_{\gamma\gamma'}.$$

From that we can build the twisted affine Hecke algebra $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$, which is like $\mathcal{H} \rtimes \Gamma$, only with $\mathbb{C}[\Gamma]$ replaced by $\mathbb{C}[\Gamma, \natural]$. These twisted algebras can also be constructed with central idempotents. Namely, let

$$1 \to Z_\Gamma^+ \to \Gamma^+ \to \Gamma \to 1 \tag{8.1}$$

be a finite central extension, such that the pullback of $\natural$ to Γ^+ splits. Then there exist a minimal idempotent $p_\natural \in \mathbb{C}[Z_\Gamma^+]$ and an algebra isomorphism

$$\phi_\natural : p_\natural \mathbb{C}[\Gamma^+] \to \mathbb{C}[\Gamma, \natural].$$

For each lift $\gamma^+ \in \Gamma^+$ of $\gamma \in \Gamma$, $\phi_\natural(p_\natural N_{\gamma^+}) \in \mathbb{C}^\times N_\gamma$. Then $p_\natural$ is also a central idempotent in $\mathcal{H} \rtimes \Gamma^+$ and

$$\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural] \cong p_\natural(\mathcal{H} \rtimes \Gamma^+) = (\mathcal{H} \rtimes \Gamma^+) p_\natural. \tag{8.2}$$

Since $p_\natural$ comes from a unitary character of Z_Γ^+, it stable under the natural *-operation on $\mathbb{C}[\Gamma^+]$. We define the * on $\mathbb{C}[\Gamma, \natural]$ by

$$\phi(p_\natural N_{\gamma^+})^* = \phi(p_\natural N_{\gamma^+}^*) = \phi(p_\natural N_{\gamma^+}^{-1}).$$

In combination with the * on $\mathcal{H}$, this endows (8.2) with a *-operation. We define the trace on $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ just like for $\mathcal{H} \rtimes \Gamma$.

To deal with parabolic induction, we use a subgroup $\Gamma_P \subset \Gamma$ for each $P \subset \Delta$.

Condition 8.1. (i) $\Gamma_P \subset \Gamma_Q$ whenever $P \subset Q$,

(ii) the action of Γ_P on T stabilizes P, T_P and T^P (and hence normalizes W_P),

(iii) Γ_P acts on T^P by multiplication with elements of the finite group $T_P \cap T^P$,

(iv) if $\gamma \in W \rtimes \Gamma, P \subset \Delta$ and $\gamma(P) \subset \Delta$, then $\gamma \Gamma_P \gamma^{-1} = \Gamma_{\gamma(P)}$,

(v) $\natural$ is trivial on $\Gamma_\emptyset^2$.

Let Γ_P^+ be the inverse image of Γ_P in Γ^+ for the map (8.1), then Condition 8.1 holds for Γ^+ as well. We say that $\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]$ is a parabolic subalgebra of $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$. Notice that $\mathcal{H}^\Delta \rtimes \mathbb{C}[\Gamma_\Delta, \natural] = \mathcal{H} \rtimes \mathbb{C}[\Gamma_\Delta, \natural]$ (but Γ_Δ need not be the whole of Γ). By Condition 8.1.iii, $\Gamma_\emptyset$ acts trivially on $T^\emptyset = T$, so $\Gamma_\emptyset$ acts trivially on $\mathcal{H}$. Together with Condition 8.1.v that implies

$$\mathcal{H}^\emptyset \rtimes \mathbb{C}[\Gamma_\emptyset, \natural] = \mathcal{A} \otimes \mathbb{C}[\Gamma_\emptyset]. \tag{8.3}$$

By Condition 8.1.ii Γ_P stabilizes $P, X \cap \mathbb{Q}P$ and $X \cap (P^\vee)^\perp$. Then Condition 8.1.iii says that Γ_P fixes $\mathbb{Q}X \cap (P^\vee)^\perp \cong \mathbb{Q}X / \mathbb{Q}P$ pointwise. Let us write the action of $\gamma \in \Gamma$ on $\mathcal{A} \cong \mathbb{C}[X]$ as

$$\gamma(\theta_x) = z_\gamma(x) \theta_{\gamma(x)} \qquad \text{where } z_\gamma \in T.$$

For $t \in T^P = \mathrm{Hom}(X / X \cap \mathbb{Q}P, \mathbb{C}^\times), w \in W_P, x \in X$ we compute

$$\begin{aligned} \gamma(\psi_t(\theta_x N_w)) &= \gamma(t(x)\theta_x N_w) = t(x) z_\gamma(x) \theta_{\gamma(x)} N_{\gamma(w)}, \\ \psi_t(\gamma(\theta_x N_w)) &= \psi_t(z_\gamma(x)\theta_{\gamma(x)} N_{\gamma(w)}) = t(\gamma(x)) z_\gamma(x) \theta_{\gamma(x)} N_{\gamma(w)}. \end{aligned} \tag{8.4}$$

These two lines are equal because γ fixes $X / X \cap \mathbb{Q}P$ pointwise, so that $t(\gamma(x)) = t(x)$. Thus $\psi_t \in \mathrm{Aut}(\mathcal{H}^P)$ is Γ_P-equivariant and extends to an automorphism of $\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]$. That enables us to define $\pi \otimes t$ for $\pi \in \mathrm{Mod}(\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural])$ and

$t \in T^P$.

Assuming all the above, we will check what is needed to make the results from the previous sections valid for $\mathcal{H} \rtimes \Gamma$ and for $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$. To ease the notation we will sometimes write things down for $\mathcal{H} \rtimes \Gamma$ and then indicate how they can be generalized to $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$. Of course this means that everywhere we should also replace $\mathcal{H}^P$ by $\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]$ and $\mathcal{H}(W, q^\lambda)$ by $\mathcal{H}(W, q^\lambda) \rtimes \mathbb{C}[\Gamma, \natural]$. The role of W^P can be played by $\Gamma^P W^P$, where $\Gamma^P \subset \Gamma$ is a set of representatives for Γ / Γ_P. Notice that $\Gamma^P W^P$ is a set of shortest length representatives for $W \rtimes \Gamma / W_P \rtimes \Gamma_P$, because

$$\Gamma^P W^P (P) \subset \Gamma^P (R^+) = R^+ .$$

In Lemma 1.1 we replace $h = N_w \theta_x$ by $N_{\gamma w} \theta_x$ and $N_{w^{-1}}$ by $N^*_{\gamma w} = N_{w^{-1}} N^*_\gamma \in \mathbb{C}^\times N_{w^{-1}} N_{\gamma^{-1}}$. For $\gamma \in \Gamma \setminus \Gamma_P$ both $(h^*)^P_e$ and $(h^P_e)^{*_P}$ are zero, while for $\gamma \in \Gamma_P$ the calculations from the proof of Lemma 1.1 remain valid with an extra factor N^*_γ at the right.

In Section 2 and Theorem 3.1 there are few additional complications, almost everything holds just as well for $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$. Only in Lemma 2.2 we need to be careful: the same argument works for $\mathcal{H} \rtimes \Gamma^+$, and from there we can restrict to $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ via (8.2).

To generalize Proposition 3.2 we need some preparations. Let $P, Q \subset \Delta$ and let $D^{P,Q}$ be a set of shortest length representatives for $W_P \Gamma_P \backslash W \Gamma / W_Q \Gamma_Q$. In contrast with $W^{P,Q}$, $D^{P,Q}$ need not be unique. Like in (3.9), every $d \in D^{P,Q}$ gives rise to an algebra isomorphism

$$\begin{array}{cccc} \psi_d : & \mathcal{H}^{d^{-1}(P) \cap Q} \rtimes (d^{-1} \Gamma_P d \cap \Gamma_Q) & \to & \mathcal{H}^{P \cap d(Q)} \rtimes (\Gamma_P \cap d \Gamma_Q d^{-1}) \\ & \theta_x N_w N_\gamma & \mapsto & \theta_{d(x)} N_{dwd^{-1}} N_{d\gamma d^{-1}} \end{array} .$$

Kilmoyer's result (3.11) can be generalized as follows:

Lemma 8.2. *Let $d \in D^{P,Q}$.*

(a) $d^{-1} W_P d \cap W_Q$ equals $W_{d^{-1}(P) \cap Q}$.

(b) $d^{-1}(W_P \rtimes \Gamma_P) d \cap (W_Q \rtimes \Gamma_Q)$ equals $W_{d^{-1}(P) \cap Q} \rtimes (d^{-1} \Gamma_P d \cap \Gamma_Q)$.

Proof. Write $d = \gamma_d w_d$ with $\gamma_d \in \Gamma$ and $w_d \in W$. For $\alpha \in P$ we have $\ell(d s_\alpha) < \ell(d)$, so $d(\alpha) \in R^+$. As $\gamma_d(R^+) = R^+$, also $w_d(\alpha) \in R^+$. For $\alpha \in P$ we have $\ell(s_\beta d) < \ell(d)$, so $R^+ \ni d^{-1}(\beta) = w_d^{-1} \gamma_d^{-1}(\beta)$. Thus $w_d(Q) \subset R^+$ and $w_d^{-1}(\gamma_d^{-1} P) \subset R^+$, which means that $w_d \in W^{\gamma_d^{-1}(P),Q}$.

(a) We compute

$$d^{-1} W_P d \cap W_Q = w_d^{-1} \gamma_d^{-1} W_P \gamma_d w_d \cap W_Q = w_d^{-1} W_{\gamma_d^{-1}(P)} w_d \cap W_Q .$$

By (3.11) the right hand side equals $W_{w_d^{-1} \gamma_d^{-1}(P) \cap Q} = W_{d^{-1}(P) \cap Q}$.

(b) First we note that by Condition 8.1.iv

$$\begin{aligned} d^{-1}(W_P \rtimes \Gamma_P) d &= w_d^{-1} \gamma_d^{-1} (W_P \rtimes \Gamma_P) \gamma_d w_d = w_d^{-1} (\gamma_d^{-1} W_P \gamma_d \rtimes \gamma_d^{-1} \Gamma_P \gamma_d) w_d \\ &= w_d^{-1} (W_{\gamma_d^{-1}(P)} \rtimes \Gamma_{\gamma_d^{-1}(P)}) w_d . \end{aligned}$$

Consider $w_1 \in W_Q, \gamma_1 \in \Gamma_Q, w_2 \in W_{\gamma_d^{-1}(P)}, \gamma_2 \in \Gamma_{\gamma_d^{-1}(P)}$ such that

$$w_1 \gamma_1 = w_d^{-1} w_2 \gamma_2 w_d . \tag{8.5}$$

Via the isomorphism $W \rtimes \Gamma / W \cong \Gamma$ we see that $\gamma_1 = \gamma_2 \in \Gamma_Q \cap \Gamma_{\gamma_d^{-1}(P)}$. Then

$$\gamma_2 w_d \gamma^{-1}(Q) = \gamma_2 w_d(Q) \subset \gamma_2(R^+) = R^+,$$
$$(\gamma_2 w_d \gamma_1)^{-1}(P) = \gamma_1 w_d^{-1} \gamma_2^{-1}(P) = \gamma_1 w_d^{-1}(P) \subset \gamma_1(R^+) = R^+,$$

so $\gamma_2 w_d \gamma_1^{-1} = \gamma_1 w_d \gamma^{-1} \in W^{P,Q}$. Now

$$w_1 = w_d^{-1} w_2 (\gamma_2 w_d \gamma_1^{-1}) \in W_Q \cap w_d^{-1} w_d^{-1} W_P D^{P,Q},$$

which by [Car, Lemma 2.7.2] is only possible when $\gamma_2 w_d \gamma^{-1} = w_d$. Hence

$$w_1 = w_d^{-1} w_2 w_d \in W_Q \cap w_d^{-1} W_{\gamma_d^{-1}(P)} w_d, \tag{8.6}$$

and from (3.11) we know that the right hand side equals $W_{Q \cap d^{-1}(P)}$. From (8.5) and (8.6) we obtain $\gamma_1 = w_d^{-1} \gamma_2 w_d$, so

$$\begin{aligned}(W_Q \rtimes \Gamma_Q) \cap w_d^{-1} (W_{\gamma_d^{-1}(P)} \rtimes \Gamma_{\gamma_d^{-1}(P)}) w_d &= W_{Q \cap \gamma_d^{-1}(P)} \rtimes (\Gamma_Q \cap w_d^{-1} \Gamma_{\gamma_d^{-1}(P)} w_d) \\ = W_{Q \cap d^{-1}(P)} \rtimes (\Gamma_Q \cap w_d \gamma_d^{-1} \Gamma_P \gamma_d w_d) &= W_{Q \cap d^{-1}(P)} \rtimes (\Gamma_Q \cap d^{-1} \Gamma_P d). \quad \square\end{aligned}$$

Let $(\pi, V_\pi) \in \mathrm{Mod}(\mathcal{H}^Q \rtimes \Gamma_Q)$. Analogous to (3.10), $\mathrm{ind}_{\mathcal{H}^Q \rtimes \Gamma_Q}^{\mathcal{H} \rtimes \Gamma}(V_\pi)$ has linear subspaces

$$(\mathrm{Res}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}^P \rtimes \Gamma_P} \mathrm{ind}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}^Q \rtimes \Gamma_Q})_{\leq d}(V_\pi) = \bigoplus_{d' \in D^{P,Q}, d' \leq d} \mathcal{H}(W_P \Gamma_P d' W_Q \Gamma_Q) \mathcal{A} \otimes_{\mathcal{H}^Q \rtimes \Gamma_Q} V_\pi.$$

With Lemma 8.2 at hand, the proof of Proposition 3.2 becomes valid for $\mathcal{H} \rtimes \Gamma$. The above also works for $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$, that is only a notational difference. The result is:

Proposition 8.3. *For each $d \in D^{P,Q}$, $(\mathrm{Res}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}^P \rtimes \Gamma_P} \mathrm{ind}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}^Q \rtimes \Gamma_Q})_{\leq d}(V_\pi)$ is an $\mathcal{H}^P \rtimes \Gamma_P$-submodule of $\mathrm{ind}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}^Q \rtimes \Gamma_Q}(V_\pi)$. There is an isomorphism of $\mathcal{H}^P \rtimes \Gamma_P$-modules*

$$\begin{aligned}&\big(\mathrm{Res}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}^P \rtimes \Gamma_P} \mathrm{ind}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}^Q \rtimes \Gamma_Q}\big)_{\leq d}(V_\pi) / \big(\mathrm{Res}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}^P \rtimes \Gamma_P} \mathrm{ind}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}^Q \rtimes \Gamma_Q}\big)_{< d}(V_\pi) \cong \\ &\qquad \mathrm{ind}^{\mathcal{H}^P \rtimes \Gamma_P}_{\mathcal{H}^{P \cap d(Q)} \rtimes (\Gamma_P \cap d \Gamma_Q d^{-1})} \big(\psi_{d*} \, \mathrm{Res}^{\mathcal{H}^Q \rtimes \Gamma_Q}_{\mathcal{H}^{d^{-1}(P) \cap Q} \rtimes (d^{-1} \Gamma_P d \cap \Gamma_Q)}(V_\pi)\big).\end{aligned}$$

Thus the functor $\mathrm{Res}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}^P \rtimes \Gamma_P} \mathrm{ind}^{\mathcal{H} \rtimes \Gamma}_{\mathcal{H}^Q \rtimes \Gamma_Q}$ has a filtration with successive subquotients

$$\mathrm{ind}^{\mathcal{H}^P \rtimes \Gamma_P}_{\mathcal{H}^{P \cap d(Q)} \rtimes (\Gamma_P \cap d \Gamma_Q d^{-1})} \circ \psi_{d*} \circ \mathrm{Res}^{\mathcal{H}^Q \rtimes \Gamma_Q}_{\mathcal{H}^{d^{-1}(P) \cap Q} \rtimes (d^{-1} \Gamma_P d \cap \Gamma_Q)},$$

where d runs through $D^{P,Q}$.

The same holds with $\mathbb{C}[\Gamma, \natural]$ instead of $\mathbb{C}[\Gamma]$.

In Section 4 the elementary Lemmas 4.1 and 4.2 also hold for $\mathcal{H} \rtimes \Gamma$. However, the Langlands classification and its variations (Theorem 4.3 and Propositions 4.4, 4.8) are just not valid any more in this form. An extension of Theorem 4.3 to $\mathcal{H} \rtimes \Gamma$ was established in [Sol2, Corollary 2.2.5], but it is more involved.

The main issue with the Langlands classification for $\mathcal{H} \rtimes \Gamma$ is the uniqueness, as witnessed by the following example. Let $R = A_2, \Delta = \{\alpha, \beta\}, X = \mathbb{Z}R$ and let $\Gamma = \{e, \gamma\}$ with γ the unique nontrivial automorphism of (X, Δ). The parabolic subalgebras of $\mathcal{H} \rtimes \Gamma$ are $\mathcal{A}, \mathcal{H}^{\{\alpha\}}, \mathcal{H}^{\{\beta\}}$ and $\mathcal{H} \rtimes \Gamma$. Pick a $t \in T^{\emptyset +}$ which is fixed by γ. Then $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}}(t)$ has a unique irreducible quotient but $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H} \rtimes \Gamma}(t)$ has two inequivalent irreducible quotients.

Lemma 4.5 and its proof still work with our standard modifications. However, to generalize Lemma 4.6 and Theorem 4.7 we first have to extend the notion of W,P-regularity. We say that an $\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural]$-representation π is $W\Gamma$,P-regular if $wt \notin \mathrm{Wt}(\pi)$ for all $t \in \mathrm{Wt}(\pi)$ and all $w \in W_P \Gamma_P D_+^{P,P}$, where

$$D_+^{P,P} = \{d \in W\Gamma : d(P) \subset R^+, d^{-1}(P) \subset R^+, d \notin \Gamma_P\}.$$

Lemma 8.4. *Let $P, Q \subset \Delta$ and $\gamma \in W\Gamma$ such that $\gamma(P) = Q$ and let $\pi \in \mathrm{Irr}(\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P, \natural])$ be $W\Gamma$,P-regular.*

(*a*) *The representation π appears with multiplicity one in $\mathrm{Res}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma,\natural]}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P,\natural]} \mathrm{ind}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma,\natural]}_{\mathcal{H}^Q \rtimes \mathbb{C}[\Gamma_Q,\natural]} (\psi_{\gamma *}\pi)$, as a direct summand.*

(*b*) $\dim \mathrm{Hom}_{\mathcal{H} \rtimes \mathbb{C}[\Gamma,\natural]} \big(\mathrm{ind}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma,\natural]}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P,\natural]} (\pi), \mathrm{ind}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma,\natural]}_{\mathcal{H}^Q \rtimes \mathbb{C}[\Gamma_{Q},,\natural]} (\psi_{\gamma *}\pi) \big) = 1.$

Proof. (a) Since $\gamma(P) \subset R^+$ and $\gamma^{-1}(Q) \subset R^+$, γ^{-1} has minimal length in $W_P \Gamma_P \gamma^{-1} W_Q \Gamma_Q$. Hence we may choose $D^{P,Q}$ so that it contains γ^{-1}. We follow the proof of Lemma 4.5 with $d' = \gamma^{-1}$. Instead of (4.6) we find $w_1, w_2 \in W_P \rtimes \Gamma_P$ and $t \in \mathrm{Wt}(\pi)$ such that $w_2^{-1} w_1 d \gamma t \in \mathrm{Wt}(\pi)$. The $W\Gamma$,P-regularity of π says that

$$w_3 w_2^{-1} w_1 d\gamma \notin W_P \Gamma_P D_+^{P,P} \qquad \text{for all } w_3 \in W_P \Gamma_P. \tag{8.7}$$

Notice that $d\gamma(P) = d(Q) \subset R^+$, which means that $d\gamma \in \Gamma W^P$. Suppose that $d\gamma$ does not have minimal length in $W_P d\gamma$. There exists $\alpha \in P$ with $\gamma^{-1} d^{-1}(\alpha) \in -R^+$. Then $\gamma^{-1} d^{-1} s_\alpha(\alpha) \in R^+$ and

$$\ell(s_\alpha d\gamma) = \ell(\gamma^{-1} d^{-1} s_\alpha) < \ell(\gamma^{-1} d^{-1}) = \ell(d\gamma).$$

As $\gamma^{-1}(Q) \subset R^+$, $d^{-1}(\alpha) \notin Q$ and $\alpha \notin d^{-1}(Q)$. That gives

$$(s_\alpha d\gamma)(P) = s_\alpha d(Q) \subset s_\alpha(R^+ \setminus \{\alpha\}) \subset R^+.$$

The reasoning can be applied to $s_\alpha d\gamma$. Repeating that if necessary, we find $w_4 \in W_P$ such that $w_4 d\gamma(P) \subset R^+$ and $w_4 d\gamma$ has minimal length in $W_P d\gamma$. Thus $(w_4 d\gamma)^{-1}(P) \subset R^+$, $w_4 d\gamma \in D_+^{P,P} \cup \Gamma_P$ and $d\gamma \in W_P(D_+^{P,P} \cup \Gamma_P)$. Combining that with (8.7), we find $d\gamma \in W_P \Gamma_P = \Gamma_P W_P$. Also $d\gamma \in \Gamma W^P$, so in fact $d\gamma \in \Gamma_P$. Then $\Gamma_P d = \Gamma_P \gamma^{-1}$, and using $d, \gamma^{-1} \in D^{P,Q}$ we obtain $d = \gamma^{-1}$. From this point on, we can conclude in the same way as in the proof of Lemma 4.6.a.

(b) This can be shown exactly as in the proof of Lemma 4.6. □

Lemma 8.4.c yields a nonzero intertwining operator

$$I(\gamma, P, \pi) : \mathrm{ind}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma,\natural]}_{\mathcal{H}^P \rtimes \mathbb{C}[\Gamma_P,\natural]} (\pi) \to \mathrm{ind}^{\mathcal{H} \rtimes \mathbb{C}[\Gamma,\natural]}_{\mathcal{H}^Q \rtimes \mathbb{C}[\Gamma_Q,\natural]} (\psi_{\gamma *}\pi),$$

unique up to scalars. With those operators Theorem 4.7 and its proof become valid for $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$. That provides $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ with a substitute for the uniqueness of Langlands quotients and Langlands representations (for $\mathcal{H}$). We warn that Proposition 4.8 fails for $\mathcal{H} \rtimes \Gamma$: $\pi \in \mathrm{Irr}(\mathcal{H}^P \rtimes \Gamma_P)$ tempered and $t \in T^{P+}$ does not enforce $W\Gamma$,P-regularity of $\pi \otimes t$.

We may relax Condition 5.1 by replacing $\mathcal{H}$ with $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$, let us call that Condition 5.1'. The advantage is that it becomes valid for more Bernstein components of representations of p-adic groups. For instance, Condition 5.1' applies to all smooth representations of classical groups [Hei, AMS] and in those cases Condition 8.1 follows from the same checks as in [Sol3, §5]. Under Condition 5.1', the indecomposability of $\mathrm{Rep}(M)^{\mathfrak{s}_M}$ forces $\mathrm{Mod}(\mathcal{H}^\emptyset \rtimes \mathbb{C}[\Gamma_\emptyset, \natural])$ to be indecomposable. Hence the algebra (8.3) is also decomposable, which forces $\Gamma_\emptyset = \{1\}$. All the arguments and results in Section 5 remain valid with $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ instead of $\mathcal{H}$, no further adjustments are necessary.

In the setting of Section 6, Condition 5.1' turns out to hold automatically with trivial 2-cocycle, see Theorem A.1. The crucial part of the proof of Theorem 6.1 is the reference to [Sol7, §2]. Since that work was conceived for algebras of the form $\mathcal{H} \rtimes \Gamma$, Theorem 6.1 applies to all extended affine Hecke algebras that satisfy Condition 5.1' with $\natural = 1$. More precisely, we extend the Steinberg representation of $\mathcal{H}(W, q^\lambda)$ to $\mathcal{H}(W, q^\lambda) \rtimes \Gamma$ by

$$\mathrm{St}(N_w N_\gamma) = \mathrm{St}(N_w) \mathrm{det}_X(\gamma) \qquad w \in W, \gamma \in \Gamma,$$

where det_X means the determinant of the action of γ on X. The more general version of Theorem 6.1.b says:

$$\mathrm{Hom}_G(\Pi_{\mathfrak{s}}, \mathrm{ind}_U^G(\xi)) \cong \mathrm{ind}_{\mathcal{H}(W,q^\lambda) \rtimes \Gamma}^{\mathcal{H} \rtimes \Gamma}(\mathrm{St}) \qquad \text{as } \mathcal{H} \rtimes \Gamma\text{-representations.} \tag{8.8}$$

That and Proposition 6.2 prompt us to define:

(8.9) an $\mathcal{H} \rtimes \Gamma$-module V is generic if and only if $\mathrm{Res}_{\mathcal{H}(W,q^\lambda) \rtimes \Gamma}^{\mathcal{H} \rtimes \Gamma}(V)$ contains St.

With this definition, the part from Proposition 6.2 up to and including Lemma 7.4 generalizes readily to $\mathcal{H} \rtimes \Gamma$. For representations of $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$ with $\natural$ nontrivial in $H^2(\Gamma, \mathbb{C}^\times)$, genericity is not defined. In such cases $\mathbb{C}[\Gamma, \natural]$ does not possess onedimensional representations, so we do not have a good analogue of det_X.

Let us discuss the relation between generic representations of $\mathcal{H}$ and of $\mathcal{H} \rtimes \Gamma$. The definition of the Steinberg representation of $\mathcal{H}(W, q^\lambda)$ shows that $\gamma(\mathrm{St}) = \mathrm{St}$ for all $\gamma \in \Gamma$. It follows that

(8.10) the action of Γ on $\mathrm{Mod}(\mathcal{H})$ preserves genericity.

Suppose now that (π, V_π) is an irreducible generic $\mathcal{H}$-representation. By Lemma 7.2.b there exists a unique (up to scalars) vector $v_{\mathrm{St}} \in V_\pi \setminus \{0\}$ on which $\mathcal{H}(W, q^\lambda)$ acts according to St. Let Γ_π be the stabilizer (in Γ) of $\pi \in \mathrm{Irr}(\mathcal{H})$. Schur's lemma says there exists a unique (up to scalars) linear bijection

$$\pi(\gamma) : V_\pi \to V_\pi \text{ such that } \pi(\gamma(h)) = \pi(\gamma)\pi(h)\pi(\gamma)^{-1} \text{ for all } h \in \mathcal{H}.$$

As $\gamma(\mathrm{St}) = \mathrm{St}$, $\pi(\gamma) v_{\mathrm{St}}$ must belong to $\mathbb{C} v_{\mathrm{St}}$. We normalize $\pi(\gamma)$ by the condition $\pi(\gamma) v_{\mathrm{St}} = v_{\mathrm{St}}$. In this way (π, V_π) extends to a representation of $\mathcal{H} \rtimes \Gamma_\pi$.

Clifford theory [RaRa, Appendix] tells us how any irreducible $\mathcal{H} \rtimes \Gamma$-representation containing π can be constructed. Namely, let $(\rho, V_\rho) \in \mathrm{Irr}(\Gamma_\pi)$ and let $\mathcal{H} \rtimes \Gamma_\pi$ act on $V_\pi \otimes V_\rho$ by

$$(h N_\gamma (v_1 \otimes v_2) = \pi(h N_\gamma) v_1 \otimes \rho(\gamma) v_2.$$

Then $\pi \rtimes \rho := \mathrm{ind}_{\mathcal{H} \rtimes \Gamma_\pi}^{\mathcal{H} \rtimes \Gamma}(V_\pi \otimes V_\rho)$ is irreducible and

$$\begin{array}{ccc} \mathrm{Irr}(\Gamma_\pi) & \to & \mathrm{Irr}(\mathcal{H} \rtimes \Gamma), \\ \rho & \mapsto & \pi \rtimes \rho \end{array} \tag{8.11}$$

is injection with as image

$$\{V \in \mathrm{Irr}(\mathcal{H} \rtimes \Gamma) : \mathrm{Res}_{\mathcal{H}}^{\mathcal{H} \rtimes \Gamma}(V) \text{ contains } V_\pi\}. \tag{8.12}$$

As for the genericity of $\pi \rtimes \rho$:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{H}(W,q^\lambda)\rtimes\Gamma}(\pi \rtimes \rho, \mathrm{St}) &= \mathrm{Hom}_{\mathcal{H}(W,q^\lambda)\rtimes\Gamma}(\mathrm{ind}_{\mathcal{H}\rtimes\Gamma_\pi}^{\mathcal{H}\rtimes\Gamma}(V_\pi \otimes V_\rho), \mathrm{St}) \\ &\cong \mathrm{Hom}_{\mathcal{H}(W,q^\lambda)\rtimes\Gamma_\pi}(\pi \otimes \rho, \mathrm{St}).\end{aligned}$$

By Lemma 7.2.b and because $\pi(\Gamma_\pi)$ fixes v_{St}, the last expression is isomorphic with $\mathrm{Hom}_{\Gamma_\pi}(\rho, \det_X)$. We conclude that

$$\pi \rtimes \rho \text{ is } \begin{cases} \text{generic if } \rho = \det_X, \\ \text{not generic otherwise.} \end{cases} \tag{8.13}$$

Conversely, consider an irreducible generic $\mathcal{H} \rtimes \Gamma$-representation (σ, V_σ). Let π be an irreducible $\mathcal{H}$-subrepresentation of σ. Then $\mathrm{ind}_{\mathcal{H}}^{\mathcal{H}\rtimes\Gamma}(\pi)$ surjects onto π, so every irreducible $\mathcal{H}$-subquotient of σ is isomorphic to $\gamma(\pi)$ for some $\gamma \in \Gamma$. As $\mathrm{Res}_{\mathcal{H}(W,q^\lambda)}^{\mathcal{H}\rtimes\Gamma}\sigma$ contains St, at least one of the $\gamma(\pi)$ is generic. In view of (8.8), actually all of them are generic, and in particular π. Then (8.11)–(8.13) show that

$$\sigma \cong \pi \rtimes \det_X = \mathrm{ind}_{\mathcal{H}\rtimes\Gamma_\pi}^{\mathcal{H}\rtimes\Gamma}(\pi \otimes \det_X). \tag{8.14}$$

Next we generalize Theorem 7.5 and Proposition 7.6. Since the statements really change, we formulate them as new results.

Theorem 8.5. *Assume that $\lambda(\alpha) \geq \lambda^*(\alpha) \geq 0$ for all $\alpha \in R$. Let $t \in T^P$ and let $\pi \in \mathrm{Irr}(\mathcal{H}^P \rtimes \Gamma_P)$ be tempered, anti-tempered and generic. The unique generic irreducible constituent of $\mathrm{ind}_{\mathcal{H}^P\rtimes\Gamma_P}^{\mathcal{H}\rtimes\Gamma}(\pi \otimes t)$:*

(a) is a quotient when $t^{-1} \in T^{P+}$,
(b) is a subrepresentation when $t \in T^{P+}$.

Proof. With (8.14) we can write

$$\pi \cong \mathrm{ind}_{\mathcal{H}^P\rtimes\Gamma_{P,\tau}}^{\mathcal{H}^P\rtimes\Gamma_P}(\tau \otimes \det_X),$$

where τ is an irreducible generic $\mathcal{H}^P$-subrepresentation of π. Notice that $\mathrm{Wt}(\tau) \subset \mathrm{Wt}(\pi)$, so that τ is also tempered and anti-tempered. By Condition 8.1.ii,iii:

$$\begin{aligned}\mathrm{ind}_{\mathcal{H}^P\rtimes\Gamma_P}^{\mathcal{H}\rtimes\Gamma}(\pi \otimes t) \;&\cong\; \mathrm{ind}_{\mathcal{H}^P\rtimes\Gamma_P}^{\mathcal{H}\rtimes\Gamma}\big(\mathrm{ind}_{\mathcal{H}^P\rtimes\Gamma_{P,\tau}}^{\mathcal{H}^P\rtimes\Gamma_P}(\tau \otimes \det_X) \otimes t\big) \\ &\cong\; \mathrm{ind}_{\mathcal{H}^P\rtimes\Gamma_P}^{\mathcal{H}\rtimes\Gamma}\big(\mathrm{ind}_{\mathcal{H}^P\rtimes\Gamma_{P,\tau}}^{\mathcal{H}^P\rtimes\Gamma_P}(\tau \otimes t \otimes \det_X)\big) \\ &\cong\; \mathrm{ind}_{\mathcal{H}^P\rtimes\Gamma_{P,\tau}}^{\mathcal{H}\rtimes\Gamma}(\tau \otimes t \otimes \det_X) \;\cong\; \mathrm{ind}_{\mathcal{H}\rtimes\Gamma_{P,\tau}}^{\mathcal{H}\rtimes\Gamma}\big(\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\tau \otimes t) \otimes \det_X\big).\end{aligned} \tag{8.15}$$

(a) Theorem 7.5.a says that the quotient $L(P, \tau \otimes t)$ of $\mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H}}(\tau \otimes t)$ is generic. By the uniqueness in Theorem 4.3.b, $\Gamma_{L(P,\tau,t)} \cap \Gamma_P = \Gamma_{P,\tau\otimes t}$, which by the remarks following (8.8) equals $\Gamma_{P,\tau}$. From (8.13) we know that

$$L(P, \tau, t) \rtimes \det_X = \mathrm{ind}_{\mathcal{H}\rtimes\Gamma_{P,\tau}}^{\mathcal{H}\rtimes\Gamma}(L(P, \tau, t) \otimes \det_X) \quad \text{is generic.} \tag{8.16}$$

Clearly (8.16) is a quotient of the final term in (8.15).
(b) This is analogous to part (a), instead of $L(P, \tau, t)$ we use $\tilde{L}(P, \tau, t)$ from Proposition 4.4. □

Proposition 8.6. *Assume that $\lambda(\alpha) \geq \lambda^*(\alpha) \geq 0$ for all $\alpha \in R$, and let $t \in T$. The unique generic irreducible constituent of $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}\rtimes\Gamma}(t)$:*

(a) is a quotient if $|t^{-1}|$ lies in the closure of $T^{\emptyset+}$,
(b) is a subrepresentation if $|t|$ lies in the closure of $T^{\emptyset+}$.

Proof. (a) Proposition 7.6.a says that $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}}(t)$ has a generic irreducible quotient, say π. By (8.13), $\pi \rtimes \det_X$ is the unique generic irreducible $\mathcal{H} \rtimes \Gamma$-representation that contains π. With Frobenius reciprocity we compute

$$(8.17)\quad \mathrm{Hom}_{\mathcal{H}\rtimes\Gamma}\big(\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}\rtimes\Gamma}(t), \pi \rtimes \det_X\big) \cong \mathrm{Hom}_{\mathcal{A}}(t, \pi \rtimes \det_X) = \mathrm{Hom}_{\mathcal{A}}(t, \mathrm{ind}_{\mathcal{A}\rtimes\Gamma_\pi}^{\mathcal{A}\rtimes\Gamma} \pi).$$

The right hand side of (8.17) contains

$$\mathrm{Hom}_{\mathcal{A}}(t, \pi) \cong \mathrm{Hom}_{\mathcal{H}}(\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}}(t), \pi) \neq 0.$$

Hence (8.17) is nonzero, which means that $\pi \rtimes \det_X$ is a quotient of $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}\rtimes\Gamma}(t)$.
(b) Proposition 7.6.b yields a generic irreducible subrepresentation of $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}}(t)$, say σ. From (8.13) is a generic irreducible $\mathcal{H} \rtimes \Gamma$-representation. We compute

$$\begin{aligned}(8.18)\quad \mathrm{Hom}_{\mathcal{H}\rtimes\Gamma}\big(\sigma \rtimes \det_X, \mathrm{ind}_{\mathcal{A}}^{\mathcal{H}\rtimes\Gamma}(t)\big) &= \mathrm{Hom}_{\mathcal{H}\rtimes\Gamma}\big(\mathrm{ind}_{\mathcal{H}\rtimes\Gamma_\sigma}^{\mathcal{H}\rtimes\Gamma}(\sigma \otimes \det_X), \mathrm{ind}_{\mathcal{A}}^{\mathcal{H}\rtimes\Gamma}(t)\big) \\ \cong \mathrm{Hom}_{\mathcal{H}\rtimes\Gamma_\sigma}\big(\sigma \otimes \det_X, \mathrm{ind}_{\mathcal{A}}^{\mathcal{H}\rtimes\Gamma}(t)\big) &\supset \mathrm{Hom}_{\mathcal{H}\rtimes\Gamma_\sigma}\big(\sigma \otimes \det_X, \mathrm{ind}_{\mathcal{A}}^{\mathcal{H}\rtimes\Gamma_\sigma}(t)\big).\end{aligned}$$

It follows from Clifford theory, in the version [Sol1, Theorem 11.2], that

$$\mathrm{ind}_{\mathcal{H}}^{\mathcal{H}\rtimes\Gamma_\sigma}(\sigma) \cong \bigoplus\nolimits_{\rho\in\mathrm{Irr}(\Gamma_\sigma)} (\sigma \otimes \rho)^{\oplus \dim \rho}.$$

Hence there exist injective $\mathcal{H} \rtimes \Gamma_\sigma$-homomorphisms

$$\sigma \otimes \det_X \to \mathrm{ind}_{\mathcal{H}}^{\mathcal{H}\rtimes\Gamma_\sigma}(\sigma) \to \mathrm{ind}_{\mathcal{H}}^{\mathcal{H}\rtimes\Gamma}(\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}} t) = \mathrm{ind}_{\mathcal{A}}^{\mathcal{H}}(t).$$

Thus all terms in (8.18) are nonzero, which by irreducibility means that $\sigma \rtimes \det_X$ is a subrepresentation of $\mathrm{ind}_{\mathcal{A}}^{\mathcal{H}\rtimes\Gamma}(t)$. $\square$

Let us summarize the findings of this section.

Corollary 8.7. *Suppose that Γ is as at the start of Section 8, and assume in particular Condition 8.1. All the results of Sections 2–5 generalize to $\mathcal{H} \rtimes \mathbb{C}[\Gamma, \natural]$, except Theorem 4.3 and Propositions 4.4, 4.8. Sections 6 and 7 generalize to $\mathcal{H} \rtimes \Gamma$.*

Appendix A. Hecke algebras for simply generic Bernstein blocks

Let G be a reductive p-adic group and let U be the unipotent radical of a minimal parabolic subgroup of G. Let ξ be a nondegenerate character of U. Let $P = MU_P$ be a parabolic subgroup of G containing U. Let (σ, E) be an irreducible unitary supercuspidal M-representation which is simply $(U \cap M, \xi)$-generic, that is,

$$\dim \mathrm{Hom}_{U\cap M}(\sigma, \xi) = 1.$$

We call $\mathrm{Rep}(G)^{\mathfrak{s}}$ with $\mathfrak{s} = [M, \sigma]$ a simply generic Bernstein block for G, because most irreducible representations in there are simply (U, ξ)-generic. In this appendix we show that $\mathrm{Rep}(G)^{\mathfrak{s}}$ is equivalent to the module category of an extended affine Hecke algebra.

Let (σ_1, E_1) be the unique irreducible generic M^1-subrepresentation from (6.4). Recall from [Ren, §VI.10.1] that $\Pi_{\mathfrak{s}} = I_P^G(\mathrm{ind}_{M^1}^M(\sigma_1))$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$. By abstract category theory [Roc, Theorem 1.8.2.1], $\mathrm{Rep}(G)^{\mathfrak{s}}$ is naturally equivalent with $\mathrm{Mod}(\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op})$.

Theorem A.1. *In the above simply generic setting, $\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}$ is isomorphic to an extended affine Hecke algebra $\mathcal{H} \rtimes \Gamma$ with q-parameters in $\mathbb{R}_{\geq 1}$. Conditions 5.1' and 8.1 are satisfied.*

Proof. We follow [Sol6, §10], with some improvements that are made possible by the simple genericity of σ. Notice that [Sol6, Working hypothesis 10.2] holds by (6.4). On the supercuspidal level with $\Pi_{\mathfrak{s}_M} = \mathrm{ind}_{M^1}^M(\sigma_1)$, [Sol6, Lemma 10.1] says that

$$\mathrm{End}_M(\Pi_{\mathfrak{s}_M}) = \mathrm{End}_M(\Pi_{\mathfrak{s}_M})^{op} = \mathbb{C}[\mathcal{O}_3] = \mathbb{C}[M_\sigma/M^1], \tag{A.1}$$

where M_σ is stabilizer of E_1 in M. In [Sol6, Lemma 10.3] the multiplicity one of σ_1 in σ implies that the operator $\rho_{\sigma,w} : E \to E$ automatically stabilizes E_1. Therefore we may choose as the element $m_w \in M$ from [Sol6, Lemma 10.3.a] just the identity element. We do that that for all w in the group

$$W(M,\mathcal{O}) = W(\Sigma_{\mathcal{O},\mu}) \rtimes R(\mathcal{O})$$

from [Sol6], which will play the role of $W \rtimes \Gamma$. With that simplification, the 2-cocycle $\natural_J : W(M,\mathcal{O})^2 \to \mathbb{C}^\times \times M_\sigma/M^1$ takes values in $\mathbb{C}^\times$. Then [Sol6, Theorem 10.9] gives:

- an affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{O}, G)$, with lattice $M_\sigma/M^1 = X^*(\mathcal{O}_3)$ and a reduced root system $\Sigma_{\mathcal{O},\mu}$,
- parameters $q_\alpha = q_F^{(\lambda(\alpha)+\lambda^*(\alpha))/2}$ and $q_{\alpha*} = q_F^{(\lambda(\alpha)-\lambda^*(\alpha))/2}$ with $1 \neq q_\alpha \geq q_{\alpha*} \geq 1$ for all $\alpha \in \Sigma_{\mathcal{O},\mu}$,
- elements T'_r for $r \in R(\mathcal{O})$, such that as vector spaces

$$\mathrm{End}_G(\Pi_{\mathfrak{s}}) = \bigoplus\nolimits_{r \in R(\mathcal{O})} \mathcal{H}\, T'_r.$$

From [Sol6, (10.20) and Lemma 10.4.a] we see that these T'_r multiply as in the twisted group algebra $\mathbb{C}[R(\mathcal{O}), \natural_J]$. Conjugation by T'_r is an automorphism of $\mathcal{H}(\mathcal{O}, G)$, which by [Sol6, Theorem 10.6.a] has the desired effect on $\mathcal{A} \cong \mathbb{C}[\mathcal{O}_3]$. For a simple root α, [Sol6, (10.24)] shows that

$$T_r'^{-1} T'_{s_\alpha} T_r \in \mathbb{C}1 + \mathbb{C}T'_{r^{-1}s_\alpha r}.$$

From that and the quadratic relations that T_{s_α} and $T'_{r^{-1}s_\alpha r} = T'_{s_{r^{-1}\alpha}}$ satisfy, we deduce that $T_r'^{-1} T'_{s_\alpha} T_r$ must equal $T'_{r^{-1}s_\alpha r}$. Altogether this shows that $\mathrm{End}_G(\Pi_{\mathfrak{s}})$ is the twisted affine Hecke algebra $\mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_J]$. There is an isomorphism

$$(\mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_J])^{op} \to \mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_J^{-1}]$$

which is the identity on $\mathcal{A}$ and sends each T'_w with $w \in W(M,\mathcal{O})$ to $T_w'^{-1}$. Thus

$$\mathrm{End}_G(\Pi_{\mathfrak{s}})^{op} \cong \mathcal{H}(\mathcal{O}, G) \rtimes \mathbb{C}[R(\mathcal{O}), \natural_J^{-1}]. \tag{A.2}$$

By (6.3), (6.5) and (A.1)

$$\mathrm{Hom}_M(\Pi_{\mathfrak{s}_M}, \mathrm{ind}_U^G(\xi)) \cong \mathrm{End}_M(\mathrm{ind}_{M^1}^M(\sigma_1)) \cong \mathbb{C}[M_\sigma/M^1]. \tag{A.3}$$

That brings us almost to the setting of [Sol7, §2], with (A.2) and (A.3) the arguments from there work. In particular the Whittaker datum (U, ξ) can be used to normalize the operators T'_w with $w \in W(M,\mathcal{O})$, and [Sol7, Theorem 2.7] provides canonical algebra isomorphisms

$$\mathrm{End}_G(\Pi_{\mathfrak{s}}) \cong \mathcal{H}(\mathcal{O}, G) \rtimes R(\mathcal{O}) \cong (\mathcal{H}(\mathcal{O}, G) \rtimes R(\mathcal{O}))^{op} \cong \mathrm{End}_G(\Pi_{\mathfrak{s}})^{op}.$$

That also finishes the verification of Condition 5.1'. Condition 8.1 was checked in [Sol3, §5]. □

We specialize to the cases where G is quasi-split. It turns out that the q-parameters from Theorem A.1 have an interesting property, which means that $\mathcal{H}$ is close to an affine Hecke algebra with equal parameters.

We may assume that σ corresponds to the basepoint of $\mathcal{O}_3$ in the proof of Theorem A.1, so that all $\alpha \in \Sigma_{\mathcal{O},\mu}$ take the value 1 at σ. Let $\sigma' = \sigma \otimes \chi$ be a twist of σ by a unitary unramified character of χ of M. Via $M_\sigma \subset M$ we can consider χ as a character of the lattice M_σ / M^1 involved in $\mathcal{H}$. We define a set of roots (in fact a root system) $\Sigma_{\sigma'} \subset \Sigma_{\mathcal{O},\mu}$ and a parameter function $k^{\sigma'}$ by

- if $s_\alpha(\sigma') = \sigma'$ and $\chi(\alpha) = 1$, then $\alpha \in \Sigma_{\sigma'}$ and $k^{\sigma'}_\alpha = \log(q_\alpha)/\log(q_F)$,
- if $s_\alpha(\sigma') = \sigma'$, $\chi(\alpha) = -1$ and $q_{\alpha*} \neq 1$, then $\alpha \in \Sigma_{\sigma'}$ and $k^{\sigma'}_\alpha = \log(q_{\alpha*})/\log(q_F)$,
- $\alpha \notin \Sigma_{\sigma'}$ for other $\alpha \in \Sigma_{\mathcal{O},\mu}$.

With [Lus, Lemma 3.15] is not difficult to see that

$$\Sigma^e_{\sigma'} = \{\alpha \in \Sigma_{\mathcal{O},\mu} : s_\alpha(\sigma') = \sigma'\}$$

is a root system and that $\chi(\alpha) \in \{\pm 1\}$ for every $\alpha \in \Sigma^e_{\sigma'}$. By the $W(\Sigma_{\mathcal{O},\mu})$-invariance of λ and λ^*, the function $k^{\sigma'}$ is $W(\Sigma^e_{\sigma'})$-invariant. The set $\Sigma_{\sigma'}$ is obtained from $\Sigma^e_{\sigma'}$ by omitting the $W(\Sigma^e_{\sigma'})$-stable collection of roots with $\chi(\alpha) = -1$ and $q_{\alpha*} = 1$. All such roots are short in a type B irreducible component of $\Sigma^e_{\sigma'}$. Thus, for each irreducible component R^e of $\Sigma^e_{\sigma'}$, the part in $\Sigma_{\sigma'}$ is either R^e or the set of long roots in R^e. This shows that $\Sigma_{\sigma'}$ is really a root system.

By $W(\Sigma_{\sigma'})$-invariance, the function $k^{\sigma'}$ takes the same value on all roots of a fixed length in one irreducible component.

Proposition A.2. *Let G be quasi-split and recall the notations from Theorem A.1 and above. Let R be an irreducible component of $\Sigma_{\sigma'}$, let $\alpha \in R$ be short and let $\beta \in R$ be long. Then $k^{\sigma'}_\alpha / k^{\sigma'}_\beta$ equals either 1 or the square of the ratio of the lengths of the coroots $\alpha^\vee$ and $\beta^\vee$ (so equals 1, 2 or 3).*

Proof. We recall from [Sol6, (3.7)] that the parameters q_α and $q_{\alpha*}$ in the proof of Theorem A.1 come from poles of Harish-Chandra's function μ^α. In the notation from [Sol6], μ^α has factors

$$\frac{(1 - X_\alpha)}{(1 - q_\alpha^{-1} X_\alpha)} \frac{(1 + X_\alpha)}{(1 + q_{\alpha*}^{-1} X_\alpha)}, \tag{A.4}$$

where X_α corresponds to evaluation at a certain element $h^\vee_\alpha \in M/M^1$. In [HeOp] one specializes to twists of σ' by unramified characters with values in $\mathbb{R}_{>0}$, which means the only the left half or the right half in (A.4) remains interesting, the other half is put in a holomorphic function and then ignored. Which of the two halves to chose agrees with how we selected q_α or $q_{\alpha*}$ for $k^{\sigma'}$. Thus, in the notation of [HeOp, §3], (A.4) becomes a factor

$$\big(1 - q_F^{\langle \nu, \alpha^\vee \rangle}\big) / \big(1 - q_F^{-1/\epsilon_{\bar\alpha} + \langle \nu, \alpha^\vee \rangle}\big). \tag{A.5}$$

Hence q_α or $q_{\alpha*}$ from [Sol6] equals $q_F^{1/\epsilon_{\bar\alpha}}$ from [HeOp], and $k^{\sigma'}_\alpha = 1/\epsilon_{\bar\alpha}$. Now we need to prove that $\epsilon_{\bar\alpha}/\epsilon_{\bar\beta}$ equals 1 or the square of the ratio of the lengths of α and β. That is precisely the condition needed in [HeOp, Theorem 4.1]. It was shown to hold for all generic Bernstein blocks of quasi-split reductive p-adic groups in [HeOp, §5–6]. □

Proposition A.2 enables one to reduce the representation theory of $\mathcal{H} \rtimes \Gamma$ (as in Theorem A.1) to extended graded Hecke algebras with equal parameters, via [Lus, §8–9] or [Sol2, §2.1].

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